

ON THE HARDY–CARLEMAN INEQUALITY FOR A NEGATIVE EXPONENT

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Abstract. In this paper we settle an open problem raised by B. Yang (2005, *Taiwanese Journal of Mathematics* 9, 469–475), by using Hölder’s and Bernoulli’s inequalities. We give a strengthened Hardy–Carleman inequality for a negative exponent.

1. Introduction

The following inequality of Hardy’s is well known [2, Chap. 9.12]:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n a_k^{\frac{1}{p}} \right)^p < \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n. \quad (1)$$

Here $p > 1$, $a_n \geq 0$ ($n \in \mathbf{N}$) and $0 < \sum_{n=1}^{\infty} a_n < \infty$.

The constant $\left(\frac{p}{p-1}\right)^p$ in (1) is the best possible. As p tends to infinity the inequality (1) reduces to Carleman’s inequality

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n, \quad (2)$$

where the constant e in (2) is still the best possible [2, Chap. 9.12]. The inequalities (1) and (2) are important in analysis and its applications [3].

In [8], we proved the following strengthened version of (2).

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 + \frac{1}{n + \frac{1}{5}} \right)^{-\frac{1}{2}} a_n. \quad (3)$$

Some other strengthened versions of (2) and related results can be found in [1, 7, 8, 9, 11].

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If we set $p = \frac{1}{r}$ in (1), then we have $0 < r < 1$, and (1) is equivalent to the inequality

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n a_k^r \right)^{1/r} < \left(\frac{1}{1-r} \right)^{1/r} \sum_{n=1}^{\infty} a_n, \tag{4}$$

where the constant $\left(\frac{1}{1-r}\right)^{1/r}$ is the best possible.

Thanh et al. [6] discussed (4) for $r \in (-\infty, 0)$, and proved the following result: If $a_n \geq 0$ for $n \in \mathbf{N}$ and $0 < \sum_{n=1}^{\infty} a_n < \infty$, then

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n a_k^r \right)^{1/r} < \left(\frac{1}{1-r} \right)^{1/r} \sum_{n=1}^{\infty} a_n \tag{5}$$

if $-1 \leq r < 0$ and

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n a_k^r \right)^{1/r} < \frac{r}{r-1} 2^{\frac{r-1}{r}} \sum_{n=1}^{\infty} a_n \tag{6}$$

if $r < -1$.

If we replace r by $-r$, in (5) and (6) we obtain

$$\sum_{n=1}^{\infty} \left(\frac{n}{\sum_{k=1}^n a_k^{-r}} \right)^{1/r} < (1+r)^{1/r} \sum_{n=1}^{\infty} a_n \tag{7}$$

if $0 < r \leq 1$ and

$$\sum_{n=1}^{\infty} \left(\frac{n}{\sum_{k=1}^n a_k^{-r}} \right)^{1/r} < \frac{r}{1+r} 2^{\frac{1+r}{r}} \sum_{n=1}^{\infty} a_n \tag{8}$$

if $1 < r < \infty$.

Recently, Yang [10] proved that the constant $(1+r)^{1/r}$ in (7) is the best possible for $0 < r \leq 1$. At the end of paper [10], Yang posed the question:

Is the constant factor $\frac{r}{1+r} 2^{\frac{1+r}{r}}$ in (8) the best possible or not for $1 < r < \infty$?

In this paper we solve this problem. We give a strengthened Hardy-Carleman inequality for a negative exponent.

2. Main results

In this section, we prove the following theorem.

THEOREM 2.1. *Let $1 < r < \infty$, $a_n \geq 0$ ($n \in \mathbf{N}$) and $0 < \sum_{n=1}^{\infty} a_n < \infty$. Then*

$$\sum_{n=1}^{\infty} \left(\frac{n}{\sum_{k=1}^n a_k^{-r}} \right)^{1/r} < \frac{1}{r} (1+r)^{\frac{1+r}{r}} \sum_{n=1}^{\infty} a_n. \tag{9}$$

To prove Theorem 2.1, we use Hölder's inequality (with negative exponent p) (see [4, page 29]) and Bernoulli's inequality. For the convenience of the reader we start by recalling these results.

LEMMA 2.1. (Hölder's inequality) *Suppose that $p < 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(x) \geq 0$ for $x \in [a, b]$, and $f \in L^p[a, b]$, $g \in L^q[a, b]$. Then*

$$\int_a^b f(t)g(t)dt \geq \left(\int_a^b f^p(t)dt \right)^{1/p} \left(\int_a^b g^q(t)dt \right)^{1/q},$$

where equality holds only if there exist real numbers α and β , such that $\alpha^2 + \beta^2 > 0$, and $\alpha f^p(x) = \beta g^q(x)$, a.e. in $[a, b]$.

LEMMA 2.2. (Bernoulli inequality) *Suppose that $x \geq -1$ and $0 < \alpha < 1$. Then*

$$(1+x)^\alpha \leq 1 + \alpha x,$$

where equality holds if and only if $x = 0$.

We also need the following lemmas.

LEMMA 2.3. *Suppose that $0 < \alpha < 1$ and $x > 0$. Then*

$$1 + \frac{\alpha x}{1 + (1 - \alpha)x} < (1+x)^\alpha.$$

Proof. We rewrite this inequality as

$$1 + x + \alpha x(1+x)^\alpha < (1+x)^{1+\alpha}.$$

We define

$$\varphi(x) = (1+x)^{1+\alpha} - \alpha x(1+x)^\alpha - x - 1 \quad \text{for } x \geq 0.$$

Simple computation yields

$$\begin{aligned} \varphi'(x) &= (1+x)^\alpha - \alpha^2 x(1+x)^{\alpha-1} - 1, \\ \varphi''(x) &= (\alpha - \alpha^2)(1+x)^{\alpha-1} - \alpha^2(\alpha - 1)x(1+x)^{\alpha-2}. \end{aligned}$$

It follows that $\varphi''(x) > 0$ for $x > 0$ and $0 < \alpha < 1$, $\varphi'(0) = 0$ and $\varphi(0) = 0$. Thus, $\varphi(x)$ is strictly increasing and $\varphi(x) > 0$ for $x > 0$. This completes the proof of Lemma 2.3. \square

LEMMA 2.4. *Suppose that $r > 1$ and $x \geq 1$. Then*

$$(1+x)^{\frac{1+r}{r}} - x^{\frac{1+r}{r}} > \frac{1+r}{r} x^{\frac{1}{r}}.$$

Proof. We rewrite this inequality as

$$(1+x) \left(1 + \frac{1}{x}\right)^{\frac{1}{r}} - x > \frac{1+r}{r}.$$

This is true. Since by Lemma 2.3, we have

$$\left(1 + \frac{1}{x}\right)^{\frac{1}{r}} > 1 + \frac{\frac{1}{r}x}{1 + (1 - \frac{1}{r})x} > 1 + \frac{1}{r(1+x)}.$$

This completes the proof of Lemma 2.4. \square

LEMMA 2.5. *We have*

(i) $2^{\frac{1}{r}} < \frac{1+r}{r}$ for $r > 1$.

(ii) $\frac{r}{1+r} 2^{\frac{1+r}{r}} > \frac{1}{r}(1+r)^{\frac{1+r}{r}}$ for $r > \frac{26}{5}$.

Proof. (i) By Bernoulli's inequality we have

$$2^{\frac{1}{r}} = (1+1)^{\frac{1}{r}} < \frac{1+r}{r}.$$

(ii) We rewrite this inequality as

$$\frac{r^2}{1+r} > \left(\frac{1+r}{2}\right)^{\frac{1+r}{r}}.$$

By Bernoulli's inequality, it follows

$$\begin{aligned} \left(\frac{1+r}{2}\right)^{\frac{1+r}{r}} &= \frac{1+r}{2} \left(1 + \frac{r-1}{2}\right)^{\frac{1}{r}} \\ &< \frac{1+r}{2} \left(1 + \frac{r-1}{2r}\right) = -\frac{r^2(r-5) - r + 1}{4r(1+r)} + \frac{r^2}{1+r} \\ &< \frac{r^2}{1+r} \end{aligned}$$

for $r > \frac{26}{5}$.

This completes the proof of Lemma 2.5. \square

Proof of Theorem 2.1. Let $r > 1$ and set $p = -\frac{1}{r}$, $a = 1$, $b = x > 1$, $f(x) = a_n$, $g_n(x) = (x-1)^{\frac{1}{(1+r)r}}$ for $x \in [n, n+1]$ and $n \in \mathbf{N}$. Hölder's inequality then yields

$$\left(\int_1^x f(t)g(t)dt\right)^{-\frac{1}{r}} \leq \left(\int_1^x f^{-\frac{1}{r}}(t)dt\right) \left(\int_1^x g^{\frac{1}{1+r}}(t)dt\right)^{-\frac{1+r}{r}}.$$

It follows that

$$\begin{aligned} \left(\int_1^x f^{-r}(t) dt\right)^{-\frac{1}{r}} &= \left(\int_1^x ((t-1)^{1+r} f(t))^{-r} ((t-1)^{(1+r)r} dt)\right)^{-\frac{1}{r}} \\ &< \left(\int_1^x (t-1)^{1+r} f(t) dt\right) \left(\int_1^x (t-1)^r dt\right)^{-\frac{1+r}{r}} \\ &< (1+r)^{\frac{1+r}{r}} (x-1)^{-\frac{(1+r)^2}{r}} \int_1^x (t-1)^{1+r} f(t) dt. \end{aligned}$$

Then we have

$$\left(\frac{x-1}{\int_1^x f^{-r}(t) dt}\right)^{1/r} < (1+r)^{\frac{1+r}{r}} (x-1)^{-r-2} \int_1^x (t-1)^{1+r} f(t) dt.$$

Then we obtain

$$\begin{aligned} \int_1^\infty \left(\frac{x-1}{\int_1^x f^{-r}(t) dt}\right)^{1/r} dx &< (1+r)^{\frac{1+r}{r}} \int_1^\infty (x-1)^{-r-2} \int_1^x (t-1)^{1+r} f(t) dt dx \\ &= (1+r)^{\frac{1+r}{r}} \int_1^\infty \left(\int_t^\infty (x-1)^{-r-2} dx\right) (t-1)^{1+r} f(t) dt \\ &= (1+r)^{1/r} \int_1^\infty f(t) dt \\ &= (1+r)^{1/r} \sum_{n=1}^\infty a_n. \end{aligned}$$

By the definition of $f(x)$, Lemmas 2.3, 2.4 and 2.5, we have

$$\begin{aligned} \int_1^\infty \left(\frac{x-1}{\int_1^x f^{-r}(t) dt}\right)^{1/r} dx &> \int_1^2 \left(\frac{x-1}{\int_1^x f^{-r}(t) dt}\right)^{1/r} dx + \int_2^\infty \left(\frac{x-1}{\int_1^x f^{-r}(t) dt}\right)^{1/r} dx \\ &> \frac{\int_1^2 (x-1)^{1/r} dx}{a_1^{-1}} + \sum_{n=2}^\infty \frac{\int_n^{n+1} (x-1)^{1/r} dx}{(\sum_{k=1}^n a_k^{-r})^{1/r}} \\ &> \frac{r}{a_1^{-1}} + \sum_{n=2}^\infty \frac{r}{1+r} \frac{(n^{\frac{1+r}{r}} - (n-1)^{\frac{1+r}{r}})}{(\sum_{k=1}^n a_k^{-r})^{1/r}} \\ &> \frac{r}{a_1^{-1}} + \sum_{n=2}^\infty \frac{(n-1)^{1/r}}{(\sum_{k=1}^n a_k^{-r})^{1/r}} \\ &> \frac{r}{a_1^{-1}} + \left(\frac{1}{2}\right)^{1/r} \sum_{n=2}^\infty \frac{n^{1/r}}{(\sum_{k=1}^n a_k^{-r})^{1/r}} \\ &> \frac{r}{1+r} \sum_{n=1}^\infty \left(\frac{n}{\sum_{k=1}^n a_k^{-r}}\right)^{1/r}. \end{aligned}$$

This completes the proof of Theorem 2.1. \square

REMARK 1. By Theorem 2.1 and Lemma 2.5 (ii), we know that the constant factor $\frac{r}{1+r} 2^{\frac{1+r}{r}}$ in (8) is not the best possible for $r > \frac{26}{5}$. We give a strengthened Hardy-Carleman inequality (9) for a negative exponent.

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