Journal of Mathematical Inequalities
Volume 11, Number 4 (2017), 983–995 doi:10.7153/jmi-2017-11-74

THE LOG–MINKOWSKI INEQUALITIES FOR QUERMASSINTEGRALS

WEI WANG AND MING FENG

(*Communicated by J. Pečarić*)

Abstract. Recently, Stancu established the log-Minkowski inequality for non-symmetric convex bodies. In this article, we establish the log-Minkowski inequality for quermassintegrals, which is more general than Stancu's results.

1. Introduction

Böröczky et al. [1] established the plane log-Brunn-Minkowski inequality and the plane log-Minkowski inequality for origin-symmetric convex bodies. For $n \geqslant 3$, they [1] conjectured that there exists the log-Brunn-Minkowski inequality and log-Minkowski inequality for origin-symmetric convex bodies in $\mathbb{R}^n$, and showed that these two inequalities are equivalent. Saroglou [13] established the log-Minkowski inequality for unconditional convex bodies. Ma [12] gave a new proof of the plane log-Minkowski inequality for origin-symmetric convex bodies.

Böröczky et al. [1] also pointed out that while the log-Minkowski inequality holds for two origin-symmetric cubes, there exists a translate of one of the cubes which makes the inequality false. It means that the log-Minkowski inequality cannot hold for all convex bodies.

Recently, Stancu [15] established the log-Minkowski inequality for general convex bodies without the symmetry assumption.

THEOREM A. *Let K and L be two convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors. Then*

$$\int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{v}_L \geqslant \frac{\left(\frac{h_K}{h_L}\right)_{average}}{\left(\frac{h_K}{h_L}\right)_{\max}} \cdot \frac{1}{n} \ln \frac{V(K)}{V(L)} + \ln\left[\left(\frac{h_K}{h_L}\right)_{\min}\right] \cdot \left[1 - \frac{\left(\frac{h_K}{h_L}\right)_{average}}{\left(\frac{h_K}{h_L}\right)_{\max}}\right],$$

Mathematics subject classification (2010): 52A40, 53A15.

Keywords and phrases: Star body, radial function, dual mixed volumes, dual Minkowski inequality, log-Minkowski inequality.

A project supported by Scientific Research of Hunan Provincial Education Department (No. 16C0635 and 2017JJ3085) and China Postdoctoral Science Foundation (No. 2016M601644).

Paper JMI-11-74

with equality if and only if K is homothetic to L. Here $d\bar{v}_L$ denotes the cone-volume probability measure of L. $\left(\frac{h_K}{h_L}\right)_{average} := \frac{\int_{S^{n-1}} \frac{h_K}{h_L} dv_L}{\int_{S^{n-1}} dv_L}$, $\left(\frac{h_K}{h_L}\right)_{\max} := \max_{u \in supp v_L} \frac{h_K}{h_L}$, $\left(\frac{h_K}{h_L}\right)_{\min} := \min_{u \in supp v_K} \frac{h_K}{h_L}$. *Here supp v_K and supp v_L will be denoted by the support of the cone-volume measure of v_K and v_L, respectively.*

By adapting the proofs in [15], we will generalize Stancu's results. The main purpose of this paper is to establish the log-Minkowski inequality for quermassintegrals as follows.

THEOREM 1.1. *Let K and L be two convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors. For $i = 0, \ldots, n-1$, then*

$$\int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{v}_{i,L} \geqslant \frac{\left(\frac{h_K}{h_L}\right)_{i,average}}{\left(\frac{h_K}{h_L}\right)_{i,\max}} \cdot \frac{1}{n-i} \ln \frac{W_i(K)}{W_i(L)} + \ln \left[\left(\frac{h_K}{h_L}\right)_{i,\min}\right] \cdot \left[1 - \frac{\left(\frac{h_K}{h_L}\right)_{i,average}}{\left(\frac{h_L}{h_K}\right)_{i,\max}}\right],$$

with equality if and only if K is homothetic to L. Here $d\bar{v}_{i,L}$ denotes the mixed cone-volume probability measure of L. $\left(\frac{h_K}{h_L}\right)_{i,average} := \frac{\int_{S^{n-1}} \frac{h_K}{h_L} dv_{i,L}}{\int_{S^{n-1}} dv_{i,L}}$, $\left(\frac{h_K}{h_L}\right)_{i,\max} := \max_{u \in supp v_{i,L}} \frac{h_K}{h_L}$, $\left(\frac{h_K}{h_L}\right)_{i,\min} := \min_{u \in supp v_{i,K}} \frac{h_K}{h_L}$.

2. Notation and background material

For general reference for the theory of convex (star) bodies the reader may wish to consult the books of Gardner [3], Gruber [7], and Schneider [14].

The setting for this paper is the n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathscr{K}^n$ denote the set that consists of all convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$, and let $\mathscr{K}_0^n$ denote the set of convex bodies that contain the origin in their interiors. The unit ball and its surface in $\mathbb{R}^n$ are denoted by B and S^{n-1}, respectively. We write $V(K)$ for the volume of the compact set K in $\mathbb{R}^n$. As usual, $V(B) = \omega_n$. The support function of $K \in \mathscr{K}^n, h_K(\cdot)$, is defined on S^{n-1} by

$$h_K(u) = \max\{u \cdot x : \ x \in K\}. \tag{2.1}$$

If $K \in \mathscr{K}_0^n$, then the polar body of K, K^*, is defined by

$$K^* := \{x \in \mathbb{R}^n : \ x \cdot y \leqslant 1, \forall y \in K\}. \tag{2.2}$$

The radial function $\rho_L : S^{n-1} \to [0, \infty)$ of a compact star-shaped set about the origin, $L \in \mathbb{R}^n$, is defined, for $u \in S^{n-1}$, by

$$\rho_L(u) = \max\{\lambda \geqslant 0 : \ \lambda u \in L\}. \tag{2.3}$$

If $\rho_L(\cdot)$ is positive and continuous, then L is called a star body about the origin. The set of star bodies about the origin in $\mathbb{R}^n$ is denoted by $\mathscr{S}^n$.

Obviously, for $K,L \in \mathscr{S}^n$,

$$K \subseteq L \Leftrightarrow \rho_K(u) \leqslant \rho_L(u), \ \ \forall\, u \in S^{n-1}. \tag{2.4}$$

If $\dfrac{\rho_K(u)}{\rho_L(u)}$ is independent of $u \in S^{n-1}$, then we say that star bodies K and L are dilates.

If $s > 0$, we have

$$\rho_{sK}(u) = s\rho_K(u), \ \text{ for all } u \in S^{n-1}. \tag{2.5}$$

If $K \in \mathscr{K}_0^n$, then, for $\forall u \in S^{n-1}$, we have

$$\rho_K(u)^{-1} = h_{K^*}(u). \tag{2.6}$$

Let $K_1, K_2 \in \mathscr{K}_0^n$. For $0 \leqslant i \leqslant n-1$, we write $W_i(K_1,K_2)$ for the mixed volume $V(K_1,\ldots,K_1,K_2,B,\ldots,B)$, where K_1 appears $n-i-1$ times, the Euclidean unit ball B appears i times, and K_2 appears once. The mixed volume $W_i(K_1,K_2)$ has the following integral representation:

$$W_i(K_1,K_2) = \frac{1}{n}\int_{S^{n-1}} h_{K_2}\, dS_{i,K_1}, \tag{2.7}$$

where dS_{i,K_1} is the ith surface area measure of K. In particular, $dS_{0,K_1} = dS_{K_1}$ is called the surface area measure of K, and $dS_{n-1,K_1} = dS$ is called the Lebesgue measure on S^{n-1}.

The mixed volume $W_i(K_1,K_1)$ will be written as $W_i(K_1)$. It is called the ith quermassintegral of K_1 and has the following integral representation:

$$W_i(K_1) = \frac{1}{n}\int_{S^{n-1}} h_{K_1}\, dS_{i,K_1}. \tag{2.8}$$

The Minkowski inequality for mixed quermassintegrals states (see [11]): For $K_1, K_2 \in \mathscr{K}^n$ and $0 \leqslant i < n-1$,

$$W_i(K_1,K_2)^{n-i} \geqslant W_i(K_1)^{n-i-1} W_i(K_2), \tag{2.9}$$

with equality if and only if K_1 and K_2 are homothetic.

For $L_1, L_2 \in \mathscr{S}^n$ and $i = 0,\ldots,n-1$. The dual mixed quermassintegral $\widetilde{W}_i(L_1,L_2)$ has the following integral representation:

$$\widetilde{W}_i(L_1,L_2) = \frac{1}{n}\int_{S^{n-1}} \rho_{L_1}^{n-i-1}\rho_{L_2}\, dS. \tag{2.10}$$

The dual quermassintegral $\widetilde{W}_i(L_1)$ has the following integral representation:

$$\widetilde{W}_i(L_1) = \frac{1}{n}\int_{S^{n-1}} \rho_{L_1}^{n-i}\, dS. \tag{2.11}$$

If $i=0$, then $\widetilde{W}_0(L_1)=V(L_1)$.

By using Minkowski's integral inequality, we can obtain the dual Minkowski inequality for dual mixed quermassintegrals (see [10]): For $L_1,L_2 \in \mathscr{S}^n$ and $0 \leqslant i < n-1$,

$$\widetilde{W}_i(L_1,L_2)^{n-i} \leqslant \widetilde{W}_i(L_1)^{n-i-1}\widetilde{W}_i(L_2), \tag{2.12}$$

with equality if and only if L_1 and L_2 are dilates.

The dual Minkowski inequality for dual quermassintegrals states that (see [10]): For $L \in \mathscr{S}^n$ and $0 < i < n-1$,

$$\widetilde{W}_i(L)^n \leqslant V(L)^{n-i}\omega_n^i, \tag{2.13}$$

with equality if and only if L is an origin-symmetric ball.

Applying the above dual Minkowski inequality (2.13) and Blaschke-Santaló inequality, we can obtain the following dual Blaschke-Santaló inequality: For $L \in \mathscr{K}_0^n$ and $0 \leqslant i \leqslant n-1$,

$$\widetilde{W}_i(L)\widetilde{W}_i(L^*) \leqslant \omega_n^2, \tag{2.14}$$

with equality if and only if L is an origin-symmetric ball.

Suppose that μ is a probability measure on a space X and $g: X \to I \subset \mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. Jessen's inequality (see [9]) states that if $\phi: X \to I \subset \mathbb{R}$ is a strictly convex function, then

$$\int_X \phi(g(x))d\mu(x) \geqslant \phi\left(\int_X g(x)d\mu(x)\right), \tag{2.15}$$

with equality if and only if $g(x)$ is a constant for μ-almost all $x \in X$. If ϕ is a strictly concave function, then the inequality (2.15) is reversed.

3. Main results

For $K,L \in \mathscr{K}_0^n$ and $i=0,\ldots,n-1$, since the mixed quermassintegral $W_i(L,K) = \frac{1}{n}\int_{S^{n-1}} h_K dS_{i,L}$, we denote the mixed quermassintegral measure by

$$dw_{i,L} = \frac{1}{n}h_L dS_{i,L}. \tag{3.1}$$

We write the mixed quermassintegral probability measure on S^{n-1} by

$$d\bar{w}_{i,L} = \frac{1}{W_i(L,K)}dw_{i,L}. \tag{3.2}$$

If $K=L$, then we write $dv_{i,L}$ for $dw_{i,L}$, and call it the mixed cone-volume probability measure of L. It is easy to check that

$$dv_{i,L} = \frac{h_L}{h_K}dw_{i,L}. \tag{3.3}$$

In particular, the measure $dv_{i,B}$ is just the Lebesgue measure on S^{n-1}. We write the quermassintegral probability measure on S^{n-1} by

$$d\bar{v}_{i,L} = \frac{1}{W_i(L)} dv_{i,L}. \tag{3.4}$$

The measure $d\bar{v}_{0,L}$ will be called the cone-volume probability measure, and it will be written simply as $d\bar{v}_L$.

PROPOSITION 3.1. *Let $K, L \in \mathscr{K}_0^n$ and $i = 0, \ldots, n-1$. Then*

$$\int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{w}_{i,L} \geqslant \ln \frac{W_i(L,K)}{W_i(L)} \geqslant \frac{1}{n-i} \ln \frac{W_i(K)}{W_i(L)},$$

with equality if and only if K is homothetic to L.

Proof. By Lebesgue's dominated convergence theorem, as $p \to \infty$,

$$\int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^{\frac{p}{n+p}} dv_{i,L} \to W_i(L,K) \quad \text{and}$$

$$\int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^{\frac{p}{n+p}} \ln \frac{h_K}{h_L} dv_{i,L} \to \int_{S^{n-1}} \frac{h_K}{h_L} \ln \frac{h_K}{h_L} dv_{i,L} = \int_{S^{n-1}} \ln \frac{h_K}{h_L} dw_{i,L}$$

We define the function $f_{K,L} : [1,\infty] \to \mathbb{R}$ by

$$f_{K,L}(p) = \frac{1}{W_i(L,K)} \int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^{\frac{p}{n+p}} dv_{i,L}. \tag{3.5}$$

Note that

$$\begin{aligned} \lim_{p\to\infty} \ln(f_{K,L}(p))^{p+n} &= \lim_{p\to\infty} \frac{\frac{nW_i(L,K)}{(p+n)^2} \int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^{\frac{p}{p+n}} \ln \frac{h_K}{h_L} dv_{i,L}}{-\frac{f_{K,L}(p)}{(p+n)^2}} \\ &= -\frac{n}{W_i(L,K)} \int_{S^{n-1}} \frac{h_K}{h_L} \ln \frac{h_K}{h_L} dv_{i,L}. \end{aligned} \tag{3.6}$$

By (3.2), (3.3), (3.6), and (3.5), we have

$$\begin{aligned} \exp\left[-n \int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{w}_{i,L}\right] &= \exp\left[-\frac{n}{W_i(L,K)} \int_{S^{n-1}} \ln \frac{h_K}{h_L} dw_{i,L}\right] \\ &= \exp\left[-\frac{n}{W_i(L,K)} \int_{S^{n-1}} \frac{h_K}{h_L} \ln \frac{h_K}{h_L} dv_{i,L}\right] \\ &= \lim_{p\to\infty} \left[\frac{1}{W_i(L,K)} \int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^{\frac{p}{n+p}} dv_{i,L}\right]^{p+n}. \end{aligned}$$

Then the first claim follows from Hölder's inequality

$$\left(\int_{S^{n-1}}\left(\frac{h_K}{h_L}\right)^{\frac{p}{n+p}}dv_{i,L}\right)^{\frac{p+n}{p}}\cdot\left(\int_{S^{n-1}}dv_{i,L}\right)^{-\frac{n}{p}}\leqslant\int_{S^{n-1}}\frac{h_K}{h_L}dv_{i,L}=W_i(L,K),$$

as $\int_{S^{n-1}}dv_{i,L}=W_i(L)$.

Applying Minkowski's inequality (2.9) to the first inequality of Proposition 3.1, we obtain

$$\int_{S^{n-1}}\ln\frac{h_K}{h_L}d\bar{w}_{i,L}\geqslant\frac{1}{n-i}\ln\frac{W_i(K)}{W_i(L)}. \tag{3.7}$$

By the equality condition of Minkowski's inequality (2.9), we know that equality in (3.7) holds if and only if K is homothetic to L. □

REMARK 3.1. The case $i=0$ of Proposition 3.1 was obtained by Stancu [15].

An immediate consequence of Proposition 3.1 is:

COROLLARY 3.1. *Let $K,L\in\mathscr{K}_0^n$ and $i=0,\dots,n-1$. If $L\subseteq K$, then*

$$\int_{S^{n-1}}\frac{h_K}{h_L}\ln\frac{h_K}{h_L}d\bar{v}_{i,L}\geqslant\frac{1}{n-i}\left(\frac{W_i(K)}{W_i(L)}\right)^{\frac{1}{n-i}}\ln\frac{W_i(K)}{W_i(L)},$$

with equality if and only if K is homothetic to L.

Proof. From Proposition 3.1 and Minkowski's inequality (2.9), it follows that

$$\begin{aligned}\int_{S^{n-1}}\frac{h_K}{h_L}\ln\frac{h_K}{h_L}d\bar{v}_{i,L}&=\frac{1}{W_i(L)}\int_{S^{n-1}}\frac{h_K}{h_L}\ln\frac{h_K}{h_L}dv_{i,L}\\&=\frac{W_i(L,K)}{W_i(L)}\int_{S^{n-1}}\ln\frac{h_K}{h_L}d\bar{w}_{i,L}\\&\geqslant\frac{W_i(L,K)}{W_i(L)}\ln\frac{W_i(L,K)}{W_i(L)}\\&\geqslant\frac{1}{n-i}\left(\frac{W_i(K)}{W_i(L)}\right)^{\frac{1}{n-i}}\ln\frac{W_i(K)}{W_i(L)},\end{aligned}$$

with equality if and only if K is homothetic to L. □

REMARK 3.2. The case $i=0$ of Corollary 3.1 was firstly obtained by Gardner, Hug, and Weil [4].

LEMMA 3.1. [2] *If p,q are probability density functions on a measure space (X,ν), then*

$$\int p\ln p\,d\nu\geqslant\int p\ln q\,d\nu.$$

PROPOSITION 3.2. *Let* $K, L \in \mathscr{K}_0^n$ *and* $i = 0, \ldots, n-1$. *Then*

$$\int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{v}_{i,L} \leqslant \ln \frac{W_i(L,K)}{W_i(L)} \leqslant \int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{w}_{i,L}.$$

Proof. By taking $p d\nu = \frac{h_L}{h_K} \frac{1}{W_i(L)} dw_{i,L}$ and $q d\nu = \frac{1}{W_i(L,K)} dw_{i,L}$ in Lemma 3.1, we obtain that

$$\begin{aligned}
\int_{S^{n-1}} \frac{h_L}{h_K} \frac{1}{W_i(L)} \ln \left(\frac{h_L}{h_K} \frac{1}{W_i(L)} \right) dw_{i,L} &= \int_{S^{n-1}} \frac{h_L}{h_K} \frac{1}{W_i(L)} \left(\ln \frac{h_L}{h_K} + \ln \frac{1}{W_i(L)} \right) dw_{i,L} \\
&= \int_{S^{n-1}} \frac{1}{W_i(L)} \left(\ln \frac{h_L}{h_K} + \ln \frac{1}{W_i(L)} \right) dv_{i,L} \\
&= \int_{S^{n-1}} \left(\ln \frac{h_L}{h_K} + \ln \frac{1}{W_i(L)} \right) d\bar{v}_{i,L} \\
&= \int_{S^{n-1}} \ln \frac{h_L}{h_K} d\bar{v}_{i,L} + \ln \frac{1}{W_i(L)} \\
&\geqslant \int_{S^{n-1}} \frac{h_L}{h_K} \frac{1}{W_i(L)} \ln \frac{1}{W_i(L,K)} dw_{i,L} \\
&= \ln \frac{1}{W_i(L,K)}.
\end{aligned}$$

Thus,

$$\int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{v}_{i,L} \leqslant \ln \frac{W_i(L,K)}{W_i(L)}.$$

On the other hand, by taking $p d\nu = \frac{1}{W_i(L,K)} dw_{i,L}$ and $q d\nu = \frac{h_L}{h_K} \frac{1}{W_i(L)} dw_{i,L}$ in Lemma 3.1, we obtain the second inequality of Proposition 3.2. □

REMARK 3.3. The case $i = 0$ of Proposition 3.2 was firstly obtained by Stancu [15].

There exists another proof of the first inequality of Proposition 3.2. Note that for $K, L \in \mathscr{K}_0^n$ and $i = 0, \ldots, n-1$,

$$\lim_{p \to \infty} \left(\frac{1}{W_i(L)} \int_{S^{n-1}} \left(\frac{h_K}{h_L} \right)^{\frac{1}{p+n}} dv_{i,L} \right)^{p+n} = \exp \left[\int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{v}_{i,L} \right]. \tag{3.8}$$

From Hölder's inequality and (3.8), it follows that

$$\left(\int_{S^{n-1}} \left(\frac{h_K}{h_L} \right)^{\frac{1}{p+n}} dv_{i,L} \right)^{p+n} \cdot \left(\int_{S^{n-1}} dv_{i,L} \right)^{1-(p+n)} \leqslant \int_{S^{n-1}} \frac{h_K}{h_L} dv_{i,L} = W_i(L,K),$$

thus

$$\int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{v}_{i,L} \leqslant \ln \frac{W_i(L,K)}{W_i(L)}. \tag{3.9}$$

We will denote the support of the mixed cone-volume measure of $v_{i,K}$ and $v_{i,L}$ by supp $v_{i,K}$ and supp $v_{i,L}$, respectively. $\left(\frac{h_K}{h_L}\right)_{i,average} := \frac{\int_{S^{n-1}} \frac{h_K}{h_L} dv_{i,L}}{\int_{S^{n-1}} dv_{i,L}}$, $\left(\frac{h_K}{h_L}\right)_{i,\max} = \max_{u\in \mathrm{supp} v_{i,L}} \frac{h_K}{h_L}$, $\left(\frac{h_K}{h_L}\right)_{i,\min} = \min_{u\in \mathrm{supp} v_{i,K}} \frac{h_K}{h_L}$. In particular, $\left(\frac{h_K}{h_L}\right)_{0,average} = \left(\frac{h_K}{h_L}\right)_{average}$, $\left(\frac{h_K}{h_L}\right)_{0,\max} = \left(\frac{h_K}{h_L}\right)_{\max}$, and $\left(\frac{h_K}{h_L}\right)_{0,\min} = \left(\frac{h_K}{h_L}\right)_{\min}$.

THEOREM 3.1. *Let $K,L \in \mathscr{K}_0^n$ and $i = 0,\dots,n-1$. If $L \subseteq K$, then*

$$\int_{S^{n-1}} \ln \frac{h_L}{h_K} d\bar{v}_{i,K} \geqslant \frac{\left(\frac{h_L}{h_K}\right)_{i,average}}{\left(\frac{h_L}{h_K}\right)_{i,\max}} \frac{1}{n-i} \ln \frac{W_i(L)}{W_i(K)}, \tag{3.10}$$

with equality if and only if $K = L$.

Proof. We define the non-negative function

$$G(q) := \int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^q \ln \frac{h_K}{h_L} dv_{i,L}.$$

If $u \mapsto \ln\left(\frac{h_K}{h_L}\right)(u)$ is zero on the support of the mixed cone-volume measure $dv_{i,L}$, then G is identically zero. Assume, for now, that this is not the case which also implies that $G(1) \geqslant G(0) > 0$. If $G(1) = G(0)$, then conclusion is trivial (as using (3.1) to obtain $K = L$), so we can assume $G(1) > G(0)$.

A simple verification shows that $G(q)$ is a log-convex function. By a Hadamard type inequality for positive log-convex function (see [6]), we have that

$$\frac{G(1)-G(0)}{\ln(G(1)/G(0))} \geqslant \int_0^1 \left[\int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^q \ln \frac{h_K}{h_L} dv_{i,L}\right] dq$$

and, since $G(1) > G(0)$, by Fubini-Tonelli's theorem,

$$G(0) \geqslant G(1) \cdot \exp\left[-\frac{G(1)-G(0)}{\int_{S^{n-1}} (\frac{h_K}{h_L} - 1) dv_{i,L}}\right].$$

Note that

$$\frac{G(1)-G(0)}{\int_{S^{n-1}} (\frac{h_K}{h_L} - 1) dv_{i,L}} = \frac{\int_{S^{n-1}} \ln \frac{h_K}{h_L} \cdot (\frac{h_K}{h_L} - 1) dv_{i,L}}{\int_{S^{n-1}} (\frac{h_K}{h_L} - 1) dv_{i,L}} \leqslant \ln \left(\frac{h_K}{h_L}\right)_{i,\max},$$

Then

$$\int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{v}_{i,L} \geqslant \exp\left[-\ln\left(\frac{h_K}{h_L}\right)_{i,\max}\right] \cdot \frac{W_i(L,K)}{W_i(L)} \int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{v}_{i,L},$$

from which (3.10) follows from (3.7).

Assuming that G is identically zero, then $h_K(u) = h_L(u)$ for all u's almost everywhere with respect to the mixed cone measure of L, or equivalently with respect to the mixed surface area measure of L. This implies $W_i(L,K) = W_i(L)$, and since $L \subseteq K$ are convex bodies, K and L must coincide. □

REMARK 3.4. The case $i = 0$ of Theorem 3.1 was firstly obtained by Stancu [15].

Proof of Theorem 1.1. If L is not included in K, there exists a λ, $0 < \lambda < 1$, such that $\tilde{L} := \lambda L \subseteq K$ and apply (3.10) for $\tilde{L}$ and K.

Thus,

$$\int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{v}_{i,L} \geqslant \frac{\left(\frac{h_K}{h_L}\right)_{i,average}}{\left(\frac{h_K}{h_L}\right)_{i,\max}} \frac{1}{n-i} \ln \frac{W_i(K)}{W_i(L)} + \ln\lambda \cdot \left(1 - \frac{\left(\frac{h_K}{h_L}\right)_{i,average}}{\left(\frac{h_K}{h_L}\right)_{i,\max}}\right).$$

By taking $\lambda = \min_{u \in \operatorname{supp} v_{i,K}} \frac{h_K}{h_L}(u)$, we obtain the desired inequality. □

A direct consequence of Theorem 1.1 is:

COROLLARY 3.2. *Let $K, L \in \mathscr{K}_0^n$ such that there exists a positive constant $c > 0$ with $h_K(u) = c h_L(u)$ for each u in the support of the mixed cone volume measure of L. For $i = 0, \ldots, n-1$, then*

$$\int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{v}_{i,L} \geqslant \frac{1}{n-i} \ln \frac{W_i(K)}{W_i(L)},$$

with equality if and only if $K = cL$.

COROLLARY 3.3. *For any $L \in \mathscr{K}_0^n$ whose support function restricted to supp $v_{i,L}$ is constant and $i = 0, \ldots, n-1$, we have that*

$$\int_{S^{n-1}} \ln \frac{h_{L^*}}{h_L} d\bar{v}_{i,L} \geqslant \frac{2}{n-i} \ln \frac{\omega_n}{W_i(L)},$$

with equality if and only if K is an origin-symmetric ball.

Proof. Taking $K = B$ in Corollary 3.2, we have

$$\int_{S^{n-1}} \ln \frac{h_{L^*}}{h_L} d\bar{v}_{i,L} = \int_{S^{n-1}} \ln \frac{1}{\rho_L h_L} d\bar{v}_{i,L} \geqslant \int_{S^{n-1}} \ln \frac{1}{h_L^2} d\bar{v}_{i,L}$$

$$= 2\int_{S^{n-1}} \ln \tfrac{1}{h_L} d\bar{v}_{i,L} \geqslant \frac{2}{n-i} \ln \frac{\omega_n}{W_i(L)},$$

with equality if and only if L is an origin-symmetric ball. $\square$

REMARK 3.5. The case $i=0$ of Corollary 3.2 and Corollary 3.3 were obtained by Stancu [15].

For $K \in \mathscr{S}^n$, we denote the dual mixed cone-volume measure by

$$d\widetilde{v}_{i,K} = \frac{1}{n}\rho_K^{n-i}dS. \tag{3.11}$$

Since $\dfrac{1}{n\widetilde{W}_i(K)}\displaystyle\int_{S^{n-1}}\rho_K^{n-i}dS = 1$, we write the dual mixed cone-volume probability measure of K on S^{n-1} by

$$d\bar{\widetilde{v}}_{i,K} = \frac{1}{\widetilde{W}_i(K)}d\widetilde{v}_{i,K}. \tag{3.12}$$

If $i=0$, the measure $d\bar{\widetilde{v}}_{0,K}$ will be denoted by the dual cone-volume probability measure, and it will be written simply as $d\bar{\widetilde{v}}_K$.

Recently, Gardner et al. [5] established the dual log-Minkowski inequality (also see [16]). Next, we will establish the following double inequalities which are more general than the dual log-Minkowski inequality.

PROPOSITION 3.3. *Let $K,L \in \mathscr{S}^n$. For $i=0,\dots,n-1$, we have*

$$\int_{S^{n-1}}\ln\frac{\rho_K}{\rho_L}d\bar{\widetilde{v}}_{i,K} \geqslant \frac{1}{n-i}\ln\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)} \geqslant \int_{S^{n-1}}\ln\frac{\rho_K}{\rho_L}d\bar{\widetilde{v}}_{i,L}.$$

Both equalities hold if and only if K and L are dilates.

Proof. Applying (2.10), (3.12), and Jensen's inequality, we obtain

$$\begin{aligned}
\widetilde{W}_i(L,K) &= \frac{1}{n}\int_{S^{n-1}}\rho_L^{n-i-1}\rho_K dS \\
&= \widetilde{W}_i(K)\int_{S^{n-1}}\left(\frac{\rho_L}{\rho_K}\right)^{n-i-1}d\bar{\widetilde{v}}_{i,K} \\
&= \widetilde{W}_i(K)\int_{S^{n-1}}\exp\left(\ln\left(\frac{\rho_L}{\rho_K}\right)^{n-i-1}\right)d\bar{\widetilde{v}}_{i,K} \\
&\geqslant \widetilde{W}_i(K)\exp\int_{S^{n-1}}\ln\left(\frac{\rho_L}{\rho_K}\right)^{n-i-1}d\bar{\widetilde{v}}_{i,K}.
\end{aligned} \tag{3.13}$$

Taking the natural logarithm of both sides of (3.13) and using the dual Minkowski inequality (2.12), one can obtain

$$\int_{S^{n-1}}\ln\frac{\rho_K}{\rho_L}d\bar{\widetilde{v}}_{i,K} \geqslant \frac{1}{n-i}\ln\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}. \tag{3.14}$$

By the equality condition of the dual Minkowski inequality (2.12), we know that equality in (3.14) holds if and only if K and L are dilates.

On the other hand,

$$\widetilde{W}_i(L,K)=\frac{1}{n}\int_{S^{n-1}}\rho_L^{n-i-1}\rho_K dS_n=\widetilde{W}_i(L)\int_{S^{n-1}}\frac{\rho_K}{\rho_L}d\bar{\widetilde{v}}_{i,L}.$$

Similarly, we can obtain

$$\int_{S^{n-1}}\ln\frac{\rho_K}{\rho_L}d\bar{\widetilde{v}}_{i,L}\leqslant\frac{1}{n-i}\ln\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}.\quad\square$$

REMARK 3.6. The case $i=0$ of Proposition 3.3 was firstly obtained by Garnder, Hug and Weil [4].

PROPOSITION 3.4. *Let K be an arbitrary convex body in $\mathscr{K}_0^n$ with its Santaló point at the origin and $i=0,\dots,n-1$. Then*

$$\int_{S^{n-1}}\ln h_K d\bar{\widetilde{v}}_{i,B}\geqslant\frac{1}{n-i}\ln\frac{\widetilde{W}_i(K)}{\omega_n}\geqslant\int_{S^{n-1}}\ln\rho_K d\bar{\widetilde{v}}_{i,B},$$

with equality if and only if K is an origin-symmetric ball.

Proof. By (2.11), (2.6), (3.12), and Jessen's inequality, we have

$$\begin{aligned}\widetilde{W}_i(K^*)&=\frac{1}{n}\int_{S^{n-1}}\rho_{K^*}^{n-i}dS=\frac{1}{n}\int_{S^{n-1}}h_K^{-(n-i)}dS\\&=\omega_n\int_{S^{n-1}}\exp(\ln(h_K^{-(n-i)}))d\bar{\widetilde{v}}_{i,B}\\&\geqslant\omega_n\exp\int_{S^{n-1}}(\ln(h_K^{-(n-i)}))d\bar{\widetilde{v}}_{i,B}.\end{aligned}$$

Thus,

$$\int_{S^{n-1}}\ln h_K d\bar{\widetilde{v}}_{i,B}\geqslant\frac{1}{n-i}\ln\frac{\omega_n}{\widetilde{W}_i(K^*)}\geqslant\frac{1}{n-i}\ln\frac{\widetilde{W}_i(K)}{\omega_n},$$

where the last step is due to the dual Blaschke-Santaló's inequality (2.14).

On the other hand, by taking $L=B$ in the second inequality of Proposition 3.3, we obtain

$$\int_{S^{n-1}}\ln\rho_K d\bar{\widetilde{v}}_{i,B}\leqslant\frac{1}{n-i}\ln\frac{\widetilde{W}_i(K)}{\omega_n},$$

with equality if and only if K is an origin-symmetric ball. $\square$

REMARK 3.7. The case $i=0$ of Proposition 3.4 was firstly obtained by Guan and Ni [8].

The dual form of Corollary 3.3 will be established.

PROPOSITION 3.5. *For any $L \in \mathscr{K}_0^n$ and $i = 0, \ldots, n-1$, we have that*

$$\int_{S^{n-1}} \ln \frac{h_{L^*}}{h_L} d\bar{\widetilde{v}}_{i,L} \leqslant \frac{2}{n-i} \ln \frac{\omega_n}{\widetilde{W}_i(L)},$$

with equality if and only if K is an origin-symmetric ball.

Proof. By (2.6), note that $h_L \geqslant \rho_L$ for $\forall L \in \mathscr{K}_0^n$, and take $K = B$ in Proposition 3.3, then we have

$$\begin{aligned}\int_{S^{n-1}} \ln \frac{\rho_{L^*}}{\rho_L} d\bar{\widetilde{v}}_{i,L} &= \int_{S^{n-1}} \ln \frac{1}{\rho_L h_L} d\bar{\widetilde{v}}_{i,L} \\ &\leqslant \int_{S^{n-1}} \ln \frac{1}{\rho_L^2} d\bar{\widetilde{v}}_{i,L} \\ &= 2\int_{S^{n-1}} \ln \frac{1}{\rho_L} d\bar{\widetilde{v}}_{i,L} \\ &\leqslant \frac{2}{n-i} \ln \frac{\omega_n}{\widetilde{W}_i(L)},\end{aligned}$$

with equality if and only if L is an origin-symmetric ball. □

REMARK 3.8. The case $i = 0$ of Proposition 3.5 was firstly obtained by Stancu [15].

Acknowledgement. We are grateful to the referee for the suggested improvement.

REFERENCES

[1] K. J. BÖRÖCZKY, E. LUTWAK, D. YANG, G. ZHANG, *The log-Brunn-Minkowski inequality*, Adv. Math. **231** (2012), 1974–1997.
[2] T. M. COVER, J. A. THOMAS, *Elements of Information Theory*, second edition, Wiley-Interscience, Hoboken, NJ, 2006.
[3] R. J. GARDNER, *Geometric Tomography*, second ed., Cambridge Univ. Press, NewYork, 2006.
[4] R. J. GARDNER, D. HUG, W. WEIL, *The Orlicz-Brunn-Minkowski theory: a general framework, additions, and inequalities*, J. Differential Geom. **97** (2014), 427–476.
[5] R. J. GARDNER, D. HUG, W. WEIL, AND D. YE, *The dual Orlicz-Brunn-Minkowski theory*, J. Math. Anal. Appl. **430** (2015), 810–829.
[6] P. M. GILL, C. E. M. PEARCE, J. PEČARIĆ, *Hadamard's inequality for r-convex functions*, J. Math. Anal. Appl. **215** (1997), 461–470.
[7] P. M. GRUBER, *Convex and Discrete Geometry*, Grundlehren Math. Wiss., vol. 336, Springer, Berlin, 2007.
[8] P. GUAN, L. NI, *Entropy and a convergence theorem for Gauss curvature flow in high dimension*, preprint, arXiv:1306.0625, 2013.
[9] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, second ed., Cambridge University Press, Cambridge, 1988.
[10] E. LUTWAK, *Dual mixed volumes*, Pac. J. Math. **58** (1975), 531–538.
[11] E. LUTWAK, *The Brunn-Minkowski-Firey theory I: mixed volumes and the Minkowski problem*, J. Differential Geom. **38** (1993), 131–150.
[12] L. MA, *A new proof of log-Brunn-Minkowski inequality*, Geom. Dedicata **177** (2015), 75–82.

[13] C. SAROGLOU, *Remarks on the conjectured log-Brunn-Minkowski inequality*, Geom. Dedicata **177** (2015), 353–365.

[14] R. SCHNEIDER, *Convex Bodies: The Brunn-Minkowski Theory*, second ed., Cambridge University Press, Cambridge, 2014.

[15] A. STANCU, *The logarithmic Minkowski inequality for non-symmetric convex bodies*, Adv. Appl. Math. **73** (2016), 43–58.

[16] W. WANG, L. J. LIU, *The dual log-Brunn-Minkowski inequality*, Taiwanese J. Math. **20** (2016), 909–919.

(Received January 10, 2016)

Wei Wang
Department of Mathematics, Tongji University
Siping Road, Shanghai, 200092, P. R. China
and
School of Mathematics and Computational Science
Hunan University of Science and Technology
Xiangtan, 411201, P. R. China
e-mail: wwang@hnust.edu.cn

Ming Feng
Xiangtan No. 2 Middle School
Xiangtan, 411100