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IMPROVEMENTS OF BOUNDS FOR THE q–GAMMA AND THE q–POLYGAMMA FUNCTIONS

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Abstract. In this paper, the complete monotonicity property of functions involving the q-gamma function is proven and used to establish sharp inequalities for the q-gamma and the q-polygamma functions for all $q > 0$. These bounds for the q-gamma and the q-polygamma functions refine those given by Salem [17].

1. Introduction

The q-gamma function is defined as [1]

$$\Gamma_q(x) = |1-q|^{1-x} q^{\frac{x(x-1)}{2}H(q-1)} \prod_{n=0}^{\infty} \frac{1-\hat{q}^{n+1}}{1-\hat{q}^{n+x}}, \qquad 1 \neq q > 0 \tag{1.1}$$

where $|\cdot|$ is the absolute value, $H(\cdot)$ denotes the Heaviside step and

$$\hat{q} = \begin{cases} q & \text{if} \quad 0 < q \leqslant 1, \\ q^{-1} & \text{if} \quad q \geqslant 1. \end{cases}$$

The close connection between two branches of the q-gamma function when $0 < q < 1$ and $q \geqslant 1$ is given by

$$\Gamma_q(x) = q^{\frac{(x-1)(x-2)}{2}} \Gamma_{q^{-1}}(x), \qquad q \geqslant 1. \tag{1.2}$$

The logarithmic derivative of the q-gamma function is called the q-digamma function $\psi_q(x)$ defined as

$$\psi_q(x) = \frac{d}{dx}(\log \Gamma_q(x)) = \frac{\Gamma_q'(x)}{\Gamma_q(x)}. \tag{1.3}$$

The nth derivatives of the q-digamma function are the so-called the q-polygamma functions denoted by $\psi_q^{(n)}(x)$; $n \in \mathbb{N}$. The q-digamma function $\psi_q(x)$ appeared in the work of Krattenthaler and Srivastava [2] when they studied the summations for

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basic hypergeometric series. Some of its properties presented and proved in their work and also in [3]. Recently, within the framework of quantum statistical mechanics (the problem of Bose-Einstein condensation in harmonically trapped, one-dimensional and ideal atoms (ideal photons)), it is found that number of atoms (the photon number) of vapor is characterized by an analytical function, which involves a q-digamma function in mathematics [4, 5, 6].

In the recent past, numerous papers were published presenting remarkable inequalities involving the q-gamma and the q-polygamma functions (see [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24] and the extensive list of references given therein).

The results obtained in [17] have given bounds for the q-gamma and the q-polygamma functions by means of proving the complete monotonicity property of an infinite class of functions related to the q-gamma function. Some of these results stat:

- The q-gamma function inequality

$$[x]_q^x q^{\frac{1}{2}H(q-1)} S_{\hat{q}} \sqrt{\frac{2\pi}{[x]_q}} \exp\left(\frac{\mathrm{Li}_2(1-q^x)}{\log q}\right) < \Gamma_q(x)$$
$$< [x]_q^x q^{\frac{5}{12}H(q-1)} S_{\hat{q}} \sqrt{\frac{2\pi}{[x]_q}} \exp\left(\frac{\mathrm{Li}_2(1-q^x)}{\log q} - \frac{1}{12}\frac{q^x \log q}{1-q^x}\right) \tag{1.4}$$

holds true for all positive real numbers x and q where $[x]_q = (1-q^x)/(1-q)$, $\mathrm{Li}_2(z)$ is the dilogarithm function defined for complex argument z as [31]

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}dt, \qquad z \notin (1,\infty)$$

and

$$S_q = q^{\frac{-1}{24}} \sqrt{\frac{q-1}{\ln q}} \sum_{m=-\infty}^{\infty} \left(r^{m(6m+1)} - r^{(2m+1)(3m+1)}\right), \quad r = e^{\frac{4\pi^2}{\ln q}}.$$

- The q-digamma function inequality

$$\log[x]_q + \frac{1}{2}\frac{q^x \log q}{1-q^x} - \frac{1}{12}\frac{q^x \log^2 q}{(1-q^x)^2} < \psi_q(x) < \log[x]_q + \frac{1}{2}\frac{q^x \log q}{1-q^x} \tag{1.5}$$

holds true for all positive real numbers x and q.

- The q-polygamma functions inequalities

$$(-1)^{r+1}\beta_{r+1}(1) < (-1)^r \psi_q^{(r)}(x) < (-1)^{r+1}\beta_{r+1}(0), \qquad r \in \mathbb{N} \tag{1.6}$$

hold true for all positive real numbers x and q, where

$$\beta_{r+1}(n) = \left(\frac{\log q}{1-q^x}\right)^r q^x P_{r-2}(q^x) - \frac{1}{2}\left(\frac{\log q}{1-q^x}\right)^{r+1} q^x P_{r-1}(q^x)$$
$$+ \sum_{i=1}^{n} \frac{B_{2i}}{(2i)!}\left(\frac{\log q}{1-q^x}\right)^{2i+r} q^x P_{2i+r-2}(q^x), \quad r = 1,2,\cdots \tag{1.7}$$

where P_k is a polynomial of degree k satisfying

$$P_k(z) = (z - z^2)P'_{k-1}(z) + (kz+1)P_{k-1}(z), \qquad P_0 = P_{-1} = 1, \qquad k \in \mathbb{N}.$$

The main objective of this paper is to present the improvements of the bounds in the above inequalities by means of proving the complete logarithmic monotonicity property of the function

$$F_a(x;q) = [x]_q^{\frac{1}{2}-x} \exp\left(\frac{aq^x \log q}{1-q^x} - \frac{\mathrm{Li}_2(1-q^x)}{\log q}\right)\Gamma_q(x) \tag{1.8}$$

for all real a and positive real numbers x and q.

A positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\log f$ satisfies

$$(-1)^n[\log f(x)]^{(n)} \geqslant 0, \qquad n \in \mathbb{N};\ x \in I. \tag{1.9}$$

Also, a real-valued function f, defined on an interval I, is called completely monotonic, if f has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x) \geqslant 0, \qquad n \in \mathbb{N}_0 = \{0,1,2,\cdots\};\ x \in I. \tag{1.10}$$

These functions have numerous applications in various branches, like, for instance, numerical analysis and probability theory.

The notion of logarithmically completely monotonic functions was recovered by Feng Qi and Bai-Ni Guo [26]. It has been proven once again in [27, 28, 29] that the class of logarithmically completely monotonic functions is a subclass of the completely monotonic functions. For more information, see ([30], p.134, Section 1.3) and the references given therein.

Ismail and Muldoon [23] provided the equivalent Stieltjes integral representation for the q-digamma function as

$$\psi_q(x) = -\log(1-q) - \int_0^\infty \frac{e^{-xt}}{1-e^{-t}}\,d\gamma_q(t), \qquad x>0,\ 0<q<1 \tag{1.11}$$

where $\gamma_q(t)$ is a discrete measure with positive masses $-\log q$ at the positive points $-k\log q, k \in \mathbb{N}$. For completeness, and economy of later statements, they include the value $q=1$ in the definition of $\gamma_q(t)$:

$$\gamma_q(t) = \begin{cases} -\log q \displaystyle\sum_{k=1}^{\infty} \delta(t + k\log q), & 0<q<1, \\ t, \qquad q=1. \end{cases}$$

Their proof depended on the identities

$$\frac{q^x \log q}{1-q^x} = -\int_0^\infty e^{-xt}\,d\gamma_q(t) \quad \text{and} \quad \log(1-q^x) = -\int_0^\infty \frac{e^{-xt}}{t}\,d\gamma_q(t) \tag{1.12}$$

which follow easily from the definition of $d\gamma_q(t)$ for all $x > 0$ and $0 < q < 1$.

With the Euler-Maclaurin formula, Moak [25] obtained the following q-analogue of Stirling formula (see also [13])

$$\log\Gamma_q(x) \sim \left(x - \frac{1}{2}\right)\log[x]_q + \frac{\mathrm{Li}_2(1-q^x)}{\log q} + \frac{1}{2}H(q-1)\log q + C_{\hat{q}}, \quad x \to \infty \quad (1.13)$$

where $C_q = \log\sqrt{2\pi} + \log S_q$.

2. The complete monotonicity property

In this section, we prove that the reciprocal of the function $F_a(x;q)$ defined by (1.8) is completely monotonic on $(0,\infty)$ for all positive real q if and only if $a \geqslant g(\hat{q})$ where

$$g(q) = -\frac{(1+q)\log q + 2(1-q)}{2(1-q)\log^2 q}. \quad (2.1)$$

Moreover, the function $F_b(x;q)$ is completely monotonic on $(0,\infty)$ for all positive real q if and only if $b \leqslant 0$. These complete monotonicity properties are exploited to establish sharp two sided inequalities for the q-gamma and q-polygamma functions. It is worth mentioning that some of new bounds for the q-gamma and q-polygamma functions are improvements of those obtained in [17].

LEMMA 2.1. *Let a be a real, t be a positive real and the function*

$$f(a,t) = 2(1-e^{-t}) - t(1+e^{-t}) + 2at^2(1-e^{-t}). \quad (2.2)$$

Then, there exists a unique root of the function $a \mapsto f(a,t)$ depends on t at $a = a(t)$ where $a(t)$ is defined as

$$a(t) = \frac{t(e^t+1) - 2(e^t-1)}{2t^2(e^t-1)}. \quad (2.3)$$

Furthermore, the function $a(t)$ is decreasing on $(0,\infty)$ onto $(0,1/12)$.

Proof. It is clear that the function $a \mapsto f(a,t)$ is increasing on $\mathbb{R}$ for all $t > 0$. The function $f(a,t)$ can be represented as

$$f(a,t) = e^{-t}\sum_{n=3}^{\infty}\frac{t^n}{n!}(2an(n-1) + 2 - n)$$

Thus $f(0,t) < 0$ and $f(1,t) > 0$ for all fixed $t > 0$ which mean that the function $f(a,t)$ has a unique root at $a = a(t)$ where

$$a(t) = \frac{t(e^t+1) - 2(e^t-1)}{2t^2(e^t-1)}$$

Differentiation gives $2t^3(e^t-1)^2 a'(t) = h(t)$ where

$$h(t) = t + 4 - e^{2t}(t-4) - 2e^t(t^2+4)$$

which can be represented as

$$h(t) = -\sum_{n=2}^{\infty} \frac{t^n}{n!}\left(2^{n-1}(n-2)+2n(n-1)+8\right) < 0$$

Therefore, $a(t)$ is decreasing on $(0,\infty)$. By using L'Hospital rule, we find $\lim_{t\to 0} a(t) = 1/12$ and $\lim_{t\to\infty} a(t) = 0$. $\square$

THEOREM 2.2. *Let x and q be positive real. Then, the function $1/F_a(x;q)$ defined by (1.8) is logarithmically completely monotonic on $(0,\infty)$ for all positive real q if and only if $a \geqslant g(\hat{q})$ where $g(q)$ defined as in (2.1). Moreover, the function $F_b(x;q)$ is logarithmically completely monotonic on $(0,\infty)$ for all positive real q if and only if $b \leqslant 0$.*

Proof. Logarithmic derivative for (1.8) gives

$$\frac{d}{dx}(\log F_a(x;q)) = \psi_q(x) - \log[x]_q - \frac{1}{2}\frac{q^x \log q}{1-q^x} + \frac{aq^x \log^2 q}{(1-q^x)^2} \tag{2.4}$$

which can be rewritten when $0 < q < 1$ in the integral form as

$$\frac{d}{dx}(\log F_a(x;q)) = \frac{1}{2}\int_0^{\infty} \frac{e^{-xt}}{t(1-e^{-t})} f(a,t)\,d\gamma_q(t), \qquad x > 0$$

and for all positive integer n, we find

$$(-1)^n \frac{d^n}{dx^n}(\log F_a(x;q)) = -\frac{1}{2}\int_0^{\infty} \frac{t^{n-2}e^{-xt}}{1-e^{-t}} f(a,t)\,d\gamma_q(t), \qquad x > 0 \tag{2.5}$$

where $f(a,t)$ defined as in (2.2). According to the last formula and the definition of the discrete measure $d\gamma_q(t)$, the function $F_a(x;q)$ is logarithmically completely monotonic on $(0,\infty)$ if $f(a,t)d\gamma_q(t) \leqslant 0$ on $(0,\infty)$, that is if $f(a,t) \leqslant 0$ at the points $t = -k\log q, k \in \mathbb{N}$ and also the function $1/F_a(x;q)$ is logarithmically completely monotonic on $(0,\infty)$ if $f(a,t) \geqslant 0$ at the points $t = -k\log q, k \in \mathbb{N}$.

In view of Lemma 2.1, the function $a \mapsto f(a,t)$ is increasing on $\mathbb{R}$ and has a unique root at $a = a(t)$, then the function $a \mapsto f(a,-k\log q)$ is increasing on $\mathbb{R}$ and has a unique root at $a = a(-k\log q)$, $k \in \mathbb{N}$ and $0 < q < 1$. Since $a(t)$ is decreasing on $(0,\infty)$ and so the function $k \mapsto a(-k\log q)$ is decreasing for all $k \in \mathbb{N}$ which reveals that

$$\lim_{t\to\infty} a(t) = 0 < a(-k\log q) < a(-\log q) = g(q), \qquad 0 < q < 1$$

where $g(q)$ is defined as in (2.1). Therefore, $f(a,t) < 0$ if $a \leqslant 0$ and $f(a,t) > 0$ if $a \geqslant a(-\log q) = g(q)$ which conclude that $F_b(x;q)$ is logarithmically completely monotonic on $(0,\infty)$ if $b \leqslant 0$ and $1/F_a(x;q)$ is logarithmically completely monotonic on $(0,\infty)$ if $a \geqslant g(q)$.

It is easy from (1.2) and the identity

$$\mathrm{Li}_2\left(\frac{z-1}{z}\right) = -\mathrm{Li}_2(1-z) - \frac{1}{2}\log^2 z$$

to prove that $F_a(x;q) = q^{\frac{1}{2}-a}F_a(x;q^{-1})$ for all $q \geqslant 1$ which concludes that $F_a(x;q)$ is logarithmically completely monotonic on $(0,\infty)$ if $a \leqslant 0$ and $1/F_b(x;q)$ is logarithmically completely monotonic on $(0,\infty)$ if $b \geqslant g(\hat{q})$ for all $q > 0$.

Conversely, suppose the function $F_b(x;q)$ is logarithmically completely monotonic (with $b > 0$) on $(0,\infty)$ for all $q > 0$, then $(d/dx)(\log F_b(x;q)) < 0$ for all $x > 0$. But, this contradicts

$$\lim_{x\to 0}\left(\frac{d}{dx}\log F_b(x;q\right) = \infty.$$

Now, suppose the function $1/F_a(x;q)$ is logarithmically completely monotonic on $(0,\infty)$ for all $q > 0$, then we find

$$\lim_{x\to\infty}\left(\hat{q}^{-x}\frac{d}{dx}\log F_a(x;q)\right) \geqslant 0, \qquad q > 0$$

which is equivalent, from (2.4), to

$$\lim_{x\to\infty}\hat{q}^{-x}\left(\psi_q(x) - \log[x]_q - \frac{1}{2}\frac{q^x\log q}{1-q^x} + \frac{aq^x\log^2 q}{(1-q^x)^2}\right) \geqslant 0, \qquad q > 0$$

By inserting the identities (2.18) and (2.19) in [16] and the fact

$$\lim_{x\to\infty}\hat{q}^{-x}\frac{q^x\log^2 q}{(1-q^x)^2} = \log^2 q, \qquad q > 0$$

into the above inequality, we get

$$a \geqslant -\frac{(1+\hat{q})\log\hat{q} + 2(1-\hat{q})}{2(1-\hat{q})\log^2\hat{q}} = g(\hat{q}), \qquad q > 0.$$

This ends the proof. $\square$

From the fact that the logarithmically completely monotonic function is a subclass of completely monotonic function we provide the following theorem:

THEOREM 2.3. *Let x and q be positive real. Then, the reciprocal of the function $F_a(x;q)$ defined by (1.8) is completely monotonic on $(0,\infty)$ for all positive real q if and only if $a \geqslant g(\hat{q})$ where $g(q)$ defined as in (2.1). Moreover, the function $F_b(x;q)$ is completely monotonic on $(0,\infty)$ for all positive real q if and only if $b \leqslant 0$.*

Furthermore, we can present the following theorem:

THEOREM 2.4. *Let x and q be positive real. Then, the function*

$$G_a(x;q) = C_{\hat{q}} + \left(\frac{1}{2} - a\right) H(q-1)\log q - \log F_a(x;q) \tag{2.6}$$

where $F_a(x;q)$ defined by (1.8) is completely monotonic on $(0,\infty)$ for all positive real q if and only if $a \geqslant g(\hat{q})$ where $g(q)$ defined as in (2.1). Moreover, the function $-G_b(x;q)$ is completely monotonic on $(0,\infty)$ for all positive real q if and only if $b \leqslant 0$.

Proof. With the formula (2.5), we get

$$(-1)^n G_a^{(n)}(x;q)) = \frac{1}{2}\int_0^\infty \frac{t^{n-2}e^{-xt}}{1-e^{-t}} f(a,t)\, d\gamma_q(t), \qquad n \in \mathbb{N}$$

With the Moak formula (1.13) and the fact

$$\lim_{x\to\infty}\left(\frac{aq^x \log q}{1-q^x} + aH(q-1)\log q\right) = 0, \qquad q > 0$$

we get $\lim_{x\to\infty} G_a(x;q) = 0$ for all $q > 0$.

In view of the above and the proof of Theorem 2.1, we get the desired results. □

3. Two sided inequalities

As a consequence of the Theorems obtained in the section above, the following two-sided inequalities will be established as an application:

3.1. Ratio of the q-gamma functions

The monotonicity of the function $F_a(x;q)$ and its reciprocal as in Theorem 2.3 give

$$F_0(x;q) < F_0(y;q), \qquad \forall x > y > 0, q > 0$$

and

$$F_{g(\hat{q})}(x;q) > F_{g(\hat{q})}(y;q), \qquad \forall x > y > 0, q > 0$$

which can be read as

$$\frac{[x]_q^{x-\frac{1}{2}}}{[y]_q^{y-\frac{1}{2}}} \exp\left(\frac{\mathrm{Li}_2(1-q^x) - \mathrm{Li}_2(1-q^y)}{\log q} + \frac{g(\hat{q})q^y(1-q^{x-y})\log q}{(1-q^x)(1-q^y)}\right)$$
$$< \frac{\Gamma_q(x)}{\Gamma_q(y)} < \frac{[x]_q^{x-\frac{1}{2}}}{[y]_q^{y-\frac{1}{2}}} \exp\left(\frac{\mathrm{Li}_2(1-q^x) - \mathrm{Li}_2(1-q^y)}{\log q}\right), \quad \forall x > y > 0, q > 0 \tag{3.1}$$

3.2. The q-gamma function

The monotonicity of the function $G_a(x;q)$ as in Theorem 2.4 ($G_b(x;q) < 0 < G_a(x;q)$ for all $a \geqslant g(\hat{q})$ and $b \leqslant 0$) gives a class of inequalities for the q-gamma function. For all positive real x and q, the inequality

$$\sqrt{2\pi}S_{\hat{q}}q^{(\frac{1}{2}-b)H(q-1)}[x]_q^{x-\frac{1}{2}}\exp\left(\frac{\mathrm{Li}_2(1-q^x)}{\log q}-\frac{bq^x\log q}{1-q^x}\right) < \Gamma_q(x) < \sqrt{2\pi}S_{\hat{q}}q^{(\frac{1}{2}-a)H(q-1)}[x]_q^{x-\frac{1}{2}}\exp\left(\frac{\mathrm{Li}_2(1-q^x)}{\log q}-\frac{aq^x\log q}{1-q^x}\right) \tag{3.2}$$

holds true for all $a \geqslant g(\hat{q})$ and $b \leqslant 0$ with the best possible constants $a = g(\hat{q})$ and $b = 0$. It is clear that the left bound of inequality (1.4) is the same left bound of inequality (3.2) at the best constant $b = 0$. To compare the right bounds at the best constant $a = g(\hat{q})$, notice that the ratio of right bounds of (3.2) and (1.4) is

$$R = q^{(\frac{1}{12}-g(\hat{q}))H(q-1)}\exp\left(-\frac{q^x\log q}{1-q^x}\left(g(\hat{q})-\frac{1}{12}\right)\right)$$

which can be read as

$$R = \exp\left(-\frac{[q^x+(1-q^x)H(q-1)]\log q}{1-q^x}\left(g(\hat{q})-\frac{1}{12}\right)\right).$$

Lemma 2.1 tells that $0 < g(\hat{q}) < 1/12$ for all positive real q which reveals that $R < 1$ for all $q > 0$. Therefore, the right bound of (3.2) is less (better) than the right bound of (1.4).

3.3. The q-digamma function

The monotonicity of the function $G'_a(x;q)$ as in Theorem 2.4 ($G'_a(x;q) < 0 < G'_b(x;q)$ for all $a \geqslant g(\hat{q})$ and $b \leqslant 0$) gives a class of inequalities for the q-digamma function. For all positive real x and q, the inequality

$$\log[x]_q + \frac{1}{2}\frac{q^x\log q}{1-q^x} - \frac{aq^x\log^2 q}{(1-q^x)^2} < \psi_q(x) < \log[x]_q + \frac{1}{2}\frac{q^x\log q}{1-q^x} - \frac{bq^x\log^2 q}{(1-q^x)^2} \tag{3.3}$$

holds true for all $a \geqslant g(\hat{q})$ and $b \leqslant 0$ with the best possible constants $a = g(\hat{q})$ and $b = 0$. It is clear that the right bound of inequality (1.5) is the same right bound of inequality (3.3) at the best constant $b = 0$. Since $g(\hat{q}) < 1/12, \forall q > 0$, then the left bound of (3.3) is greater (better) than the left bound of (1.5).

3.4. The q-polygamma function

The monotonicity of the function $G_a^{(n)}(x;q), n \geqslant 2$ as in Theorem 2.4 ($(-1)^n G_b^{(n)}(x;q) < 0 < (-1)^n G_a^{(n)}(x;q), n \geqslant 2$ for all $a \geqslant g(\hat{q})$ and $b \leqslant 0$) gives a class of inequalities for the q-digamma function. For all positive real x and q, the inequality

$$(-1)^{r-1}\frac{d^{r-1}}{dx^{r-1}}\left[1-\frac{1}{2}\frac{d}{dr}+a\frac{d^2}{dx^2}\right]\left[\frac{q^x\log q}{1-q^x}\right]<(-1)^r\psi_q^{(r)}(x)$$
$$<(-1)^{r-1}\frac{d^{r-1}}{dx^{r-1}}\left[1-\frac{1}{2}\frac{d}{dr}+b\frac{d^2}{dx^2}\right]\left[\frac{q^x\log q}{1-q^x}\right], \qquad r\in\mathbb{N}$$

which by the identity [25]

$$\frac{d^r}{dx^r}\left[\frac{q^x\log q}{1-q^x}\right]=\left[\frac{\log q}{1-q^x}\right]^{r+1}q^xP_{r-1}(q^x), \qquad r\in\mathbb{N}$$

can be rewritten as

$$(-1)^{r+1}\beta_{r+1}(0)-a(-1)^r\left[\frac{\log q}{1-q^x}\right]^{r+2}q^xP_r(q^x)<(-1)^r\psi_q^{(r)}(x)$$
$$<(-1)^{r+1}\beta_{r+1}(0)-b(-1)^r\left[\frac{\log q}{1-q^x}\right]^{r+2}q^xP_r(q^x), \qquad r\in\mathbb{N} \tag{3.4}$$

holds true for all $a\geqslant g(\hat{q})$ and $b\leqslant 0$ with the best possible constants $a=g(\hat{q})$ and $b=0$. It is clear that the right bound of inequality (1.6) is the same right bound of inequality (3.4) at the best constant $b=0$. Since $g(\hat{q})<1/12, \forall q>0$, then the left bound of (3.4) is better than the left bound of (1.6).

In conclusion, the bounds here for the q-gamma and the q-polygamma functions are the same or better than the bounds obtained in [17].

3.5. The gamma and polygamma functions

In the case of the ordinary gamma and polygamma functions, $\lim_{q\to 1} d\gamma_q(t)=dt$ in (2.5) and the proof will be the same, thus we do not drew our attention for this part. The results above, in particular, can be rewritten for the gamma and polygamma functions by taking $q\to 1$ as follow

$$\frac{x^{x-\frac{1}{2}}}{y^{y-\frac{1}{2}}}\exp\left(\frac{y-x}{12xy)}\right)<\frac{e^x\Gamma(x)}{e^y\Gamma(y)}<\frac{x^{x-\frac{1}{2}}}{y^{y-\frac{1}{2}}}, \quad \forall x>y>0, \tag{3.1}$$

$$\sqrt{\frac{2\pi}{x}}x^x\exp\left(\frac{b}{x}-x\right)<\Gamma(x)<\sqrt{\frac{2\pi}{x}}x^x\exp\left(\frac{a}{x}-x\right) \tag{3.2}$$

$$\log x-\frac{1}{2x}-\frac{a}{x^2}<\psi(x)<\log x-\frac{1}{2x}-\frac{b}{x^2} \tag{3.3}$$

$$-\frac{(r+1)!a}{x^{r+2}}<(-1)^r\psi^{(r)}(x)+\frac{(r-1)!(2x+r)}{2x^{r+1}}<-\frac{(r+1)!b}{x^{r+2}}, \qquad r\in\mathbb{N} \tag{3.4}$$

These results hold true for all $a\geqslant 1/12$ and $b\leqslant 0$ with the best possible constants $a=1/12$ and $b=0$ and are shown to be new.

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