

## A NOTE ON GENERALIZED CAUCHY–SCHWARZ INEQUALITY

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*Abstract.* We generalize the well-known Cauchy-Schwarz inequality involving any number of real or complex matrices, and also give a necessary and sufficient condition for the equality. This is an improvement of the two recent literatures due to N. Harvey and D. Choi.

### 1. Introduction

In [1], N. Harvey generalized the Cauchy-Schwarz inequality to an inequality involving four vectors. Namely, for any  $a, b, c, d \in \mathbb{R}^n$ , it holds that

$$\|a\|^2\|b\|^2 + \|c\|^2\|d\|^2 \geq 2a^T c b^T d + (a^T b)^2 + (c^T d)^2 - (a^T d)^2 - (b^T c)^2.$$

Afterwards, the result was refined by D. Choi ([2]) to a stronger one:

$$\sum_{k=1}^m \|a_{(k)}\|^2 \|b_{(k)}\|^2 \geq \sum_{k=1}^m (a_{(k)}^T b_{(k)})^2 \tag{1.1}$$

$$+ \frac{2}{m-1} \sum_{1 \leq k < l \leq m} |a_{(k)}^T a_{(l)} b_{(k)}^T b_{(l)} - a_{(k)}^T b_{(l)} b_{(k)}^T a_{(l)}|$$

for any  $a_{(k)}, b_{(k)} \in \mathbb{R}^n, k = 1, \dots, m$ , and  $m \geq 2$ . Furthermore, he also gained the complex version of the inequality.

In this paper, we will extend them into the following:

$$\sum_{k=1}^m \|A_{(k)}\|^2 \|B_{(k)}\|^2 \geq \sum_{k=1}^m (\text{tr}(A_{(k)}^T B_{(k)}))^2 \tag{1.2}$$

$$+ \frac{2}{m-1} \sum_{1 \leq k < l \leq m} \left| \text{tr}(A_{(k)}^T A_{(l)}) \text{tr}(B_{(k)}^T B_{(l)}) - \text{tr}(A_{(k)}^T B_{(l)}) \text{tr}(B_{(k)}^T A_{(l)}) \right|$$

for any  $n \times n$  real matrix  $A_{(k)}, B_{(k)}, k = 1, \dots, m$ , and  $m \geq 2$  (Theorem 2.2). We remark that when  $\{A_{(k)}, B_{(k)}, k = 1, \dots, m\}$  are  $2m$  diagonal matrices, (1.2) becomes (1.1). The complex version of Inequality (1.2) is also obtained (Theorem 2.4). The whole proof is direct as in [1] and [2], but the calculation is a bit more tedious.

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Recall that N. Harvey and D. Choi used the Euclidean norm on  $\mathbb{R}^n$  or  $\mathbb{C}^n$ , while in this paper we generalized the inequality using the Frobenius norm on the space of  $n \times n$  real or complex matrices. Noticed that an  $n \times n$  matrix can be viewed as an  $n^2$  dimensional vector, and the Frobenius norm of an  $n \times n$  matrix is the Euclidean norm of that  $n^2$  dimensional vector. Moreover, for any  $n \times n$  matrix  $A$  and  $B$ , define inner product as  $\langle A, B \rangle =: \text{tr}(A^T B)$ . It easy to see this inner product equals to the Euclidean inner product for  $A$  and  $B$  as  $n^2$  dimensional vectors. Thus, we remark that the inequalities obtained are also true for any finite inner product space.

In [2], D. Choi gave a necessary and sufficient condition if the equality holds in (1.1):  $a_{(k),i}b_{(k),j} - a_{(k),j}b_{(k),i} = a_{(l),i}b_{(l),j} - a_{(l),j}b_{(l),i}$  for all  $k, l = 1, \dots, m$ , and  $i, j = 1, \dots, n$ . We point out that the author missed the absolute notation there. In this paper we will correct this mistake in Theorems 2.1 ~ 2.4 below.

### 2. Main results and their proofs

**THEOREM 2.1.** *Let  $A, B, C$  and  $D$  be four  $n \times n$  real matrices. Then*

$$\|A\|^2\|B\|^2 + \|C\|^2\|D\|^2 \geq (\text{tr}(A^T B))^2 + (\text{tr}(C^T D))^2 + 2|\text{tr}(A^T C)\text{tr}(B^T D) - \text{tr}(A^T D)\text{tr}(B^T C)|,$$

where the equality holds if and only if  $a_{ij}b_{kl} - a_{kl}b_{ij} = \pm(c_{ij}d_{kl} - c_{kl}d_{ij})$  for all  $i, j, k, l = 1, \dots, n$ .

*Proof.* Notice that  $\|A\|^2 = \sum_{i,j} a_{ij}^2$ ,  $\|B\|^2 = \sum_{i,j} b_{ij}^2$  and  $\text{tr}(A^T B) = \sum_{i,j} a_{ij}b_{ij}$ . Then we have

$$\begin{aligned} \|A\|^2\|B\|^2 - (\text{tr}(A^T B))^2 &= \sum_{i,j} a_{ij}^2 \sum_{k,l} b_{kl}^2 - \sum_{i,j} a_{ij}b_{ij} \sum_{k,l} a_{kl}b_{kl} \\ &= \sum_{i,j,k,l} (a_{ij}^2 b_{kl}^2 - a_{ij}b_{ij}a_{kl}b_{kl}) \\ &= \frac{1}{2} \sum_{i,j,k,l} (a_{ij}b_{kl} - a_{kl}b_{ij})^2. \end{aligned}$$

Using the formula above, we obtain

$$\begin{aligned} &\|A\|^2\|B\|^2 + \|C\|^2\|D\|^2 - (\text{tr}(A^T B))^2 - (\text{tr}(C^T D))^2 \\ &= \frac{1}{2} \sum_{i,j,k,l} [(a_{ij}b_{kl} - a_{kl}b_{ij})^2 + (c_{ij}d_{kl} - c_{kl}d_{ij})^2] \\ &= \frac{1}{2} \sum_{i,j,k,l} \left\{ [(a_{ij}b_{kl} - a_{kl}b_{ij}) \pm (c_{ij}d_{kl} - c_{kl}d_{ij})]^2 \mp 2(a_{ij}b_{kl} - a_{kl}b_{ij})(c_{ij}d_{kl} - c_{kl}d_{ij}) \right\} \\ &= \frac{1}{2} \sum_{i,j,k,l} [(a_{ij}b_{kl} - a_{kl}b_{ij}) \pm (c_{ij}d_{kl} - c_{kl}d_{ij})]^2 \mp \left( 2 \sum_{i,j} a_{ij}c_{ij} \sum_{k,l} b_{kl}d_{kl} - 2 \sum_{i,j} a_{ij}d_{ij} \sum_{k,l} b_{kl}c_{kl} \right) \\ &\geq \mp 2(\text{tr}(A^T C)\text{tr}(B^T D) - \text{tr}(A^T D)\text{tr}(B^T C)), \end{aligned}$$

which gives the desired inequality. The condition for equality is obvious. This finishes the proof.  $\square$

**THEOREM 2.2.** *Let  $A_{(1)}, \dots, A_{(m)}, B_{(1)}, \dots, B_{(m)}$  be  $n \times n$  real matrices. Then, for  $m \geq 2$ , we have*

$$\sum_{k=1}^m \|A_{(k)}\|^2 \|B_{(k)}\|^2 \geq \sum_{k=1}^m (\text{tr}(A_{(k)}^T B_{(k)}))^2 + \frac{2}{m-1} \sum_{1 \leq k < l \leq m} \left| \text{tr}(A_{(k)}^T A_{(l)}) \text{tr}(B_{(k)}^T B_{(l)}) - \text{tr}(A_{(k)}^T B_{(l)}) \text{tr}(B_{(k)}^T A_{(l)}) \right|,$$

where the equality holds if and only if  $a_{(k),ij} b_{(k),pq} - a_{(k),pq} b_{(k),ij} = \pm (a_{(l),ij} b_{(l),pq} - a_{(l),pq} b_{(l),ij})$  for all  $k, l = 1, \dots, m$  and  $i, j, p, q = 1, \dots, n$ .

*Proof.* It follows from Theorem 2.1 that

$$\begin{aligned} & \|A_{(k)}\|^2 \|B_{(k)}\|^2 + \|A_{(l)}\|^2 \|B_{(l)}\|^2 - (\text{tr}(A_{(k)}^T B_{(k)}))^2 - (\text{tr}(A_{(l)}^T B_{(l)}))^2 \\ & \geq 2 \left| \text{tr}(A_{(k)}^T A_{(l)}) \text{tr}(B_{(k)}^T B_{(l)}) - \text{tr}(A_{(k)}^T B_{(l)}) \text{tr}(B_{(k)}^T A_{(l)}) \right| \end{aligned}$$

for  $1 \leq k < l \leq m$ . Thus, we have

$$\begin{aligned} & \sum_{k=1}^m \left[ \|A_{(k)}\|^2 \|B_{(k)}\|^2 - (\text{tr}(A_{(k)}^T B_{(k)}))^2 \right] \\ & = \frac{1}{m-1} \sum_{1 \leq k < l \leq m} \left[ \|A_{(k)}\|^2 \|B_{(k)}\|^2 + \|A_{(l)}\|^2 \|B_{(l)}\|^2 - (\text{tr}(A_{(k)}^T B_{(k)}))^2 - (\text{tr}(A_{(l)}^T B_{(l)}))^2 \right] \\ & \geq \frac{2}{m-1} \sum_{1 \leq k < l \leq m} \left| \text{tr}(A_{(k)}^T A_{(l)}) \text{tr}(B_{(k)}^T B_{(l)}) - \text{tr}(A_{(k)}^T B_{(l)}) \text{tr}(B_{(k)}^T A_{(l)}) \right| \quad \square \end{aligned}$$

In what follows, we consider the complex case. The complex Frobenius inner product is defined by  $\langle A, B \rangle = \text{tr}(A^* B)$ . First, we give

**THEOREM 2.3.** *Let  $A, B, C$  and  $D$  be four  $n \times n$  complex matrices. Then*

$$\begin{aligned} \|A\|^2 \|B\|^2 + \|C\|^2 \|D\|^2 & \geq |\text{tr}(A^* B)|^2 + |\text{tr}(C^* D)|^2 \\ & \quad + 2 |\text{Re}(\text{tr}(A^* C) \text{tr}(B^* D) - \text{tr}(A^* D) \text{tr}(B^* C))|, \end{aligned}$$

where the equality holds if and only if  $a_{ij} b_{kl} - a_{kl} b_{ij} = \pm (c_{ij} d_{kl} - c_{kl} d_{ij})$  for all  $i, j, k, l = 1, \dots, n$ .

*Proof.* Since  $A, B$  are complex matrices,  $\|A\|^2 = \sum_{i,j} a_{ij} \overline{a_{ij}}$ ,  $\|B\|^2 = \sum_{i,j} b_{ij} \overline{b_{ij}}$ . Hence,

$$\begin{aligned} \|A\|^2 \|B\|^2 - |\operatorname{tr}(A^* B)|^2 &= \sum_{i,j} a_{ij} \overline{a_{ij}} \sum_{k,l} b_{kl} \overline{b_{kl}} - \sum_{i,j} \overline{a_{ij}} b_{ij} \sum_{k,l} a_{kl} \overline{b_{kl}} \\ &= \sum_{i,j,k,l} [a_{ij} \overline{a_{ij}} b_{kl} \overline{b_{kl}} - \overline{a_{ij}} b_{ij} a_{kl} \overline{b_{kl}}] \\ &= \frac{1}{2} \sum_{i,j,k,l} |a_{ij} b_{kl} - a_{kl} b_{ij}|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} &\|A\|^2 \|B\|^2 + \|C\|^2 \|D\|^2 - |\operatorname{tr}(A^* B)|^2 - |\operatorname{tr}(C^* D)|^2 \\ &= \frac{1}{2} \sum_{i,j,k,l} [|a_{ij} b_{kl} - a_{kl} b_{ij}|^2 + |c_{ij} d_{kl} - c_{kl} d_{ij}|^2] \\ &= \frac{1}{2} \sum_{i,j,k,l} [|(a_{ij} b_{kl} - a_{kl} b_{ij}) \pm (c_{ij} d_{kl} - c_{kl} d_{ij})|^2 \mp 2 \operatorname{Re}(\overline{a_{ij} b_{kl} - a_{kl} b_{ij}})(c_{ij} d_{kl} - c_{kl} d_{ij})] \\ &= \frac{1}{2} \sum_{i,j,k,l} |(a_{ij} b_{kl} - a_{kl} b_{ij}) \pm (c_{ij} d_{kl} - c_{kl} d_{ij})|^2 \\ &\quad \mp 2 \operatorname{Re} \left( \sum_{i,j} \overline{a_{ij}} c_{ij} \sum_{k,l} \overline{b_{kl}} d_{kl} - \sum_{i,j} \overline{a_{ij}} d_{ij} \sum_{k,l} \overline{b_{kl}} c_{kl} \right) \\ &\geq \mp 2 \operatorname{Re}(\operatorname{tr}(A^* C) \operatorname{tr}(B^* D) - \operatorname{tr}(A^* D) \operatorname{tr}(B^* C)). \end{aligned}$$

Then the desired inequality follows.  $\square$

By a similar argument, we further obtain

**THEOREM 2.4.** *Let  $A_{(1)}, \dots, A_{(m)}, B_{(1)}, \dots, B_{(m)}$  be  $n \times n$  complex matrices. Then, for  $m \geq 2$ , we have*

$$\begin{aligned} \sum_{k=1}^m \|A_{(k)}\|^2 \|B_{(k)}\|^2 &\geq \sum_{k=1}^m |\operatorname{tr}(A_{(k)}^* B_{(k)})|^2 \\ &\quad + \frac{2}{m-1} \sum_{1 \leq k < l \leq m} \left| \operatorname{Re}(\operatorname{tr}(A_{(k)}^* A_{(l)}) \operatorname{tr}(B_{(k)}^* B_{(l)}) - \operatorname{tr}(A_{(k)}^* B_{(l)}) \operatorname{tr}(B_{(k)}^* A_{(l)}) \right|, \end{aligned}$$

where the equality holds if and only if  $a_{(k),ij} b_{(k),pq} - a_{(k),pq} b_{(k),ij} = \pm (a_{(l),ij} b_{(l),pq} - a_{(l),pq} b_{(l),ij})$  for all  $k, l = 1, \dots, m$  and  $i, j, p, q = 1, \dots, n$ .

Finally, we unify the two Frobenius inner product (real and complex) into one representation by  $\langle \cdot, \cdot \rangle$ . Then Theorems 2.1 ~ 2.4 can be written as

**THEOREM 2.5.** *Let  $A_{(1)}, \dots, A_{(m)}, B_{(1)}, \dots, B_{(m)}$  be  $n \times n$  real or complex matri-*

ces. Then, for  $m \geq 2$ , we have

$$\sum_{k=1}^m \|A_{(k)}\|^2 \|B_{(k)}\|^2 \geq \sum_{k=1}^m |\langle A_{(k)}, B_{(k)} \rangle|^2 + \frac{2}{m-1} \sum_{1 \leq k < l \leq m} |\operatorname{Re}(\langle A_{(k)}, A_{(l)} \rangle \langle B_{(k)}, B_{(l)} \rangle - \langle A_{(k)}, B_{(l)} \rangle \langle B_{(k)}, A_{(l)} \rangle)|,$$

where the equality holds if and only if  $a_{(k),ij}b_{(k),pq} - a_{(k),pq}b_{(k),ij} = \pm(a_{(l),ij}b_{(l),pq} - a_{(l),pq}b_{(l),ij})$  for all  $k, l = 1, \dots, m$  and  $i, j, p, q = 1, \dots, n$ .

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