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SOME NEW NONLINEAR WEAKLY SINGULAR INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

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Abstract. The purpose of the present note is to establish some new weakly singular integral inequalities of Wendroff type, which generalized some known weakly singular inequalities for functions in two variables. The inequalities given here can be used in the analysis of various problems in the theory of certain classes of integral equations and evolution equations.

1. Introduction

It is well known that singular integral inequalities played very important roles in the qualitative theory of differential and integral equations. In the study of differential and integral equations, one often deal with certain integral inequalities. The Gronwall-Bellman type integral inequality and its various linear and nonlinear generalizations are crucial in the discussion of existence, uniqueness, continuation, boundedness, and stability properties of solutions. During the past few years, many papers [1–6, 8–16] have appeared in the literature which deal with integral and sum inequalities involving functions of one and more than one independent variables. Usually, the integrals concerning this type inequalities have regular or continuous kernels, but some problems of theory and practicality require us to solve integral inequalities with singular kernels. For example, Henry [7] used this type integral inequalities to prove a global existence and an exponential decay result for a parabolic Cauchy problem. Recently, Ma et al. [10] studied the inequality

$$u^p(t) \leqslant a(t) + b(t)\int_0^t (t^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} f(s) u^q(s)ds, \quad t \in R_+.$$

In 2016, Xu and Meng [11] presented Gronwall-Bellman type integral inequalities with nonlinear weakly singular integral kernel of the form

$$u^p(t) \leqslant a(t) + b(t)\int_0^t (t-s)^{\beta-1} c(s) u^m(s)ds + d(t)\int_0^t (t^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} f(s) u^q(s)ds.$$

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Cheung et al.[11] investigated the inequality in two variables

$$u^p(x,y) \leqslant a(x,y)+b(x,y)\int_0^x\int_0^y (x^\alpha - s^\alpha)^{\beta-1}s^{\gamma-1}(y^\alpha - t^\alpha)^{\beta-1}t^{\gamma-1}f(s,t)u^q(s,t)dtds,$$
$$(x,y)\in D.$$

In this paper, motivated mainly by the work of [9–12], and applying Medved's method of desingularization of weakly singular inequalities, we establish some new singular version of the Wendroff inequality for functions in two variables. An example is included to illustrate the usefulness of our results.

2. Preliminary knowledge

In what follows, R denotes the set of real numbers, $R_+ = [0,+\infty)$, $C^i(M,S)$ denotes the class of all i-times continuously differentiable functions defined on set M with range in the set S $(i = 1,2,\cdots)$, $C^0(M,S) = C(M,S)$. For convenience, we cite some useful lemmas and definitions in the discussion of our proof as follows.

LEMMA 2.1. (See [8]) *Let $a \geqslant 0$, $p \geqslant q \geqslant 0$ and $p \neq 0$, then*

$$a^{\frac{q}{p}} \leqslant \frac{q}{p}K^{\frac{q-p}{p}}a + \frac{p-q}{p}K^{\frac{q}{p}},$$

for any $K > 0$.

DEFINITION 2.2. (See [9]) Let $[x,y,z]$ be an ordered parameter group of nonnegative real numbers. The group is called belong to the first class distribution and denoted by $[x,y,z] \in I$ if conditions $x \in (0,1]$, $y \in (\frac{1}{2},1)$ and $z \geqslant \frac{3}{2} - y$ are satisfied. The group is called belong to the second class distribution and denoted by $[x,y,z] \in II$, if $x \in (0,1]$, $y \in (0,\frac{1}{2}]$ and $z > \frac{1-2y^2}{1-y^2}$ are satisfied.

LEMMA 2.3. (See [12, page 296]) *Let α, β, γ and p be positive constants, then*

$$\int_0^t (t^\alpha - s^\beta)^{p(\beta-1)}s^{p(r-1)}ds = \frac{t^\theta}{\alpha}B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right], \quad t \in R_+,$$

where $B[\xi,\eta] = \int_0^1 s^{\xi-1}(1-s)^{\eta-1}ds$ $(\Re\xi > 0, \Re\eta > 0)$ is the well-known $B-function$ and $\theta = p[\alpha(\beta-1)+\gamma-1]+1$.

LEMMA 2.4. (See [9]) *Suppose that the positive constants α, β, γ, p_1 and p_2 satisfy conditions:*

(a) if $[\alpha,\beta,\gamma] \in I$, $p_1 = \frac{1}{\beta}$;

(b) if $[\alpha,\beta,\gamma] \in II$, $p_2 = \frac{1+4\beta}{1+3\beta}$,

then $B[\frac{p_i(\gamma-1)+1}{\alpha}, p_i(\beta-1)+1] \in (0,+\infty)$ and $\theta_i = p_i[\alpha(\beta-1)+\gamma-1]+1 \geqslant 0$, $i = 1,2$.

LEMMA 2.5. (See [6, page 329]) *Let $u(x,y)$, $p(x,y)$, $q(x,y)$ and $k(x,y)$ be nonnegative continuous functions defined for $x,\ y\in R_+$. If*

$$u(x,y)\leqslant p(x,y)+q(x,y)\int_0^x\int_0^y k(s,t)u(s,t)dtds, \tag{2.1}$$

then

$$u(x,y)\leqslant p(x,y)+q(x,y)\left(\int_0^x\int_0^y k(s,t)p(s,t)dtds\right)\exp\left(\int_0^x\int_0^y k(s,t)q(s,t)dtds\right). \tag{2.2}$$

3. Main result

THEOREM 3.1. *Let $u(x,y)$, $a(x,y)$, $b(x,y)$, $f(x,y)$, $h(x,y)$ be nonnegative continuous functions for $(x,y)\in D$, $D=[0,X)\times[0,Y)$ $(0<X\leqslant\infty,\ 0<Y\leqslant\infty)$, $a(x,y)$ and $b(x,y)$ are nondecreasing in D and $u(x,y)$ satisfies the following form of integral inequality*

$$\begin{aligned}u^p(x,y)\leqslant& a(x,y)+b(x,y)\int_0^x\int_0^y(x^\alpha-s^\alpha)^{\beta-1}s^{\gamma-1}(y^\alpha-t^\alpha)^{\beta-1}t^{\gamma-1}\\&\times\left[f(s,t)u^q(s,t)+h(s,t)u^r(\sigma(s),\sigma(t))\right]dtds,\end{aligned} \tag{3.1}$$

with the initial condition

$$\begin{aligned}u(x,y)=\phi(x,y),\ (x,y)\in D'=[\mu,0]\times[\mu,0],\\ \phi(\sigma(x),\sigma(y))\leqslant(a(x,y))^{\frac{1}{p}}\ \ for\ \ (x,y)\in D\ \ with\ \ \sigma(x)\leqslant 0,\ \sigma(y)\leqslant 0,\end{aligned} \tag{3.2}$$

where $p\neq 0$, $p\geqslant q>0$, $p\geqslant r>0$, p, q, r be constants, $\sigma(x)$, $\sigma(y)\in C(R_+,\ R)$, $\sigma(t)\leqslant t$, $t\in R_+$. $-\infty<\mu=\inf\{\sigma(t),\ t\in R_+\}\leqslant 0$ and $\phi(x,y)\in C([\mu,0]\times[\mu,0],\ R_+)$. Then

(i) If $[\alpha,\beta,\gamma]\in I$,

$$\begin{aligned}u(x,y)\leqslant\Bigg\{a(x,y)+\Bigg[P_1(x,y)+Q_1(x,y)\left(\int_0^x\int_0^y(f(s,t)+h(s,t))^{\frac{1}{1-\beta}}P_1(s,t)dtds\right)\\ \times\exp\left(\int_0^x\int_0^y(f(s,t)+h(s,t))^{\frac{1}{1-\beta}}Q_1(s,t)dtds\right)\Bigg]^{1-\beta}\Bigg\}^{\frac{1}{p}},\quad(x,y)\in D,\end{aligned} \tag{3.3}$$

where

$$M_1=\frac{1}{\alpha}B\left[\frac{\beta+\gamma-1}{\alpha\beta},\frac{2\beta-1}{\beta}\right],$$
$$M=\max\left\{\frac{q}{p}K^{\frac{q-p}{p}},\frac{r}{p}K^{\frac{r-p}{p}}\right\},$$
$$A_1(x,y)=\frac{q}{p}K^{\frac{q-p}{p}}a(x,y)+\frac{p-q}{p}K^{\frac{q}{p}},\ A_2(x,y)=\frac{r}{p}K^{\frac{r-p}{p}}a(x,y)+\frac{p-r}{p}K^{\frac{r}{p}},$$
$$\mathscr{A}_1(x,y)=\int_0^x\int_0^y\left(f(s,t)A_1(s,t)+h(s,t)A_2(s,t)\right)^{\frac{1}{1-\beta}}dtds,$$
$$P_1(x,y)=2^{\frac{\beta}{1-\beta}}M_1^{\frac{2\beta}{1-\beta}}(xy)^{\frac{(\alpha+1)(\beta-1)+\gamma}{1-\beta}}b^{\frac{1}{1-\beta}}(x,y)\mathscr{A}_1(x,y),$$
$$Q_1(x,y)=2^{\frac{\beta}{1-\beta}}M_1^{\frac{2\beta}{1-\beta}}(xy)^{\frac{(\alpha+1)(\beta-1)+\gamma}{1-\beta}}b^{\frac{1}{1-\beta}}(x,y)M^{\frac{1}{1-\beta}}.$$

(ii) If $[\alpha,\beta,\gamma]\in II$,

$$u(x,y)\leqslant\left\{a(x,y)+\left[P_2(x,y)+Q_2(x,y)\left(\int_0^x\int_0^y(f(s,t)+h(s,t))^{\frac{1+4\beta}{\beta}}P_2(s,t)dtds\right)\right.\right.$$
$$\left.\left.\times\exp\left(\int_0^x\int_0^y(f(s,t)+h(s,t))^{\frac{1+4\beta}{\beta}}Q_2(s,t)dtds\right)\right]^{\frac{\beta}{1+4\beta}}\right\}^{\frac{1}{p}},\quad (x,y)\in D,\tag{3.4}$$

where

$$M_2=\frac{1}{\alpha}B\left[\frac{\gamma(1+4\beta)-\beta}{\alpha(1+3\beta)},\frac{4\beta^2}{1+3\beta}\right],$$
$$\mathscr{A}_2(x,y)=\int_0^x\int_0^y\left(f(s,t)A_1(s,t)+h(s,t)A_2(s,t)\right)^{\frac{1+4\beta}{\beta}}dtds,$$
$$P_2(x,y)=2^{\frac{1+3\beta}{\beta}}M_2^{\frac{2(1+3\beta)}{\beta}}(xy)^{\frac{(1+4\beta)[\alpha(\beta-1)+\gamma]-\beta}{\beta}}b^{\frac{1+4\beta}{\beta}}(x,y)\mathscr{A}_2(x,y),$$
$$Q_2(x,y)=2^{\frac{1+3\beta}{\beta}}M_2^{\frac{2(1+3\beta)}{\beta}}(xy)^{\frac{(1+4\beta)[\alpha(\beta-1)+\gamma]-\beta}{\beta}}b^{\frac{1+4\beta}{\beta}}(x,y)M^{\frac{1+4\beta}{\beta}}.$$

Proof. For any fixing positive number X, Y, we define a function $\omega(x,y)$ by

$$\omega^p(x,y)=a(X,Y)+b(x,y)\int_0^x\int_0^y(x^\alpha-s^\alpha)^{\beta-1}s^{\gamma-1}(y^\alpha-t^\alpha)^{\beta-1}t^{\gamma-1}$$
$$\times\left[f(s,t)u^q(s,t)+h(s,t)u^r(\sigma(s),\sigma(y))\right],\quad (x,y)\in D.\tag{3.5}$$

Then we can easily see that $\omega(x,y)$ is a nonnegative and nondecreasing function and

$$u(x,y)\leqslant\omega(x,y),\quad (x,y)\in D,\tag{3.6}$$

therefore, for $x\in[0,X)$, $y\in[0,Y)$ with $\sigma(x)\geqslant 0$, $\sigma(y)\geqslant 0$, we have

$$u(\sigma(x),\sigma(y))\leqslant\omega(\sigma(x),\sigma(y))\leqslant\omega(x,y),\quad (x,y)\in D,\tag{3.7}$$

on the other hand, for $x \in [0,X)$, $y \in [0,Y)$ with $\sigma(x) \leqslant 0$, $\sigma(y) \leqslant 0$, using the initial condition (3.2) and the nondecreasing function $a(x,y)$, we have

$$u(\sigma(x),\sigma(y)) = \phi(\sigma(x),\sigma(y)) \leqslant (a(x,y))^{\frac{1}{p}} \leqslant (a(X,Y))^{\frac{1}{p}} \leqslant \omega(x,y), \quad (x,y) \in D, \tag{3.8}$$

and it follows from (3.7) and (3.8) that

$$u(\sigma(x),\sigma(y)) \leqslant \omega(x,y), \quad (x,y) \in D, \tag{3.9}$$

from (3.5) and (3.9), there is

$$\omega^p(x,y) \leqslant a(X,Y) + b(x,y) \int_0^x \int_0^y (x^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} (y^\alpha - t^\alpha)^{\beta-1} t^{\gamma-1} \times \left[f(s,t)\omega^q(s,t) + h(s,t)\omega^r(s,t) \right] dtds. \tag{3.10}$$

If taking $x = X$, $y = Y$ in (3.10), we have

$$\omega^p(X,Y) \leqslant a(X,Y) + b(X,Y) \int_0^X \int_0^Y (X^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} (Y^\alpha - t^\alpha)^{\beta-1} t^{\gamma-1} \times \left[f(s,t)\omega^q(s,t) + h(s,t)\omega^r(s,t) \right] dtds, \tag{3.11}$$

for $0 < X \leqslant \infty$, $0 < Y \leqslant \infty$ be arbitrary constant, it proves that

$$\omega^p(x,y) \leqslant a(x,y) + b(x,y) \int_0^x \int_0^y (x^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} (y^\alpha - t^\alpha)^{\beta-1} t^{\gamma-1} \times \left[f(s,t)\omega^q(s,t) + h(s,t)\omega^r(s,t) \right] dtds, \quad (x,y) \in D, \tag{3.12}$$

set

$$v(x,y) = b(x,y) \int_0^x \int_0^y (x^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} (y^\alpha - t^\alpha)^{\beta-1} t^{\gamma-1} \times \left[f(s,t)\omega^q(s,t) + h(s,t)\omega^r(s,t) \right] dtds, \tag{3.13}$$

then

$$\omega^p(x,y) \leqslant a(x,y) + v(x,y), \tag{3.14}$$

or

$$\omega(x,y) \leqslant (a(x,y) + v(x,y))^{\frac{1}{p}}, \tag{3.15}$$

by Lemma 2.1 and inequality (3.15), for any $K > 0$, we have

$$\omega^q(s,t) \leqslant (a(s,t) + v(s,t))^{\frac{q}{p}} \leqslant \frac{q}{p} K^{\frac{q-p}{p}} (a(s,t) + v(s,t)) + \frac{p-q}{p} K^{\frac{q}{p}}, \tag{3.16}$$

$$\omega^r(s,t) \leqslant (a(s,t)+v(s,t))^{\frac{r}{p}} \leqslant \frac{r}{p}K^{\frac{r-p}{p}}(a(s,t)+v(s,t)) + \frac{p-r}{p}K^{\frac{r}{p}}, \tag{3.17}$$

it follows from (3.13), (3.16) and (3.17) that

$$\begin{aligned} v(x,y) \leqslant & b(x,y)\int_0^x\int_0^y (x^\alpha - s^\alpha)^{\beta-1}s^{\gamma-1}(y^\alpha - t^\alpha)^{\beta-1}t^{\gamma-1} \\ & \times\Bigg[f(s,t)\left(\frac{q}{p}K^{\frac{q-p}{p}}(a(s,t)+v(s,t)) + \frac{p-q}{p}K^{\frac{q}{p}}\right) \\ & + h(s,t)\left(\frac{r}{p}K^{\frac{r-p}{p}}(a(s,t)+v(s,t)) + \frac{p-r}{p}K^{\frac{r}{p}}\right)\Bigg]dtds \\ = & b(x,y)\int_0^x\int_0^y (x^\alpha - s^\alpha)^{\beta-1}s^{\gamma-1}(y^\alpha - t^\alpha)^{\beta-1}t^{\gamma-1}(f(s,t)A_1(s,t)+h(s,t)A_2(s,t)) \\ & + Mb(x,y)\int_0^x\int_0^y (x^\alpha - s^\alpha)^{\beta-1}s^{\gamma-1}(y^\alpha - t^\alpha)^{\beta-1}t^{\gamma-1}(f(s,t)+h(s,t))v(s,t)dtds, \end{aligned} \tag{3.18}$$

where

$$A_1(x,y) = \frac{q}{p}K^{\frac{q-p}{p}}a(x,y) + \frac{p-q}{p}K^{\frac{q}{p}},\ A_2(x,y) = \frac{r}{p}K^{\frac{r-p}{p}}a(x,y) + \frac{p-r}{p}K^{\frac{r}{p}},$$
$$M = \max\left\{\frac{q}{p}K^{\frac{q-p}{p}}, \frac{r}{p}K^{\frac{r-p}{p}}\right\}.$$

If $[\alpha,\beta,\gamma] \in I$, let $p_1 = \frac{1}{\beta}$, $q_1 = \frac{1}{1-\beta}$; if $[\alpha,\beta,\gamma] \in II$, let $p_2 = \frac{1+4\beta}{1+3\beta}$, $q_2 = \frac{1+4\beta}{\beta}$, then $\frac{1}{p_i} + \frac{1}{q_i} = 1$ for $i = 1,2$. By applying Hölder's inequality with indices p_i, q_i to (3.18), we get

$$\begin{aligned} v(x,y) \leqslant & b(x,y)\left[\int_0^x\int_0^y (x^\alpha - s^\alpha)^{p_i(\beta-1)}s^{p_i(\gamma-1)}(y^\alpha - t^\alpha)^{p_i(\beta-1)}t^{p_i(\gamma-1)}dtds\right]^{\frac{1}{p_i}} \\ & \times\left[\int_0^x\int_0^y (f(s,t)A_1(s,t)+h(s,t)A_2(s,t))^{q_i}dtds\right]^{\frac{1}{q_i}} \\ & + Mb(x,y)\left[\int_0^x\int_0^y (x^\alpha - s^\alpha)^{p_i(\beta-1)}s^{p_i(\gamma-1)}(y^\alpha - t^\alpha)^{p_i(\beta-1)}t^{p_i(\gamma-1)}dtds\right]^{\frac{1}{p_i}} \\ & \times\left[\int_0^x\int_0^y (f(s,t)+h(s,t))^{q_i}v^{q_i}(s,t)dtds\right]^{\frac{1}{q_i}}, \end{aligned} \tag{3.19}$$

by Lemma 2.3 and Lemma 2.4, the last inequality can be rewritten as

$$\begin{aligned} v(x,y) \leqslant & b(x,y)(M_i^2(xy)^{\theta_i})^{\frac{1}{p_i}}\mathscr{A}_i^{\frac{1}{q_i}}(x,y) + Mb(x,y)(M_i^2(xy)^{\theta_i})^{\frac{1}{p_i}} \\ & \times\left[\int_0^x\int_0^y (f(s,t)+h(s,t))^{q_i}v^{q_i}(s,t)dtds\right]^{\frac{1}{q_i}}, \quad (x,y)\in D, \end{aligned} \tag{3.20}$$

where

$$M_i = \frac{1}{\alpha} B\left[\frac{p_i(\gamma-1)+1}{\alpha}, p_i(\beta-1)+1\right],$$
$$\mathscr{A}_i(x,y) = \int_0^x \int_0^y (f(s,t)A_1(s,t)+h(s,t)A_2(s,t))^{q_i} dtds,$$

and θ_i is given as in Lemma 2.4 for $i=1,2$.

We also need the following well-know consequence of Jensen inequality

$$(A_1+A_2+\cdots+A_n)^r \leqslant n^{r-1}(A_1^r+A_2^r+\cdots+A_n^r) \quad (JI)$$

for $A_i \geqslant 0$ $(i=1,2,\cdots,n)$ and $r \geqslant 1$. Applying inequality (JI) to (3.20), we get

$$v^{q_i}(x,y) \leqslant 2^{q_i-1}(M_i^2(xy)^{\theta_i})^{\frac{q_i}{p_i}} b^{q_i}(x,y)\mathscr{A}_i(x,y) + 2^{q_i-1}M^{q_i}b^{q_i}(x,y)(M_i^2(xy)^{\theta_i})^{\frac{q_i}{p_i}}$$
$$\times \int_0^x \int_0^y (f(s,t)+h(s,t))^{q_i} v^{q_i}(s,t)dtds, \tag{3.21}$$

by Lemma 2.5 and the last inequality, we have

$$v^{q_i}(x,y) \leqslant P_i(x,y) + Q_i(x,y)\left(\int_0^x \int_0^y (f(s,t)+h(s,t))^{q_i} P_i(s,t)dtds\right)$$
$$\times \exp\left(\int_0^x \int_0^y (f(s,t)+h(s,t))^{q_i} Q_i(s,t)dtds\right), \tag{3.22}$$

where

$$P_i(x,y) = 2^{q_i-1}(M_i^2(xy)^{\theta_i})^{\frac{q_i}{p_i}} b^{q_i}(x,y)\mathscr{A}_i(x,y),$$
$$Q_i(x,y) = 2^{q_i-1}(M_i^2(xy)^{\theta_i})^{\frac{q_i}{p_i}} b^{q_i}(x,y)M^{q_i},$$

finally, substituting (3.22) into (3.15) leads to

$$\omega(x,y) \leqslant \Bigg\{a(x,y) + \Bigg[P_i(x,y) + Q_i(x,y)\left(\int_0^x \int_0^y (f(s,t)+h(s,t))^{q_i} P_i(s,t)dtds\right)$$
$$\times \exp\left(\int_0^x \int_0^y (f(s,t)+h(s,t))^{q_i} Q_i(s,t)dtds\right)\Bigg]^{\frac{1}{q_i}}\Bigg\}^{\frac{1}{p}}, \tag{3.23}$$

from (3.6), we have

$$u(x,y) \leqslant \Bigg\{a(x,y) + \Bigg[P_i(x,y) + Q_i(x,y)\left(\int_0^x \int_0^y (f(s,t)+h(s,t))^{q_i} P_i(s,t)dtds\right)$$
$$\times \exp\left(\int_0^x \int_0^y (f(s,t)+h(s,t))^{q_i} Q_i(s,t)dtds\right)\Bigg]^{\frac{1}{q_i}}\Bigg\}^{\frac{1}{p}}. \tag{3.24}$$

Considering two situations for $i=1,2$ and using parameters α, β and γ to denote p_i, q_i and θ_i in (3.24), we can get the desired estimations. $\square$

COROLLARY 3.2. *Let functions $u(x,y)$, $a(x,y)$, $b(x,y)$, $f(x,y)$ and $h(s,t)$ be defined as in Theorem 3.1, and q, r be a constant with $0<q\leqslant 1$, $0<r\leqslant 1$, suppose that*

$$u(x,y)\leqslant a(x,y)+b(x,y)\int_0^x\int_0^y (x-s)^{\beta-1}s^{\gamma-1}(y-t)^{\beta-1}t^{\gamma-1} \times\Big[f(s,t)u^q(s,t)+h(s,t)u^r(\sigma(s),\sigma(t))\Big]dtds,\quad (x,y)\in D, \tag{3.25}$$

with the initial condition of Theorem 3.1, then one has the following

(i) If $\beta\in(\frac{1}{2},1)$,

$$u(x,y)\leqslant a(x,y)+\Big[\overline{P}_1(x,y)+\overline{Q}_1(x,y)\Big(\int_0^x\int_0^y (f(s,t)+h(s,t))^{\frac{1}{1-\beta}}\overline{P}_1(s,t)dtds\Big) \times\exp\Big(\int_0^x\int_0^y (f(s,t)+h(s,t))^{\frac{1}{1-\beta}}\overline{Q}_1(s,t)dtds\Big)\Big]^{1-\beta},\quad (x,y)\in D, \tag{3.26}$$

where

$$\begin{aligned}
&\overline{M}_1=B\Big[\frac{\beta+\gamma-1}{\beta},\frac{2\beta-1}{\beta}\Big],\\
&\overline{M}=\max\Big\{qK^{q-1},rK^{r-1}\Big\},\\
&\overline{A}_1(x,y)=qK^{q-1}a(x,y)+(1-q)K^q,\ \overline{A}_2(x,y)=rK^{r-1}a(x,y)+(1-r)K^r,\\
&\overline{\mathscr{A}}_1(x,y)=\int_0^x\int_0^y (f(s,t)\overline{A}_1(s,t)+h(s,t)\overline{A}_2(s,t))^{\frac{1}{1-\beta}}dtds,\\
&\overline{P}_1(x,y)=2^{\frac{\beta}{1-\beta}}\overline{M}_1^{\frac{2\beta}{1-\beta}}(xy)^{\frac{2\beta+\gamma-2}{1-\beta}}b^{\frac{1}{1-\beta}}(x,y)\overline{\mathscr{A}}_1(x,y),\\
&\overline{Q}_1(x,y)=2^{\frac{\beta}{1-\beta}}\overline{M}_1^{\frac{2\beta}{1-\beta}}(xy)^{\frac{2\beta+\gamma-2}{1-\beta}}b^{\frac{1}{1-\beta}}(x,y)\overline{M}^{\frac{1}{1-\beta}}.
\end{aligned}$$

(ii) If $\beta\in(0,\frac{1}{2}]$,

$$u(x,y)\leqslant a(x,y)+\Big[\overline{P}_2(x,y)+\overline{Q}_2(x,y)\Big(\int_0^x\int_0^y (f(s,t)+h(s,t))^{\frac{1+4\beta}{\beta}}\overline{P}_2(s,t)dtds\Big) \times\exp\Big(\int_0^x\int_0^y (f(s,t)+h(s,t))^{\frac{1+4\beta}{\beta}}\overline{Q}_2(s,t)dtds\Big)\Big]^{\frac{\beta}{1+4\beta}},\quad (x,y)\in D, \tag{3.27}$$

where

$$\overline{M}_2 = B\left[\frac{\gamma(1+4\beta)-\beta}{1+3\beta}, \frac{4\beta^2}{1+3\beta}\right],$$
$$\overline{\mathscr{A}}_2(x,y) = \int_0^x \int_0^y (f(s,t)\overline{A}_1(s,t) + h(s,t)\overline{A}_2(s,t))^{\frac{1+4\beta}{\beta}} dtds,$$
$$\overline{P}_2(x,y) = 2^{\frac{1+3\beta}{\beta}} \overline{M}_2^{\frac{2(1+3\beta)}{\beta}} (xy)^{\frac{(4\beta+1)(\gamma-1)+4\beta^2}{\beta}} b^{\frac{1+4\beta}{\beta}}(x,y)\overline{\mathscr{A}}_2(x,y),$$
$$\overline{Q}_2(x,y) = 2^{\frac{1+3\beta}{\beta}} \overline{M}_2^{\frac{(2(1+3\beta))}{\beta}} (xy)^{\frac{(4\beta+1)(\gamma-1)+4\beta^2}{\beta}} b^{\frac{1+4\beta}{\beta}}(x,y)\overline{M}^{\frac{1+4\beta}{\beta}}.$$

Proof. Inequalities (3.26) and (3.27) follow by letting $p = \alpha = 1$ and $0 < q \leqslant 1$, $0 < r \leqslant 1$ in Theorem 3.1 and by simple computation. Details are omitted here. □

COROLLARY 3.3. *Let functions $u(x,y)$, $a(x,y)$, $b(x,y)$, $f(s,t)$ and $h(x,y)$ be defined as in Theorem 3.1, suppose that*

$$u^2(x,y) \leqslant a(x,y) + b(x,y)\int_0^x \int_0^y (x-s)^{\beta-1}s^{\gamma-1}(y-t)^{\beta-1}t^{\gamma-1} \times \Big[f(s,t)u(s,t) + h(s,t)u(\sigma(s),\sigma(t))\Big]dtds, \quad (x,y)\in D, \tag{3.28}$$

with the initial condition of Theorem 3.1, then for any $K>0$, one has the following

(i) If $\beta \in (\frac{1}{2},1)$,

$$u^2(x,y) \leqslant a(x,y) + \Bigg[\widetilde{P}_1(x,y) + \widetilde{Q}_1(x,y)\left(\int_0^x\int_0^y (f(s,t)+h(s,t)^{\frac{1}{1-\beta}}\widetilde{P}_1(s,t)dtds\right) \times \exp\left(\int_0^x\int_0^y (f(s,t)+h(s,t))^{\frac{1}{1-\beta}}\widetilde{Q}_1(s,t)dtds\right)\Bigg]^{1-\beta}, \quad (x,y)\in D, \tag{3.29}$$

where

$$\widetilde{M}_1 = B\left[\frac{\beta+\gamma-1}{\beta}, \frac{2\beta-1}{\beta}\right],$$
$$\widetilde{A}_1(x,y) = \widetilde{A}_2(x,y) = \frac{1}{2}K^{-\frac{1}{2}}a(x,y) + \frac{1}{2}K^{\frac{1}{2}},$$
$$\widetilde{\mathscr{A}_1}(x,y) = \int_0^x\int_0^y \left[(f(s,t)+h(s,t))\widetilde{A}_1(s,t)\right]^{\frac{1}{1-\beta}} dtds,$$
$$\widetilde{P}_1(x,y) = 2^{\frac{1}{1-\beta}}\widetilde{M}_1^{\frac{2\beta}{1-\beta}}(xy)^{\frac{2\beta+\gamma-2}{1-\beta}} b^{\frac{1}{1-\beta}}(x,y)\widetilde{\mathscr{A}_1}(x,y),$$
$$\widetilde{Q}_1(x,y) = 2^{\frac{\beta}{1-\beta}}\widetilde{M}_1^{\frac{2\beta}{1-\beta}}(xy)^{\frac{2\beta+\gamma-2}{1-\beta}} b^{\frac{1}{1-\beta}}(x,y)\widetilde{M}^{\frac{1}{1-\beta}}.$$

(ii) If $\beta \in (0, \frac{1}{2}]$,

$$u^2(x,y) \leqslant a(x,y) + \Bigg[\widetilde{P}_2(x,y) + \widetilde{Q}_2(x,y)\Bigg(\int_0^x \int_0^y (f(s,t)+h(s,t))^{\frac{1+4\beta}{\beta}} \widetilde{P}_2(s,t)dtds\Bigg)$$
$$\times \exp\Bigg(\int_0^x \int_0^y (f(s,t)+h(s,t))^{\frac{1+4\beta}{\beta}} \widetilde{Q}_2(s,t)dtds\Bigg)\Bigg]^{\frac{\beta}{1+4\beta}}, \quad (x,y) \in D, \tag{3.30}$$

where

$$\widetilde{M}_2 = B\left[\frac{\gamma(1+4\beta)-\beta}{1+3\beta}, \frac{4\beta^2}{1+3\beta}\right],$$
$$\widetilde{\mathscr{A}_2}(x,y) = \int_0^x \int_0^y \left[(f(s,t)+h(s,t))\widetilde{A}_2(s,t)\right]^{\frac{1+4\beta}{\beta}} dtds,$$
$$\widetilde{P}_2(x,y) = 2^{\frac{1+4\beta}{\beta}} \widetilde{M}_2^{\frac{2(1+3\beta)}{\beta}} (xy)^{\frac{(4\beta+1)(\gamma-1)+4\beta^2}{\beta}} b^{\frac{1+4\beta}{\beta}}(x,y)\widetilde{\mathscr{A}_2}(x,y),$$
$$\widetilde{Q}_1(x,y) = 2^{\frac{1+3\beta}{\beta}} \widetilde{M}_2^{\frac{2(1+3\beta)}{\beta}} (xy)^{\frac{(4\beta+1)(\gamma-1)+4\beta^2}{\beta}} b^{\frac{1+4\beta}{\beta}}(x,y)\widetilde{M}^{\frac{1+4\beta}{\beta}}.$$

Proof. Inequality (3.29) and (3.30) follow by letting $p = 2$, $q = r = \alpha = 1$ in Theorem 3.1 and by simple computation. Details are omitted. □

4. Applications

In this section, we will indicate the usefulness of our main result in the study of the boundedness of solutions for certain partial integral equations with weakly singular kernel.

Consider the partial integral equation:

$$z^p(x,y) = l(x,y) + g(x,y)\int_0^x \int_0^y (x^\alpha - s^\alpha)^{\beta-1}s^{\gamma-1}(y^\alpha - t^\alpha)^{\beta-1}t^{\gamma-1}$$
$$\times \Big[F_1(s,t,z(s,t)) + F_2(s,t)z^r(\sigma(s),\sigma(t))\Big]dtds, \tag{4.1}$$

for $(x,y) \in D = [0,X) \times [0,Y)$, where $l(x,y)$, $g(x,y)$, $z(x,y)$, $F_2(s,t) \in C(D,R)$, $F_1(s,t,z(s,t)) \in (D \times R, R)$. Suppose that

$$\begin{aligned} |l(x,y)| &\leqslant a(x,y), \\ |g(x,y)| &\leqslant b(x,y), \\ |F_1(s,t,z)| &\leqslant f(s,t)|z^q|, \\ |F_2(s,t)| &\leqslant h(s,t), \end{aligned} \tag{4.2}$$

if $z(x,y),\ (x,y)\in D$ is any solution of (4.1), we have

$$|z^p(x,y)| \leqslant a(x,y)+b(x,y)\int_0^x\int_0^y (x^\alpha-s^\alpha)^{\beta-1}s^{\gamma-1}(y^\alpha-t^\alpha)^{\beta-1}t^{\gamma-1} \times\left[f(s,t)|z^q(s,t)|+h(s,t)|z^r(\sigma(s),\sigma(t))|\right]dtds. \tag{4.3}$$

with the initial condition

$$|z(x,y)|=\phi(x,y),\ (x,y)\in D'=[\mu,0]\times[\mu,0],$$
$$\phi(\sigma(x),\sigma(y))\leqslant(a(x,y))^{\frac{1}{p}}\ for\ (x,y)\in D\ with\ \sigma(x)\leqslant 0,\ \sigma(y)\leqslant 0, \tag{4.4}$$

where $p\neq 0,\ p\geqslant q>0,\ p\geqslant r>0,\ p,\ q,\ r$ be constants, the functions $a(x,y)$, $b(x,y),\ f(x,y),\ h(s,t)$ be nonnegative continuous functions for $(x,y)\in D$ and $\sigma(x)$, $\sigma(y)\in C(R_+,\ R),\ \sigma(t)\leqslant t,\ t\in R_+$. $-\infty<\mu=\inf\{\sigma(t),\ t\in R_+\}\leqslant 0$ and $\phi(x,y)\in C([\mu,0]\times[\mu,0],\ R_+)$.

It is easy for us to find that the inequality (4.3) is similar to (3.1), and applying Theorem 3.1, we have the same results as (3.3) and (3.4).

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