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A METHOD FOR PROVING SOME INEQUALITIES ON MIXED HYPERBOLIC–TRIGONOMETRIC POLYNOMIAL FUNCTIONS

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Abstract. In this article we present a method for proving inequalities of the form

$$f(x)=\sum_{i=1}^{n}\alpha_i x^{p_i}\sinh^{q_i}x\cosh^{r_i}x>0,$$

for $x\in(\delta_1,\delta_2)$, $\delta_1\leqslant 0\leqslant\delta_2$; where $\alpha_i\in\mathbb{R}\setminus\{0\}$, $p_i,q_i,r_i\in\mathbb{N}_0$ and $n\in\mathbb{N}$. The method is based on the precise approximations of the hyperbolic sine and hyperbolic cosine functions by Maclaurin polynomials. Using this method we present new proofs of some well-known inequalities, but also we prove some new inequalities. Inequalities involving hyperbolic functions are much less studied than inequalities involving trigonometric functions. In this paper, the method described in the article [7] has been adapted to the inequalities involving hyperbolic functions.

1. Introduction

In this article we consider a method for proving inequalities of the form:

$$f(x)=\sum_{i=1}^{n}\alpha_i x^{p_i}\sinh^{q_i}x\cosh^{r_i}x>0, \tag{1}$$

for $x\in(\delta_1,\delta_2)$, $\delta_1\leqslant 0\leqslant\delta_2$; where $\alpha_i\in\mathbb{R}\setminus\{0\}$, $p_i,q_i,r_i\in\mathbb{N}_0$ and $n\in\mathbb{N}$. We transform the function $f(x)$ into the mixed hyperbolic-trigonometric polynomial function, whose monomials are mixed by some variable and hyperbolic sine or hyperbolic cosine functions applied to the same variable. Our method is compatible with the effective method from the article [7] and it is based on the direct comparison of the hyperbolic sine and hyperbolic cosine functions with the corresponding Maclaurin polynomials. From a computational point of view, automated proving of inequalities is described in both methods.

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2. The description of the method

2.1. Preliminaries

Let the function $\varphi(x)$ be approximated by Taylor polynomial $T_k(x)$ of degree k in the neighbourhood of a point a. If there is $\eta > 0$ such that in the interval $(a-\eta, a+\eta)$ it holds:

$$T_k(x) \geqslant \varphi(x),$$

then we introduce the symbol $\overline{T}_k^{\varphi,a}(x) = T_k(x)$ and we call $\overline{T}_k^{\varphi,a}(x)$ upward approximation of the function $\varphi(x)$ in the neighbourhood of the point a. Analogously, if there is $\eta > 0$ such that in the interval $(a-\eta, a+\eta)$ it holds:

$$T_k(x) \leqslant \varphi(x),$$

then we introduce the symbol $\underline{T}_k^{\varphi,a}(x) = T_k(x)$ and we call $\underline{T}_k^{\varphi,a}(x)$ downward approximation of the function $\varphi(x)$ in the neighbourhood of the point a. Further on, we consider only the cases when the function $\varphi(x)$ is either $\sinh x$ or $\cosh x$.

Observing Maclaurin approximations of $\sinh x$ and $\cosh x$, we get the following lemmas:

LEMMA 2.1. *For the polynomial* $T_n(t) = \displaystyle\sum_{i=0}^{(n-1)/2} \frac{t^{2i+1}}{(2i+1)!}$, *where* $n = 2k+1$, $k \in \mathbb{N}_0$, *it is valid:*

$$(\forall t \geqslant 0) \;\; T_n(t) \leqslant T_{n+2}(t) \leqslant \sinh t, \tag{2}$$

$$(\forall t \leqslant 0) \;\; T_n(t) \geqslant T_{n+2}(t) \geqslant \sinh t. \tag{3}$$

For $t = 0$ *the inequalities in* (2) *and* (3) *turn into equalities.*

LEMMA 2.2. *For the polynomial* $T_n(t) = \displaystyle\sum_{i=0}^{n/2} \frac{t^{2i}}{(2i)!}$, *where* $n = 2k$, $k \in \mathbb{N}_0$, *it is valid:*

$$(\forall t \in \mathbb{R}) \;\; T_n(t) \leqslant T_{n+2}(t) \leqslant \cosh t. \tag{4}$$

For $t = 0$ *the inequalities in* (4) *turn into equalities.*

Let $a = e^x$, $x \in \mathbb{R}$. Then it holds:

$$\cosh x = \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{and} \quad \sinh x = \frac{1}{2}\left(a - \frac{1}{a}\right). \tag{5}$$

Let us introduce the following functions:

$$R_k(a) = a^k + \frac{1}{a^k} \quad \text{and} \quad Q_k(a) = a^k - \frac{1}{a^k}, \tag{6}$$

for $k = 1, 2, \ldots$. Then it is:

$$R_k(a) = 2\cosh(kx) \quad \text{and} \quad Q_k(a) = 2\sinh(kx), \tag{7}$$

for $a = e^x$ and $k = 1, 2, \dots$. Hence, it is easy to prove:

$$R_n(a) \cdot R_m(a) = R_{n+m}(a) + R_{|n-m|}(a) \tag{8}$$

and

$$R_n(a) \cdot Q_m(a) = Q_{n+m}(a) + \nu \cdot Q_{|n-m|}(a), \tag{9}$$

where $\nu = \operatorname{sgn}(m-n)$. Especially, $R_0(a) = 2$ and $Q_0(a) = 0$.

According to [3] we have the representation of powers of hyperbolic functions in terms of functions of multiples of the argument (angle), depending on the parity of power n. Hence, the following auxiliary propositions are valid:

LEMMA 2.3. *For $n \in \mathbb{N}$ the following formulas are true:*
(i) *if n is odd*

$$\sinh^n x = \frac{1}{2^{n-1}} \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{k} \sinh((n-2k)x), \tag{10}$$

(ii) *if n is even*

$$\sinh^n x = \frac{(-1)^{\frac{n}{2}}}{2^n} \binom{n}{\frac{n}{2}} + \frac{1}{2^{n-1}} \sum_{k=0}^{\frac{n}{2}-1} (-1)^k \binom{n}{k} \cosh((n-2k)x). \tag{11}$$

LEMMA 2.4. *For $n \in \mathbb{N}$ the following formulas are true:*
(i) *if n is odd*

$$\cosh^n x = \frac{1}{2^{n-1}} \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} \cosh((n-2k)x), \tag{12}$$

(ii) *if n is even*

$$\cosh^n x = \frac{1}{2^n} \binom{n}{\frac{n}{2}} + \frac{1}{2^{n-1}} \sum_{k=0}^{\frac{n}{2}-1} \binom{n}{k} \cosh((n-2k)x). \tag{13}$$

Based on the previous two lemmas the following statement holds:

THEOREM 2.5. *For $n, m \in \mathbb{N}$ we have the following cases:*
(i) *if both n and m are odd*

$$\cosh^n x \cdot \sinh^m x = \tfrac{1}{2^{n+m-1}} \sum_{k=0}^{\frac{n+m}{2}-1} (-1)^k \left(\sum_{r=0}^{k} (-1)^r \binom{n}{r} \binom{m}{k-r} \sinh\left((n+m-2k)x\right) \right), \tag{14}$$

(ii) *if n is even and m is odd*

$$\cosh^n x \cdot \sinh^m x = \tfrac{1}{2^{n+m-1}} \sum_{k=0}^{\frac{n+m-1}{2}} (-1)^k \left(\sum_{r=0}^{k} (-1)^r \binom{n}{r} \binom{m}{k-r} \sinh\left((n+m-2k)x\right) \right), \tag{15}$$

(iii) *if n is odd and m is even*

$$\cosh^n x \cdot \sinh^m x = \frac{1}{2^{n+m-1}} \sum_{k=0}^{\frac{n+m-1}{2}} (-1)^k \left(\sum_{r=0}^{k} (-1)^r \binom{n}{r} \binom{m}{k-r} \cosh\left((n+m-2k)x\right) \right), \tag{16}$$

(iv) *if both n and m are even*

$$\cosh^n x \cdot \sinh^m x = \frac{1}{2^{n+m-1}} \left(\sum_{k=0}^{\frac{n+m}{2}-1} (-1)^k \sum_{r=0}^{k} (-1)^r \binom{n}{r} \binom{m}{k-r} \cosh\left((n+m-2k)x\right) \right. \\ \left. + \frac{1}{2} (-1)^{\frac{m+n}{2}} \sum_{r=0}^{\frac{n+m}{2}} (-1)^r \binom{n}{r} \binom{m}{\frac{n+m}{2}-r} \right). \tag{17}$$

Proof. Analogously to the proof of the Theorem 1.5. from the article [7], whereby for the functions R_k and Q_k we use the terms (6), (7), (8) and (9). □

2.2. The method

The aim is to present a method for proving inequalities of the form (1) for $x \in (0, \delta)$ and $\delta = \delta_2 > 0$. We will use the downward Maclaurin approximations of the hyperbolic sine and hyperbolic cosine determined in the Lemmas 2.1. and 2.2. Let us notice that for the functions $\sinh x$ and $\cosh x$ there are only downward approximations in contrast to the functions $\sin x$ and $\cos x$ which have both, downward and upward approximations [7] (Lemma 1.1. and Lemma 1.2.). However, in the article [8] the following theorem has been proved:

THEOREM 2.6. *Suppose f is a real function on (a,b), n is a positive integer such that $f^{(k)}(a+)$, $f^{(k)}(b-)$, $(k \in \{0,1,2,\ldots,n\})$ exist and $(-1)^n f^{(n)}(x)$ is increasing on (a,b), then for all $x \in (a,b)$ the following inequality holds:*

$$\sum_{k=0}^{n-1} \frac{f^{(k)}(b-)}{k!}(x-b)^k + \frac{1}{(a-b)^n}\left(f(a+) - \sum_{k=0}^{n-1} \frac{(a-b)^k f^{(k)}(b-)}{k!}\right)(x-b)^n \\ < f(x) < \sum_{k=0}^{n} \frac{f^{(k)}(b-)}{k!}(x-b)^k. \tag{18}$$

Furthermore, if $(-1)^n f^{(n)}(x)$ is decreasing on (a,b), then the reversed inequality of (18) holds.

Suppose $f^{(n)}(x)$ is increasing on (a,b), then for all $x \in (a,b)$ the following inequality holds also:

$$\sum_{k=0}^{n-1} \frac{f^{(k)}(a+)}{k!}(x-a)^k + \frac{1}{(b-a)^n}\left(f(b-) - \sum_{k=0}^{n-1} \frac{(b-a)^k f^{(k)}(a+)}{k!}\right)(x-a)^n \\ > f(x) > \sum_{k=0}^{n} \frac{f^{(k)}(a+)}{k!}(x-a)^k. \tag{19}$$

Furthermore, if $f^{(n)}(x)$ is decreasing on (a,b), then the reversed inequality of (19) holds.

By this theorem it has been proved the existence of upward approximations for the functions $\sinh x$ and $\cosh x$, but only in the finite interval (a,b).

Let us observe the addend of the sum (1), $s_i(x) = \alpha_i x^{p_i}\cosh^{q_i} x \sinh^{r_i} x$, where $\alpha_i \neq 0$ for $i = 1,\ldots,n$. Let us introduce the symbol

$$\mathfrak{m}_i = \begin{cases} \frac{q_i+r_i}{2} - 1, & \text{when } q_i \text{ and } r_i \text{ are both even or both odd,} \\ \frac{q_i+r_i-1}{2}, & \text{when } q_i \text{ and } r_i \text{ have different parity.} \end{cases} \tag{20}$$

According to the Theorem 2.5. the addends $s_i(x)$ $(i = 1,2,\ldots,n)$ are represented in four different ways depending on the cases, so the following possibilities are given in the description of the method:

1. Let r_i be odd, then it holds:

$$\begin{aligned} s_i(x) &= \alpha_i x^{p_i}\cosh^{q_i} x \sinh^{r_i} x \\ &= \frac{\alpha_i x^{p_i}}{2^{q_i+r_i-1}} \sum_{k=0}^{\mathfrak{m}_i} (-1)^k \sum_{r=0}^{k} (-1)^r \binom{q_i}{r}\binom{r_i}{k-r} \sinh((q_i+r_i-2k)x) \\ &= \frac{x^{p_i}}{2^{q_i+r_i-1}} \sum_{k=0}^{\mathfrak{m}_i} \left(\sum_{r=0}^{k} \alpha_i (-1)^{k+r} \binom{q_i}{r}\binom{r_i}{k-r} \right) \sinh((q_i+r_i-2k)x). \end{aligned} \tag{21}$$

Let us mark with $\beta_k = \sum_{r=0}^{k} \alpha_i (-1)^{k+r} \binom{q_i}{r}\binom{r_i}{k-r}$.

Then, for every sub-addend $\beta_k \sinh((q_i+r_i-2k)x)$, depending on the sign of β_k, two cases are possible:

1) if $\beta_k > 0$:

$$\beta_k \sinh((q_i+r_i-2k)x) > \beta_k \underline{T}^{\sinh,0}_{2l_k^{(i)}+1}((q_i+r_i-2k)x), \tag{22}$$

2) if $\beta_k < 0$:

There is a polynomial $p_n(x)$ in the finite interval, according to the Theorem 2.6., which is an upward approximation for $\sinh((q_i+r_i-2k)x$, so it holds:

$$\beta_k \sinh((q_i+r_i-2k)x) > \beta_k p_n(x). \tag{23}$$

2. Let q_i be odd and r_i even, then it holds:

$$\begin{aligned} s_i(x) &= \alpha_i x^{p_i}\cosh^{q_i} x \sinh^{r_i} x \\ &= \frac{\alpha_i x^{p_i}}{2^{q_i+r_i-1}} \sum_{k=0}^{\mathfrak{m}_i} (-1)^k \sum_{r=0}^{k} (-1)^r \binom{q_i}{r}\binom{r_i}{k-r} \cosh((q_i+r_i-2k)x) \\ &= \frac{x^{p_i}}{2^{q_i+r_i-1}} \sum_{k=0}^{\mathfrak{m}_i} \left(\sum_{r=0}^{k} \alpha_i (-1)^{k+r} \binom{q_i}{r}\binom{r_i}{k-r} \right) \cosh((q_i+r_i-2k)x). \end{aligned} \tag{24}$$

Then, for every sub-addend $\beta_k \cosh((q_i + r_i - 2k)x)$, depending on the sign of β_k, two cases are possible:

1) if $\beta_k > 0$:

$$\beta_k \cosh((q_i + r_i - 2k)x) > \beta_k \underline{T}^{\cosh,0}_{2l_k^{(i)}+0}((q_i + r_i - 2k)x), \tag{25}$$

2) if $\beta_k < 0$:

There is a polynomial $p_n(x)$ in the finite interval, according to the Theorem 2.6., which is an upward approximation for $\cosh((q_i + r_i - 2k)x$, so it holds:

$$\beta_k \cosh((q_i + r_i - 2k)x) > \beta_k p_n(x). \tag{26}$$

3. Let q_i and r_i be even, then it holds:

$$\begin{aligned} s_i(x) = \tfrac{x^{p_i}}{2^{q_i+r_i-1}} \Bigg(\sum_{k=0}^{m_i} \Bigg(\sum_{r=0}^{k} \alpha_i (-1)^{k+r} \binom{q_i}{r} \binom{r_i}{k-r} \Bigg) \cosh((q_i + r_i - 2k)x) \\ + \tfrac{1}{2} (-1)^{\frac{q_i+r_i}{2}} \sum_{r=0}^{\frac{q_i+r_i}{2}} (-1)^r \binom{q_i}{r} \binom{r_i}{\frac{q_i+r_i}{2} - r} \Bigg). \end{aligned} \tag{27}$$

Further, for every sub-addend $\beta_k \cosh((q_i + r_i - 2k)x)$, depending on the sign of β_k, two cases are possible (25) or (26).

Therefore, for every addend $s_i(x)$ we can find a polynomial $\tau_i(x)$, which represents a downward approximation. Comparing all the addends $s_i(x)$ $(i = 1, 2, \ldots, n)$ that appear in the sum (1), according to the above stated cases, we get the polynomial

$$P(x) = \sum_{i=1}^{n} \tau_i(x) \tag{28}$$

which is a downward approximation of the function $f(x)$ in (1); i.e. it holds:

$$f(x) > P(x). \tag{29}$$

On the basis of the previous consideration, the following statement is valid:

THEOREM 2.7. *Let the following properties of the polynomial* $P(x) = \sum_{i=1}^{n} \tau_i(x)$ *be true:*

(i) there is at least one positive real root of the polynomial $P(x)$*;*

(ii) $P(x) > 0$ *for* $x \in (0, x^*)$*, where* x^* *is the least positive real root of the polynomial* $P(x)$*.*

Then it is valid

$$f(x) > 0$$

for $x \in (0, x^*) \subseteq (0, \delta)$.

REMARK 2.8. Let us notice that hereby the proof of the inequality $f(x) > 0$ has been obtained for $x \in (0, \delta_2)$, where $\delta_2 = x^*$. The previous theorem can be applied for $x \in (\delta_1, 0)$ by introducing the substitute $t = -x$.

REMARK 2.9. If there is not at least one positive real root of the polynomial $P(x)$ and $P(x) > 0$ for $x \in (0, \infty)$, then it is valid $f(x) > 0$ for $x \in (0, \infty)$.

The main difference in comparison to the method which has been described in the article [7], is the fact that upward approximations of the functions $\sinh x$ and $\cosh x$ exist only in the finite intervals. Consequently, this method can be used either with inequalities of the type (1) in the finite interval or with inequalities of the same type in the interval $(0, \infty)$ where the coefficients preceding $\sinh x$ and $\cosh x$ are always positive, i.e. when we only need the downward approximations of these functions.

The completeness of the presented method for the function $f(x)$ from (1) is valid analogously as for the function $\sum_{i=1}^{n} \alpha_i x^{p_i} \sin^{q_i} x \cos^{r_i} x$, which has been shown in the article [7] (Section 2., part II).

REMARK 2.10. Let us emphasise that the previous method can be applied to the functions of the form $f(x) = \sum_{i=1}^{n} \alpha_i h_i(x) \cosh^{q_i} x \sinh^{r_i} x$ for $x \in (0, \delta)$, where $h_i(x)$ is a polynomial. The first possibility is when the polynomial $h_i(x)$ is of the constant sign in the given interval ($h_i(x) > 0$ or $h_i(x) < 0$), and we do that by analogy with the previously described procedure. On the other hand, it is possible that the polynomial $h_i(x)$ is not of the constant sign. Then, $\alpha_i h_i(x) \cosh^{q_i} x \sinh^{r_i} x$ can be written as a sum of addends of the form $s_i(x)$, and then we can apply the previously described method for each of those addends.

Let the indexes $l_k^{(i)}$ $\left(i \in \{1, \ldots, n\} \text{ and } k \in \{0, \ldots, \mathrm{m}_i\}\right)$, which appear in the polynomial $P(x)$, be enumerated: $l_0, l_1, \ldots, l_m$; where $m+1$ is the overall number of subaddends which come from every addend $s_i(x)$. Let us notice that depending on the index l_s it holds:

$$f(x) > P(x, l_0, l_1, \ldots, l_s + 1, \ldots, l_m) > P(x, l_0, l_1, \ldots, l_s, \ldots, l_m) \tag{30}$$

for every index $s \in \{0, 1, 2, \ldots, m\}$ and $l_s \in \mathbb{N}_0$. By increasing every index l_s, we get even better and better downward approximations of the function $f(x)$. The previously described method defines a procedure which ends when at least one $(m+1)$-tuple of the indexes $(l_0, l_1, \ldots, l_m) = (\hat{l}_0, \hat{l}_1, \ldots, \hat{l}_m)$ has been determined for which it is valid:

$$P(x, \hat{l}_0, \hat{l}_1, \ldots, \hat{l}_m) > 0 \tag{31}$$

for $x \in (0, \delta)$. By completing the procedure, we get a proof of the initial inequality (1).

REMARK 2.11. The method comes down to proving polynomial inequalities of the form $P(x) > 0$ for $x \in (0, \delta)$ which is a decidable problem according to the results by Tarski [5].

Using this method it is our aim in this article to prove some well-known results that have been considered in the lately published articles and to obtain some new results, concerning the inequalities of the form (1).

3. Some applications

In this section we consider some applications of the method based on the Theorem 2.7. in some concrete inequalities.

3.1. A proof of an inequality from the article [6]

In the article [6] R. Klen, M. Visuri and M. Vuorinen have proved the following statement (Theorem 1.2):

THEOREM 3.1. *For* $x \in (0,1)$

$$\left(\frac{1}{\cosh x}\right)^{1/2} < \frac{x}{\sinh x} < \left(\frac{1}{\cosh x}\right)^{1/4}. \tag{32}$$

Now we present a proof of the inequality (32).

Proof.

I We prove the inequality:

$$\left(\frac{1}{\cosh x}\right)^{1/2} < \frac{x}{\sinh x} \tag{33}$$

for $x \in (0,1)$. The requested inequality is equivalent to $f(x) > 0$ for $x \in (0,1)$, where

$$f(x) = x^2 \cosh x - \sinh^2 x. \tag{34}$$

According to the Theorem 2.5., the function $f(x)$ can be written in the following way:

$$f(x) = x^2 \cosh x + \frac{1}{2} - \frac{1}{2}\cosh 2x, \tag{35}$$

which is a concrete mixed hyperbolic-trigonometric polynomial.

Then, according to the Lemma 2.2. and the description of the method, the inequality $\cosh y > \underline{T}_k^{\cosh,0}(y)$ $(k=4)$ is true, $\forall y \in \mathbb{R}$ and according to the Theorem 2.6. it is valid: $\cosh 2y < 1 + 2y^2 + (2\cosh^2 1 - 4)y^4$, for $y \in (0,1)$.

For $x \in (0,1)$ it is valid:

$$f(x) > x^2 \underline{T}_4^{\cosh,0}(x) - x^2 - (\cosh^2 1 - 2)x^4 = P_6(x), \tag{36}$$

where $P_6(x)$ is the polynomial

$$P_6(x) = \frac{x^4}{24}(x^2 + 60 - 24\cosh^2 1) = \frac{x^4}{24}(x^2 + c), \tag{37}$$

where $c = 2.853\ldots$. Since $P_6(x) > 0$, we conclude that $f(x) > 0$ for $x \in (0,1)$.

II Now we prove the inequality:

$$\frac{x}{\sinh x} < \left(\frac{1}{\cosh x}\right)^{1/4} \tag{38}$$

for $x \in (0,1)$. The requested inequality is equivalent to $f(x) > 0$ for $x \in (0,1)$, where

$$f(x) = \sinh^4 x - x^4 \cosh x. \tag{39}$$

According to the Theorem 2.5., the function $f(x)$ can be written in the following way:

$$f(x) = \frac{3}{8} + \frac{1}{8}\cosh 4x - \frac{1}{2}\cosh 2x - x^4 \cosh x, \tag{40}$$

which is a concrete mixed hyperbolic-trigonometric polynomial.

Then, according to the Lemma 2.2. and the description of the method, the inequality $\cosh y > \underline{T}_k^{\cosh,0}(y)$ $(k=6)$ is true, $\forall y \in \mathbb{R}$ and according to the Theorem 2.6. it is valid: $\cosh y < 1 + \frac{y^2}{2} + \frac{y^4}{24} + \frac{1}{24}(24\cosh 1 - 37)y^6$ and $\cosh 2y < 1 + 2y^2 + \frac{2y^4}{3} + \frac{1}{24}(48\cosh^2 1 - 112)y^6$, for $y \in (0,1)$.

For $x \in (0,1)$ it is valid:

$$\begin{aligned} f(x) > {} & \tfrac{3}{8} + \tfrac{1}{8}\underline{T}_6^{\cosh,0}(4x) + \tfrac{1}{2}\Big(-1 - 2x^2 - \tfrac{2x^4}{3} - \tfrac{1}{24}(48\cosh^2 1 - 112)x^6\Big) \\ & + x^4\Big(-1 - \tfrac{x^2}{2} - \tfrac{x^4}{24} - \tfrac{1}{24}(24\cosh 1 - 37)x^6\Big) = P_{10}(x), \end{aligned} \tag{41}$$

where $P_{10}(x)$ is the polynomial

$$\begin{aligned} P_{10}(x) &= \frac{x^6}{360}((555 - 360\cosh 1)x^4 - 15x^2 + 916 - 360\cosh^2 1) \\ &= \frac{x^6}{360}P_4(x). \end{aligned} \tag{42}$$

A real numerical factorization of the polynomial $P_4(x)$, has been determined via Matlab software, and given with

$$P_4(x) = \alpha(x - x_1)(x - x_2)(x^2 + px + q), \tag{43}$$

where $\alpha = -0.509\ldots, x_1 = 1.871\ldots, x_2 = -x_1, p = 0, q = 32.971\ldots$; whereby the inequality $p^2 - 4q < 0$ is true. The polynomial $P_4(x)$ has exactly two simple real roots with a symbolic radical representation and the corresponding numerical values x_1 and x_2. Since $P_4(x) > 0$ for the values $x \in (0,1) \subset (x_2, x_1)$, finally we can conclude that

$$P_{10}(x) > 0 \text{ for } x \in (0,1) \Longrightarrow f(x) > 0 \text{ for } x \in (0,1). \tag{44}$$

Hence, the proof of the inequality (32) is completed. □

3.2. A proof of an inequality from the article [2]

In the article [2] C. Barbu and L. Piscoran have proved the following statement (Theorem 13):

THEOREM 3.2. *Let $x \in \mathbb{R} \setminus \{0\}$. Then the following inequality*

$$\frac{\sinh x}{x} > \frac{1}{\cosh \frac{x}{3}} \tag{45}$$

holds.

Now we present a proof of the inequality (45).

Proof. First we prove (45) for $x \in (0,\infty)$. The requested inequality is equivalent to $F(x) = \dfrac{\sinh x}{x} - \dfrac{1}{\cosh \dfrac{x}{3}} > 0$ which is equivalent to $f(x) > 0$ for $x \in (0,\infty)$, where

$$f(x) = \sinh x \cosh \frac{x}{3} - x. \tag{46}$$

Let us define the function

$$\varphi(x) = f(3x) = \sinh 3x \cosh x - 3x. \tag{47}$$

Now we have that $f(x) > 0$ for $x \in (0,\infty)$ is equivalent to $\varphi(x) > 0$ for $x \in (0,\infty)$. The function $\varphi(x)$ can be written in the following way:

$$\varphi(x) = \frac{1}{2}\sinh 4x + \frac{1}{2}\sinh 2x - 3x, \tag{48}$$

which is a concrete mixed hyperbolic-trigonometric polynomial.

Then, according to the Lemma 2.1. and the description of the method, the inequality $\sinh y > \underline{T}_k^{\sinh,0}(y)$ $(k=3, k=5)$ is true for $y \in (0,\infty)$.

For $x \in (0,\infty)$ it is valid:

$$\varphi(x) > \frac{1}{2}\underline{T}_3^{\sinh,0}(4x) + \frac{1}{2}\underline{T}_5^{\sinh,0}(2x) - 3x = P_5(x), \tag{49}$$

where $P_5(x)$ is the polynomial

$$P_5(x) = \frac{2x^3}{15}(x^2 + 45). \tag{50}$$

Since $P_5(x) > 0$ for $x \in (0,\infty)$, we can conclude that $\varphi(x) > 0$ for $x \in (0,\infty)$, therefore $f(x) > 0$ for $x \in (0,\infty)$.

For $x \in (-\infty, 0)$ the inequality (45) is valid on the basis of the even property of the function $F(x)$. Therefore, the inequality (45) is true for $x \in \mathbb{R} \setminus \{0\}$. □

3.3. A proof of a new double inequality

As we have already mentioned in the first paragraph of 2.2., if we use upward approximations of $\sinh x$ and $\cosh x$, then inequalities can be proved only in the finite interval (a,b). According to our knowledge, the following double inequality is new (it's not presented in literature).

THEOREM 3.3. *For* $x \in (0,1)$ *it holds:*

$$\left(\cosh \frac{x}{4}\right)^5 < \frac{\sinh x}{x} < \frac{3\cosh x + 2}{5}. \tag{51}$$

Let us prove the inequality (51).

Proof.

I We prove the inequality:

$$\left(\cosh \frac{x}{4}\right)^5 < \frac{\sinh x}{x} \tag{52}$$

for $x \in (0,1)$. The requested inequality is equivalent to $F(x) = \dfrac{\sinh x}{x} - \left(\cosh \dfrac{x}{4}\right)^5 > 0$ which is equivalent to $f(x) > 0$ for $x \in (0,1)$, where

$$f(x) = \sinh x - x\left(\cosh \frac{x}{4}\right)^5. \tag{53}$$

Let us define the function

$$\varphi(x) = f(4x) = \sinh 4x - 4x\cosh^5 x. \tag{54}$$

Now we have that $f(x) > 0$ for $x \in (0,1)$ is equivalent to $\varphi(x) > 0$ for $x \in (0, \frac{1}{4})$.

According to the Theorem 2.5., the function $\varphi(x)$ can be written in the following way:

$$\varphi(x) = \sinh 4x - \frac{x}{4}\cosh 5x - \frac{5x}{4}\cosh 3x - \frac{5x}{2}\cosh x, \tag{55}$$

which is a concrete mixed hyperbolic-trigonometric polynomial.

Then, according to the Lemma 2.1. and the description of the method, the inequality $\sinh y > \underline{T}_k^{\sinh,0}(y)$ $(k=5)$ is true, for $y \in (0,\infty)$ and according to the Theorem 2.6., for $y \in (0, \frac{1}{4})$, it is valid:

$$\begin{aligned}\cosh 5y < 1 + \frac{25}{2}y^2 + \frac{625}{24}y^4 + \frac{1}{24}\Big(&1572864\cosh^5\frac{1}{4} - 1966080\cosh^3\frac{1}{4}\\ &+491520\cosh\frac{1}{4} - 185104\Big)y^6,\end{aligned}$$

$$\cosh 3y < 1 + \frac{9}{2}y^2 + \frac{27}{8}y^4 + \frac{1}{24}\left(393216\cosh^3\frac{1}{4} - 294912\cosh\frac{1}{4} - 127248\right)y^6$$

and

$$\cosh y < 1 + \frac{y^2}{2} + \frac{y^4}{24} + \frac{1}{24}\left(98304\cosh\frac{1}{4} - 101392\right)y^6.$$

For $x \in (0, \frac{1}{4})$ it is valid:

$$\begin{aligned}\varphi(x) > \underline{T}_5^{\sinh,0}(4x) - \tfrac{x}{4}\Big(1 + \tfrac{25}{2}x^2 + \tfrac{625}{24}x^4 \\ + \tfrac{1}{24}(1572864\cosh^5\tfrac{1}{4} - 1966080\cosh^3\tfrac{1}{4} + 491520\cosh\tfrac{1}{4} - 185104)x^6\Big) \\ - \tfrac{5x}{4}\Big(1 + \tfrac{9}{2}x^2 + \tfrac{27}{8}x^4 + \tfrac{1}{24}(393216\cosh^3\tfrac{1}{4} - 294912\cosh\tfrac{1}{4} - 127248)x^6\Big) \\ - \tfrac{5x}{2}\Big(1 + \tfrac{x^2}{2} + \tfrac{x^4}{24} + \tfrac{1}{24}(98304\cosh\tfrac{1}{4} - 101392)x^6\Big) \\ = P_7(x),\end{aligned} \tag{56}$$

where $P_7(x)$ is the polynomial

$$\begin{aligned}P_7(x) &= \frac{x^3}{30}\Big((573520 - 491520\cosh^5\tfrac{1}{4})x^4 - 69x^2 + 20\Big) \\ &= \frac{x^3}{30}P_4(x).\end{aligned} \tag{57}$$

A real numerical factorization of the polynomial $P_4(x)$, has been determined via Matlab software, and given with

$$P_4(x) = \alpha(x - x_1)(x - x_2)(x^2 + px + q), \tag{58}$$

where $\alpha = -205.838\ldots, x_1 = 0.431\ldots, x_2 = -x_1, p = 0, q = 0.521\ldots$; whereby the inequality $p^2 - 4q < 0$ is true. The polynomial $P_4(x)$ has exactly two simple real roots with a symbolic radical representation and the corresponding numerical values x_1 and x_2. Since $P_4(x) > 0$ for the values $x \in (0, \frac{1}{4}) \subset (x_2, x_1)$, it follows that $P_7(x) > 0$ for $x \in (0, \frac{1}{4})$. Then we can conclude that $\varphi(x) > 0$ for $x \in (0, \frac{1}{4}) \Rightarrow f(x) > 0$ for $x \in (0, 1)$.

II Now we prove the inequality:

$$\frac{\sinh x}{x} < \frac{3\cosh x + 2}{5} \tag{59}$$

for $x \in (0, 1)$. The requested inequality is equivalent to $f(x) > 0$ for $x \in (0, 1)$, where

$$f(x) = 3x\cosh x + 2x - 5\sinh x, \tag{60}$$

which is a concrete mixed hyperbolic-trigonometric polynomial.

According to the Lemma 2.2. and the description of the method, the inequality $\cosh y > \underline{T}_k^{\cosh,0}(y)$ $(k = 4)$ is true, $\forall y \in \mathbb{R}$ and according to the Theorem 2.6. the inequality $\sinh y < y + \frac{y^3}{6} + \frac{1}{6}(6\sinh 1 - 7)y^5$ is valid, for $y \in (0, 1)$.

For $x \in (0, 1)$ it is valid:

$$f(x) > 3x\underline{T}_4^{\cosh,0}(x) + 2x - 5\Big(x + \tfrac{x^3}{6} + \tfrac{1}{6}(6\sinh 1 - 7)x^5\Big) = P_5(x), \tag{61}$$

where $P_5(x)$ is the polynomial

$$P_5(x) = -\frac{x^3}{24}\Big((120\sinh 1 - 143)x^2 - 16\Big) = 0.082x^3(x^2 + 8.097). \tag{62}$$

Since $P_5(x) > 0$ for $x \in (0, 1)$, we can conclude that $f(x) > 0$ for $x \in (0, 1)$.

Hence, the proof of the inequality (51) is completed. □

4. Conclusion

The previous method can be applied to some inequalities which correspond to univariate mixed hyperbolic-trigonometric polynomial functions. Concrete results of the presented method have been obtained in this article through the applications. Some of the inequalities found in the articles: [2] (Theorem 10, Corollary 14.), [1] (Corollaries 2.1. and 2.2.), [6] (Lemma 3.3.), [4] (Remark 1. from Section 2), can also be proved by this method. Also, combining this method with the method from the article [7] for $\sin x$ and $\cos x$, we can prove the inequalities in which sine and cosine appear together with hyperbolic sine and hyperbolic cosine, as for example in the article [2] (Theorem 1. and Corollary 2.).

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