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UNIVALENCY AND STARLIKENESS OF NORMALIZED HURWITZ–LERCH ZETA FUNCTION INSIDE UNIT DISK

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Abstract. In the present investigation we study normalized Hurwitz-Lerch Zeta function and find sufficient conditions, so that the normalized Hurwitz-Lerch Zeta function have certain geometric properties like close-to-convexity, univalency and starlikeness inside the unit disc.

1. Introduction

Let $\mathscr{H}$ denote the class of analytic functions inside the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathscr{A}$ denote the class of analytic functions inside the unit disk $\mathbb{D}$, having the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \quad z \in \mathbb{D}. \tag{1.1}$$

We denote by $\mathscr{S}$, the class of all functions $f \in \mathscr{A}$ which are univalent in $\mathbb{D}$ i. e.

$$\mathscr{S} = \{f \in \mathscr{A} |\ f \text{ is one-to-one in } \mathbb{D}\}.$$

A function $f \in \mathscr{A}$ is called starlike (with respect to 0), denoted by $f \in \mathscr{S}^*$ if $tw \in f(\mathbb{D})$ for all $w \in f(\mathbb{D})$ and $t \in [0,1]$. A function $f \in \mathscr{A}$ that maps $\mathbb{D}$ onto a convex domain is called convex function and class of such functions is denoted by $\mathscr{K}$. For a given $\alpha < 1$, a function $f \in \mathscr{A}$ is called starlike function of order α, denoted by $\mathscr{S}^*(\alpha)$, if

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad z \in \mathbb{D}.$$

For a given $\alpha < 1$, a function $f \in \mathscr{A}$ is called convex function of order α, denoted by $\mathscr{K}(\alpha)$, if

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad z \in \mathbb{D}.$$

It is well known that $\mathscr{S}^*(0) = \mathscr{S}^*$ and $\mathscr{K}(0) = \mathscr{K}$. If $f \in \mathscr{A}$ satisfies

$$\mathfrak{Re}\left\{\frac{zf'(z)}{e^{i\alpha} g(z)}\right\} > 0,\ z \in \mathbb{D} \tag{1.2}$$

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for some $g(z) \in \mathcal{S}^*$ and some $\alpha \in (-\pi/2, \pi/2)$, then $f(z)$ is said to be close-to-convex (with respect to $g(z)$) in $\mathbb{D}$ and denoted by $f(z) \in \mathcal{C}$. An univalent function $f \in \mathcal{S}$ belongs to $\mathcal{C}$ if and only if the complement E of the image-region $F = \{f(z) : |z| < 1\}$ is the union of rays that are disjoint (except that the origin of one ray may lie on another one of the rays).

The Noshiro-Warschawski theorem implies that close-to-convex functions are univalent in $\mathbb{D}$, but not necessarily the converse. It is easy to verify that $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{C}$. For more details see [5].

Recently, several researchers studied classes of analytic functions involving special functions $\mathcal{F} \subset \mathcal{A}$, to find different conditions such that the members of $\mathcal{F}$ have certain geometric properties like univalency, starlikeness and close-to-convexity in $\mathbb{D}$. In this context many results are available in the literature regarding the generalized hypergeometric functions (see [9, 10, 11]) and the Bessel functions (see [1, 2, 3]).

In the present paper we study Hurwitz-Lerch Zeta function defined by

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{1.3}$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } z \in \mathbb{D} \text{ and } \mathfrak{Re}\{s\} > 1 \text{ when } |z| = 1).$$

For $s = 1$

$$\Phi(z,1,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)} = \frac{1}{a} {}_2F_1(1,a;1+a;z). \tag{1.4}$$

Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function can be found in the recent investigations by Srivastava and Choi [12].

The study of the geometric properties such as univalency and starlikeness of $\Phi(z,s,a)$ permit us to study the geometric properties of polylogarithmic functions also. As the function $\Phi(z,s,a)$ does not belong to the class $\mathcal{A}$, so it is natural to consider the following normalization of the Hurwitz-Lerch Zeta function

$$\mathbb{H}(z,s,a) = z\,a^s \Phi(z,s,a) = z + \sum_{n=2}^{\infty} \frac{a^s}{(n-1+a)^s} z^n \tag{1.5}$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } z \in \mathbb{D} \text{ and } \mathfrak{Re}(s) > 1 \text{ when } |z| = 1).$$

From (1.5) it easy to check that for $s = 1$, we have

$$z\mathbb{H}'(z,a) = (1-a)\mathbb{H}(z,a) + a\frac{z}{1-z}. \tag{1.6}$$

and

$$\frac{\mathbb{H}(z,s,a)}{z} = 1 + \sum_{n=2}^{\infty} \frac{a^s}{(n-1+a)^s} z^{n-1}. \tag{1.7}$$

Note that polylogarithmic function is defined by

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z,s,1). \tag{1.8}$$

Thus from (1.5)

$$\mathbb{H}(z,s,1) = z\Phi(z,s,1) = Li_s(z). \tag{1.9}$$

Also using (1.4) and (1.5)

$$\mathbb{H}(z,1,a) = z_2F_1(1,a;1+a;z). \tag{1.10}$$

For our present investigation we study $\mathbb{H}(z,s,a)$ for $s>0$ only. By $co\mathscr{K}$ we denote the convex hull of the class of convex functions $\mathscr{K}$, that is the set of all convex combinations of functions belonging to the class $\mathscr{K}$. Let us recall [4] that the closure of the set $co\mathscr{K}$ is

$$\overline{co}\mathscr{K} = \left\{ f \in \mathscr{H} : f(0)=0,\ f'(0)=1,\ \mathfrak{Re}\left\{\frac{f(z)}{z}\right\} > \frac{1}{2},\ z \in \mathbb{D} \right\}. \tag{1.11}$$

We say that the $f \in \mathscr{H}$ is subordinate to $g \in \mathscr{H}$ in the unit disc $\mathbb{D}$, written $f \prec g$ if and only if there exits an analytic function $w \in \mathscr{H}$ such that $w(0)=0, |w(z)|<1$ and $f(z) = g[w(z)]$ for $z \in \mathbb{D}$.

DEFINITION 1.1. (Subordinating Factor Sequence) A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating sequence for the class $\mathscr{X} \subset \mathscr{A}$, whenever we have

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec \sum_{n=1}^{\infty} a_n z^n,\ \ z \in \mathbb{D}, \tag{1.12}$$

for all $\sum_{n=1}^{\infty} a_n z^n \in \mathscr{X}$.

To prove our main we need following results:

LEMMA 1.1. (Féjer [6]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_1 = 1$, and that for $n \geqslant 2$ the sequence $\{a_n\}$ is a convex decreasing, i.e. $0 \geqslant a_{n+2} - a_{n+1} \geqslant a_{n+1} - a_n$, for all $n \in \mathbb{N}$. Then*

$$\mathfrak{Re}\left\{\sum_{n=1}^{\infty} a_n z^{n-1}\right\} > 1/2\ (z \in \mathbb{D}). \tag{1.13}$$

Note that each convex decreasing sequence generates also a convex null sequence. Recall that the sequence $a_0,\ a_1,\dots$ of nonnegative numbers is called a convex null sequence if

$$\lim_{k\to\infty} a_k = 0 \quad \text{and} \quad a_0 - a_1 \geqslant a_1 - a_2 \geqslant \cdots \geqslant a_k - a_{k+1} \geqslant \cdots \geqslant 0.$$

For a convex null sequence $a_0 = 1,\ a_1,\dots$ we have instead of (1.13) the following inequality

$$\mathfrak{Re}\left\{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n z^n\right\} > 0\ (z \in \mathbb{D}).$$

LEMMA 1.2. (Ozaki [7]). *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. *Suppose*

$$1 \geqslant 2a_2 \geqslant \cdots \geqslant na_n \geqslant (n+1)a_{n+1} \geqslant \cdots \geqslant 0 \tag{1.14}$$

or

$$1 \leqslant 2a_2 \leqslant \cdots \leqslant na_n \leqslant (n+1)a_{n+1} \leqslant \cdots \leqslant 2. \tag{1.15}$$

then f *is close-to-convex with respect to starlike function* $z/(1-z)$.

LEMMA 1.3. (Féjer [6]). *If* $a_n \geqslant 0$, $\{na_n\}$ *and* $\{na_n - (n+1)a_{n+1}\}$ *both are nonincreasing, then the function* f *defined by* (1.1) *is in* $\mathscr{S}^*$.

LEMMA 1.4. [8]. *The function*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

is in the set $\overline{co}\mathscr{K}$ *if and only if* $b_2, b_3, \ldots$ *is a subordinating factor sequence for the class* $\mathscr{K}$.

For more results on subordinating factor sequence, we refer the source paper [14].

2. Starlikeness and Close-to-convexity

THEOREM 2.1. *Assume that* s *and* a *satisfy any one of the following conditions:*

1. $0 < a \leqslant 1$ *and* $s \geqslant 1$
2. $1 < a \leqslant 2$ *and* $0 \leqslant s \leqslant 1$.

Then the normalized Hurwitz-Lerch Zeta function $\mathbb{H}(z,s,a)$ *is close-to-convex (hence univalent) with respect to starlike function* $g(z) = z/(1-z)$.

Proof. Let

$$\mathbb{H}(z,s,a) = z + \sum_{n=2}^{\infty} \frac{a^s}{(n-1+a)^s} z^n = \sum_{n=1}^{\infty} A_n z^n$$

where $A_1 = 1$ and for $n \geqslant 2$,

$$A_n = \frac{a^s}{(n-1+a)^s}. \tag{2.1}$$

Now

$$\begin{aligned} nA_n - (n+1)A_{n+1} &= \frac{a^s}{(n-1+a)^s(n+a)^s}\left[n(n+a)^s - (n+1)(n-1+a)^s\right] \\ &= \frac{a^s}{(n-1+a)^s(n+a)^s} X(n). \end{aligned} \tag{2.2}$$

Where

$$X(n) = n(n+a)^s - (n+1)(n-1+a)^s. \tag{2.3}$$

It is easy to see that $X(n) \geqslant 0$ for all $n \geqslant 1$, provided $a \in (0,1]$ and $s \geqslant 1$. Thus, the sequence $\{nA_n\}$ is nonincreasing. Applying Lemma 1.2, we get that $\mathbb{H}(z,s,a)$ is close-to-convex with respect to starlike function $z/(1-z)$.

To prove second part, in view of (1.15), it suffices to show that $\{nA_n\}$ is an increasing sequence and that it has a limit less than or equal to 2. The inequality $(n+1)A_{n+1} - nA_n > 0$ becomes

$$\left(\frac{n-1+a}{n+a}\right)^s > \frac{n}{n+1}. \tag{2.4}$$

Because $0 \leqslant s \leqslant 1$ and $0 < (n-1+a)/(n+a) < 1$ it suffices to verify (2.4) for $s=1$. Then (2.4) becomes

$$a - 1 > 0, \tag{2.5}$$

which is true because $1 < a \leqslant 2$. Thus, the sequence $\{nA_n\}$ is an increasing sequence. To complete the proof, it is sufficient to show that the value of the limit is less than or equal to 2. So taking

$$\lim_{n\to\infty} nA_n = \lim_{n\to\infty} \left\{\frac{na}{n-1+a}\right\}^s = a^s \leqslant 2, \tag{2.6}$$

under the hypothesis of Theorem. This completes the proof. □

Putting $a = 1$ in Theorem 2.1, we get the following

COROLLARY 2.1. *For $s \geqslant 1$, polylogarithmic function $Li_s(z)$ is close-to-convex with respect to starlike function $z/(1-z)$ and hence univalent in $\mathbb{D}$.*

Again substituting $s = 1$, we get

COROLLARY 2.2. *For $0 < a \leqslant 2$, $z\,{}_2F_1(1,a;1+a;z)$ is close-to-convex with respect to starlike function $z/(1-z)$ and hence univalent in $\mathbb{D}$.*

REMARK 2.1. Corollary 2.2 gives the same result as discussed in [10], Example 3.1, Page 339.

THEOREM 2.2. *For $a > 0$ and $s \geqslant \max\{2a-1,1\}$, the normalized Hurwitz-Lerch Zeta function $\mathbb{H}(z,s,a)$ is starlike function in $\mathbb{D}$.*

Proof. In view of Lemma 1.3, it is sufficient to prove that nA_n and $\{nA_n - (n+1)A_{n+1}\}$ are nonincreasing sequences for all $n \geqslant 1$. The sequence nA_n is nonincreasing by the proof of Theorem 2.1. Therefore, it suffices to show that $nA_n - 2(n+1)A_{n+1} + (n+2)A_{n+2} \geqslant 0$ for all $n \geqslant 1$. Using (2.1) gives

$$\begin{aligned} & nA_n - 2(n+1)A_{n+1} + (n+2)A_{n+2} \geqslant 0 \\ \Leftrightarrow\ & a^s \left[\frac{n}{(n-1+a)^s} + \frac{n+2}{(n+1+a)^s} - \frac{2(n+1)}{(n+a)^s} \right] \geqslant 0 \\ \Leftrightarrow\ & a^s \left[f(n) + f(n+2) - 2f(n+1) \right] \geqslant 0 \end{aligned}$$

where

$$f(x) = \frac{x}{(x+\alpha)^s}, \quad x \geqslant 1,$$

and

$$\alpha = a-1,\ a > 0 \text{ and } s \geqslant 1.$$

To show $[f(n) + f(n+2) - 2f(n+1)] \geqslant 0$, $n = 1,2,3,4,\ldots$, it suffices to prove that $f(x)$ is a convex function in the real sense or that $f''(x) \geqslant 0$, $x \geqslant 1$. We have

$$f''(x) = \frac{xs^2 - xs - 2\alpha s}{(x+\alpha)^{s+2}}, \quad x \geqslant 1. \tag{2.7}$$

Denominator is already positive for all $x \geqslant 1$ and $\alpha > -1$. Let $\phi(x) = xs^2 - xs - 2\alpha s$. Obviously $\phi'(x) = s^2 - s \geqslant 0$ for all $s \geqslant 1$. Thus $f''(x) \geqslant 0$ provided $\phi(1) \geqslant 0$ for all $x \geqslant 1$ and $s \geqslant 1$. Now

$$\begin{aligned} & \phi(1) \geqslant 0 \\ \Leftrightarrow\ & (s-1) - 2\alpha \geqslant 0 \text{ and } s \geqslant 1 \\ \Leftrightarrow\ & s \geqslant 2\alpha + 1 \text{ and } s \geqslant 1 \\ \Leftrightarrow\ & s \geqslant \max\{2a-1, 1\}. \end{aligned}$$

This completes the proof. □

Putting $a = 1$ in Theorem 2.2, we get the following

COROLLARY 2.3. *For $s \geqslant 1$, polylogarithmic function $Li_s(z)$ is starlike function in $\mathbb{D}$.*

Again substituting $s = 1$, we get the following corollary.

COROLLARY 2.4. *For $0 < a \leqslant 1$, $z\,{}_2F_1(1,a;1+a;z)$ is starlike function in $\mathbb{D}$.*

From (1.8), it is clear that

$$z[Li_s(z)]' = Li_{s-1}(z).$$

Applying Theorem 2.2, on $Li_{s-1}(z)$, we get

COROLLARY 2.5. *For $s \geqslant 2$, polylogarithmic function $Li_s(z)$ is convex function in $\mathbb{D}$.*

THEOREM 2.3. *For $a > 0$ and $s \geqslant 0$,*

$$\mathfrak{Re}\left\{\frac{\mathbb{H}(z,s,a)}{z}\right\} > \frac{1}{2} \ (z \in \mathbb{D}). \tag{2.8}$$

Proof. We first prove that

$$\{a_n\}_{n=1}^{\infty} = \left\{\frac{a^s}{(n-1+a)^s}\right\}_{n=1}^{\infty}$$

is a decreasing sequence. Since

$$(n+a)^s \geqslant (n-1+a)^s \ \ (n \in \mathbb{N}, a > 0 \text{ and } s \geqslant 0).$$

which implies

$$\frac{a^s}{(n-1+a)^s} \geqslant \frac{a^s}{(n+a)^s} \ \ (n \in \mathbb{N}, a > 0 \text{ and } s \geqslant 0).$$

Next we prove that $\{a_n\}_{n=1}^{\infty}$ is a convex decreasing sequence for this we show

$$a_{n+2} - a_{n+1} \geqslant a_{n+1} - a_n \ \ \forall\, n \in \mathbb{N}.$$

Now

$$\begin{aligned}
&a_n - 2a_{n+1} + a_{n+2} \geqslant 0\\
\Leftrightarrow\ & a^s\left[\frac{1}{(n-1+a)^s} + \frac{1}{(n+1+a)^s} - \frac{2}{(n+a)^s}\right] \geqslant 0\\
\Leftrightarrow\ & a^s\left[f(n) + f(n+2) - 2f(n+1)\right] \geqslant 0,
\end{aligned}$$

where

$$f(x) = \frac{1}{(x+\alpha)^s} \ \ (x \geqslant 1, \alpha = a-1,\ a > 0 \text{ and } \ s \geqslant 0).$$

To show $[f(n) + f(n+2) - 2f(n+1)] \geqslant 0$, $n = 1,2,3,4,\ldots$, it suffices to prove that $f(x)$ is a convex function in the real sense or that $f''(x) \geqslant 0$, $x \geqslant 1$. We have

$$f''(x) = \frac{s(s+1)}{(x+\alpha)^{s+2}} \geqslant 0 \ \ (x \geqslant 1, s \geqslant 0, a > 0). \tag{2.9}$$

Thus $\{a_n\}_{n=1}^{\infty}$ is a convex decreasing sequence. Now applying Lemma 1.1 on $\{a_n\}_{n=1}^{\infty}$, we have

$$\mathfrak{Re}\left\{\sum_{n=1}^{\infty} a_n z^{n-1}\right\} > 1/2, \quad z \in \mathbb{D}. \tag{2.10}$$

which is equivalent to

$$\mathfrak{Re}\left\{\frac{\mathbb{H}(z,s,a)}{z}\right\} > 1/2, \quad z \in \mathbb{D}. \quad \square \tag{2.11}$$

COROLLARY 2.6. *For $a > 0$ and $s \geqslant 0$ the sequence*

$$\left\{ \frac{a^s}{(n+a)^s} \right\}_{n=1}^{\infty} \tag{2.12}$$

is a subordinating factor sequence for the class $\mathscr{K}$.

Proof. By (1.11) and (2.8), we have $\mathbb{H}(z,s,a) \in \overline{co}\mathscr{K}$ for $a > 0$ and $s \geqslant 0$. Applying Lemma 1.4, we directly obtain (2.12). □

THEOREM 2.4. *For $a > 0$ and $s \geqslant \max\{2a-1, 1\}$,*

$$\mathfrak{Re}\left\{\mathbb{H}'(z,s,a)\right\} > \frac{1}{2}\ (z \in \mathbb{D}). \tag{2.13}$$

Proof. From (1.5),

$$\mathbb{H}'(z,s,a) = 1 + \sum_{n=2}^{\infty} \frac{na^s}{(n-1+a)^s} z^{n-1}. \tag{2.14}$$

So taking

$$a_n = \frac{na^s}{(n-1+a)^s}$$

and proceeding similarly as in Theorem 2.2, we get the proof. □

COROLLARY 2.7. *For $a > 0$ and $s \geqslant \max\{2a-1, 1\}$ the sequence*

$$\left\{ \frac{(n+1)a^s}{(n+a)^s} \right\}_{n=1}^{\infty} \tag{2.15}$$

is a subordinating factor sequence for the class $\mathscr{K}$.

Proof. By (1.11) and (2.13), we have $z\mathbb{H}'(z,s,a) \in \overline{co}\mathscr{K}$ for $a > 0$ and $s \geqslant \max\{2a-1, 1\}$. Applying Lemma 1.4, we directly obtain (2.15) from (2.14). □

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