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GENERALIZATION OF JENSEN'S AND JENSEN–STEFFENSEN'S INEQUALITIES BY GENERALIZED MAJORIZATION THEOREM

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Abstract. In this paper, we use generalized majorization theorem and give the generalizations of Jensen's and Jensen-Steffensen's inequalities. We present the generalization of converse of Jensen's inequality. We give bounds for the identities related to the generalization of Jensen's inequality by using Čebyšev functionals. We also give Grüss and Ostrowski types inequalities for these functionals. We present mean value theorems and n-exponential convexity which leads to exponential convexity and log-convexity for these functionals. We give some families of functions which enable us to construct a large families of functions that are exponentially convex and also give classes of means.

1. Introduction and preliminaries

One of the most important inequality in Mathematics and Statistics is the Jensen inequality (see [19, p. 43]).

THEOREM 1. *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a convex function. Let $n \geqslant 2$, $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ and $\mathbf{w} = (w_1, \ldots, w_n)$ be a positive n-tuple. Then*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leqslant \frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i), \tag{1}$$

where

$$W_k = \sum_{i=1}^{k} w_i, \quad k = 1, \ldots, n. \tag{2}$$

If f is strictly convex, then inequality (1) is strict unless $x_1 = \cdots = x_n$.

The condition "$\mathbf{w}$ is a positive n-tuple" can be replaced by "$\mathbf{w}$ is a non-negative n-tuple and $W_n > 0$". Note that the Jensen inequality (1) can be used as an alternative definition of convexity.

It is reasonable to ask whether the condition "$\mathbf{w}$ is a non-negative n-tuple" can be relaxed at the expense of restricting $\mathbf{x}$ more severely. An answer to this question was given by Steffensen [21] (see also [19, p. 57]).

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THEOREM 2. *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a convex function. If $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ is a monotonic n-tuple and $\mathbf{w} = (w_1, \ldots, w_n)$ a real n-tuple such that*

$$0 \leqslant W_k \leqslant W_n\,, \quad k = 1, \ldots, n-1, \quad W_n > 0, \tag{3}$$

is satisfied, where W_k are as in (2), then (1) holds. If f is strictly convex, then inequality (1) is strict unless $x_1 = \cdots = x_n$.

Inequality (1) under conditions from Theorem 2 is called the Jensen-Steffensen inequality.

Now we give some basic introduction to majorization:

There is a certain intuitive appeal to the vague notion that the components of m-tuple $\mathbf{x}$ are less spread out, or more nearly equal, than are the components of m-tuple $\mathbf{y}$. The notion arises in a variety of contexts, and it can be made precise in a number of ways. But in remarkably many cases, the appropriate statement is that $\mathbf{x}$ majorizes $\mathbf{y}$ means that the sum of k largest entries of $\mathbf{y}$ does not exceed the sum of k largest entries of $\mathbf{x}$ for all $k = 1, 2, \ldots, m-1$ with equality for $k = m$ and we write as $\mathbf{y} \prec \mathbf{x}$. A mathematical origin of majorization is illustrated by the work of Schur [20] on Hadamard's determinant inequality. Many mathematical characterization problems are known to have solutions that involve majorization. A complete and superb reference on the subject are the books [10], [17]. The comprehensive survey by Ando [9] provides alternative derivations, generalizations and a different viewpoint. The following theorem known as the majorization theorem and its convenient proof is given by Marshall and Olkin in [17].

THEOREM 3. *Let $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ be two m-tuples such that $x_i, y_i \in [a, b]$, for $i = 1, 2, \ldots, m$. Then for any continuous convex function $f : [a, b] \to \mathbb{R}$ the inequality*

$$\sum_{i=1}^{m} f(y_i) \leqslant \sum_{i=1}^{m} f(x_i)$$

holds if and only if $\mathbf{y} \prec \mathbf{x}$.

The following theorem can be regarded as the generalization of Theorem 3, known as weighted majorization theorem and is proved by Fuchs in [13].

THEOREM 4. *Let $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ be two decreasing real m-tuples with $x_i, y_i \in [a, b]$, for $i = 1, 2, \ldots, m$. Let $\mathbf{w} = (w_1, \ldots, w_m)$ be real m-tuple such that*

$$\sum_{i=1}^{l} w_i y_i \leqslant \sum_{i=1}^{l} w_i x_i \text{ for } l = 1, 2, \ldots, m-1 \tag{4}$$

and

$$\sum_{i=1}^{m} w_i y_i = \sum_{i=1}^{m} w_i x_i. \tag{5}$$

Then for every continuous convex function $f : [a,b] \to \mathbb{R}$*, we have*

$$\sum_{i=1}^{m} w_i f(y_i) \leqslant \sum_{i=1}^{m} w_i f(x_i).$$

The following theorem is a consequence of Theorem 4.

THEOREM 5. *Let* $x, y : [a,b] \to [\alpha, \beta]$ *be decreasing and* $w : [a,b] \to \mathbb{R}$ *be continuous functions. If*

$$\int_a^v w(t)y(t)\,dt \leqslant \int_a^v w(t)x(t)\,dt \ \text{ for every } v \in [a,b], \tag{6}$$

and

$$\int_a^b w(t)y(t)\,dt = \int_a^b w(t)x(t)\,dt \tag{7}$$

hold, then for every continuous convex function $f : [\alpha, \beta] \to \mathbb{R}$*, we have*

$$\int_a^b w(t) f(y(t))\,dt \leqslant \int_a^b w(t) f(x(t))\,dt. \tag{8}$$

In our main results we will use generalized result for n-convex function, therefore here we recall the definition of n-convexity (see for example [19]).

DEFINITION 1. The divided difference of order n, $n \in \mathbb{N}$, of the function $f : [a,b] \to \mathbb{R}$ at mutually different points $x_0, x_1, \ldots, x_n \in [a,b]$ is defined recursively by

$$[x_i; f] = f(x_i), \quad i = 0, \ldots, n$$
$$[x_0, \ldots, x_n; f] = \frac{[x_1, \ldots, x_n; f] - [x_0, \ldots, x_{n-1}; f]}{x_n - x_0}.$$

The value $[x_0, \ldots, x_n; f]$ is independent of the order of the points $x_0, \ldots, x_n$.

This definition may be extended to include the case in which some or all the points coincide. Assuming that $f^{(j-1)}(x)$ exists, we define

$$[\underbrace{x, \ldots, x}_{j\text{-times}}; f] = \frac{f^{(j-1)}(x)}{(j-1)!}.$$

DEFINITION 2. A function $f : [a,b] \to \mathbb{R}$ is n-convex, $n \geqslant 0$, if for all choices of $(n+1)$ distinct points $x_i \in [a,b]$, $i = 0, \ldots, n$, the inequality

$$[x_0, x_1, \ldots, x_n; f] \geqslant 0$$

holds.

THEOREM 6. [19, p. 16] *Let* $f:[\alpha,\beta]\to\mathbb{R}$ *be a function such that* $f^{(n)}$ *exists, then* f *is* n*-convex if and only if* $f^{(n)}\geqslant 0$.

From Definition 2, it follows that 2-convex functions are just convex functions. Furthermore, 1-convex functions are increasing functions and 0-convex functions are nonnegative functions. To complete this section we give some generalized majorization theorems from [7], which we will use in our main results.

The following generalized Montgomery identity via Taylor's formula given in [6, 8].

PROPOSITION 1. *Let* $n\in\mathbb{N}$, $f:I\to\mathbb{R}$ *be such that* $f^{(n-1)}$ *is absolutely continuous,* $I\subset\mathbb{R}$ *an open interval,* $a,b\in I$ *and* $a<b$. *Then the following identity holds*

$$\begin{aligned} f(x) &= \frac{1}{b-a}\int_a^b f(t)\,dt+\sum_{k=0}^{n-2}\frac{f^{(k+1)}(a)}{k!\,(k+2)}\frac{(x-a)^{k+2}}{b-a}-\sum_{k=0}^{n-2}\frac{f^{(k+1)}(b)}{k!\,(k+2)}\frac{(x-b)^{k+2}}{b-a}\\ &\quad+\frac{1}{(n-1)!}\int_a^b T_n(x,s)\,f^{(n)}(s)\,ds, \end{aligned} \tag{9}$$

where

$$T_n(x,s)=\begin{cases} -\frac{(x-s)^n}{n(b-a)}+\frac{x-a}{b-a}(x-s)^{n-1}, & a\leqslant s\leqslant x,\\ -\frac{(x-s)^n}{n(b-a)}+\frac{x-b}{b-a}(x-s)^{n-1}, & x<s\leqslant b. \end{cases} \tag{10}$$

In case $n=1$ the sum $\sum_{k=0}^{n-2}\ldots$ is empty, so the identity (9) reduces to the well-known Montgomery identity

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+\int_a^b P(x,s)\,f'(s)\,ds,$$

where $P(x,s)$ is the Peano kernel, defined by

$$P(x,s)=\begin{cases} \frac{s-a}{b-a}, & a\leqslant s\leqslant x,\\ \frac{s-b}{b-a}, & x<s\leqslant b. \end{cases}$$

The following generalizations of majorization theorem by Montgomery identity are given in [7].

THEOREM 7. ([7]) *Suppose all the assumptions of Proposition 1 hold. Additionally, suppose that* $x_i, y_i\in[a,b]$ *and* $w_i\in\mathbb{R}$ *for* $i=1,2,\ldots,m$. *Then*

$$\begin{aligned} &\sum_{i=1}^m w_i f(y_i)-\sum_{i=1}^m w_i f(x_i)\\ &\quad=\frac{1}{b-a}\sum_{k=0}^{n-2}\frac{1}{k!(k+2)}\sum_{i=1}^m w_i\left[f^{(k+1)}(a)\left\{(y_i-a)^{k+2}-(x_i-a)^{k+2}\right\}\right. \end{aligned}$$

$$- f^{(k+1)}(b)\left\{(y_i-b)^{k+2}-(x_i-b)^{k+2}\right\}\Bigg]$$
$$+\frac{1}{(n-1)!}\int_a^b\left[\sum_{i=1}^m w_i\left(T_n(y_i,s)-T_n(x_i,s)\right)\right]f^{(n)}(s)ds, \tag{11}$$

where $T_n(.,s)$ is as defined in Proposition 1.

COROLLARY 1. ([7]) *Suppose all the assumptions of Theorem 7 hold. Additionally, suppose that $\mathbf{x}=(x_1,\dots x_m)$, $\mathbf{y}=(y_1,\dots y_m)\in[a,b]^m$ are two decreasing m-tuples and $\mathbf{w}=(w_1,\dots,w_m)\in\mathbb{R}^m$ which satisfy conditions (4) and (5). If f is $2n$-convex function then the following inequality holds:*

$$\sum_{i=1}^m w_i f(y_i)-\sum_{i=1}^m w_i f(x_i)$$
$$\geqslant \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{1}{k!(k+2)}\sum_{i=1}^m w_i\Bigg[f^{(k+1)}(a)\left\{(y_i-a)^{k+2}-(x_i-a)^{k+2}\right\}$$
$$- f^{(k+1)}(b)\left\{(y_i-b)^{k+2}-(x_i-b)^{k+2}\right\}\Bigg]. \tag{12}$$

Moreover, if $f^{(j)}(a)\geqslant 0$ and $(-1)^j f^{(j)}(b)\geqslant 0$ for $j=1,\dots,2n-1$, then

$$\sum_{i=1}^m w_i f(y_i)\geqslant\sum_{i=1}^m w_i f(x_i).$$

THEOREM 8. ([7]) *Let $x,y:[\alpha,\beta]\to\mathbb{R}$ be two functions and $w:[\alpha,\beta]\to\mathbb{R}$ be continuous function. Let $f:I\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\in\mathbb{N}$, $I\subset\mathbb{R}$ an open interval, $a,b\in I$, $a<b$, then we have the following identity:*

$$\int_\alpha^\beta w(t)f(y(t))dt-\int_\alpha^\beta w(t)f(x(t))dt$$
$$=\frac{1}{b-a}\sum_{k=0}^{n-2}\frac{1}{k!(k+2)}\int_\alpha^\beta w(t)\Bigg[f^{(k+1)}(a)\left\{(y(t)-a)^{k+2}-(x(t)-a)^{k+2}\right\}$$
$$- f^{(k+1)}(b)\left\{(y(t)-b)^{k+2}-(x(t)-b)^{k+2}\right\}\Bigg]dt$$
$$+\frac{1}{(n-1)!}\int_a^b\left(\int_\alpha^\beta w(t)\left(T_n(y(t),s)-T_n(x(t),s)\right)dt\right)f^{(n)}(s)ds, \tag{13}$$

where $T_n(.,s)$ is as defined in Proposition 1.

COROLLARY 2. ([7]) *Suppose all the assumptions of Theorem 8 hold. Additionally, suppose that x and y are decreasing functions which satisfy conditions (6) and (7). If f is $2n$-convex function then the following inequality holds:*

$$\begin{aligned}
&\int_{\alpha}^{\beta} w(t)f(y(t))dt - \int_{\alpha}^{\beta} w(t)f(x(t))dt \\
&\geqslant \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{1}{k!(k+2)}\int_{\alpha}^{\beta} w(t)\left[f^{(k+1)}(a)\left\{(y(t)-a)^{k+2}-(x(t)-a)^{k+2}\right\}\right. \\
&\left.-f^{(k+1)}(b)\left\{(y(t)-b)^{k+2}-(x(t)-b)^{k+2}\right\}\right]dt. \qquad (14)
\end{aligned}$$

Moreover, if $f^{(j)}(a) \geqslant 0$ and $(-1)^j f^{(j)}(b) \geqslant 0$ for $j = 1,\ldots,2n-1$, then

$$\int_{\alpha}^{\beta} w(t)f(y(t))dt \geqslant \int_{\alpha}^{\beta} w(t)f(x(t))dt.$$

For some more recent results, related to generalizations and refinements of majorization theorem, see [1]–[5], [7, 14] and some of the references in them.

In this paper we utilize generalized majorization theorem and establish generalization of Jensen's and Jensen-Steffensen's inequalities for the class of $2n$-convex functions. We also discuss the generalization of converse of Jensen's inequality. We use inequalities for Čebyšev functional to obtain bounds for the identities related to the generalization of Jensen's inequalities. We present mean value theorems and n-exponential convexity for the functional obtained from the generalized Jensen's and Jensen-Steffensen's inequalities which leads to exponential convexity and log-convexity for these functionals. Finally, we discuss the results for particular families of functions.

2. Generalization of Jensen's inequality

First we give generalization of Jensen's inequality associated with Montgomery identity.

THEOREM 9. *Let $n \in \mathbb{N}$, $f : I \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a,b \in I$, $a < b$. Let $\mathbf{x} = (x_1,\ldots,x_m) \in [a,b]^m$ be m-tuple and $\mathbf{w} = (w_1,\ldots w_m)$ be positive m-tuple, $W_m = \sum_{i=1}^{m} w_i$ and $\bar{x} = \frac{1}{W_m}\sum_{i=1}^{m} w_i x_i$.*

(i) If $\mathbf{x}$ is decreasing m-tuple and $f : [a,b] \to \mathbb{R}$ is $2n$-convex function, then we have

$$\begin{aligned}
\frac{1}{W_m}\sum_{i=1}^{m} w_i f(x_i) - f(\bar{x}) \geqslant \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{1}{k!(k+2)}\left[f^{(k+1)}(a)\left(\frac{\sum_{i=1}^{m} w_i(x_i-a)^{k+2}}{W_m}-(\bar{x}-a)^{k+2}\right)\right. \\
\left.-f^{(k+1)}(b)\left(\frac{\sum_{i=1}^{m} w_i(x_i-b)^{k+2}}{W_m}-(\bar{x}-b)^{k+2}\right)\right]. \qquad (15)
\end{aligned}$$

(ii) If the inequality (15) holds and the function F defined by

$$F(.)=\sum_{k=0}^{2n-2}\frac{f^{(k+1)}(a)}{k!\,(k+2)}\frac{(.-a)^{k+2}}{b-a}-\sum_{k=0}^{2n-2}\frac{f^{(k+1)}(b)}{k!\,(k+2)}\frac{(.-b)^{k+2}}{b-a}, \tag{16}$$

is convex, then the right hand side of (15) is non-negative and

$$f(\overline{x})\leqslant\frac{1}{W_m}\sum_{i=1}^{m}w_if(x_i). \tag{17}$$

Proof.

(i) Let k be the largest number from $\{1,\ldots,m\}$ such that $x_k\geqslant\overline{x}$, then as $\mathbf{x}$ is decreasing m-tuple so we have $x_l\geqslant\overline{x}$ for $l=1,2,\ldots,k$ and $x_l\leqslant\overline{x}$ for $l=k+1,k+2,\ldots,m$.

Now as $x_l\geqslant\overline{x}$ for $l=1,2,\ldots,k$, so we have

$$\sum_{i=1}^{l}w_i\overline{x}\leqslant\sum_{i=1}^{l}w_ix_i \quad\text{for } l=1,2,\ldots,k. \tag{18}$$

Similarly as $x_l\leqslant\overline{x}$ for $l=k+1,k+2,\ldots,m$, so we have

$$\sum_{i=k+1}^{j}w_ix_i\leqslant\sum_{i=k+1}^{j}w_i\overline{x} \quad\text{for } j=k+1,k+2,\ldots,m.$$

Hence

$$\sum_{i=1}^{j}w_ix_i=\sum_{i=1}^{m}w_ix_i-\sum_{i=j+1}^{m}w_ix_i\geqslant\sum_{i=1}^{m}w_i\overline{x}-\sum_{i=j+1}^{m}w_i\overline{x}=\sum_{i=1}^{j}w_i\overline{x}, \tag{19}$$
$$\text{for } j=k,k+1,\ldots,m.$$

Using (18) and (19) we get that

$$\sum_{i=1}^{l}w_i\overline{x}\leqslant\sum_{i=1}^{l}w_ix_i, \text{ for all } l=1,2,\ldots,m-1$$

and obviously

$$\sum_{i=1}^{m}w_i\overline{x}=\sum_{i=1}^{m}w_ix_i.$$

Since the conditions (4) and (5) are satisfied. Therefore using Corollary 1 for $\mathbf{y}=(x_1,\ldots,x_m)$ and $\mathbf{x}=(\overline{x},\ldots,\overline{x})$, we get (15).

(ii) We may write the right hand side of (15) as

$$\frac{1}{W_m}\sum_{i=1}^{m}w_iF(x_i)-F(\overline{x}).$$

Since F is convex so by Jensen's inequality, we have

$$\frac{1}{W_m}\sum_{i=1}^{m} w_i F(x_i) - F(\overline{x}) \geqslant 0.$$

Hence (17) holds. □

In the following theorem we give integral version of Theorem 9.

THEOREM 10. *Let $n \in \mathbb{N}$, $f:[a,b] \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous, $x:[\alpha,\beta] \to \mathbb{R}$ be continuous function such that $x([\alpha,\beta]) \subseteq [a,b]$, $\lambda:[\alpha,\beta] \to \mathbb{R}$ be increasing, bounded function with $\lambda(\alpha) \neq \lambda(\beta)$ and $\overline{x} = \frac{\int_\alpha^\beta x(t)d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}$.*

(i) If x is decreasing function and $f:[a,b] \to \mathbb{R}$ is $2n$-convex function, then we have

$$\begin{aligned} &\frac{\int_\alpha^\beta f(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - f(\overline{x}) \\ &\quad \geqslant \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{1}{k!(k+2)}\left[f^{(k+1)}(a)\left\{\frac{\int_\alpha^\beta (x(t)-a)^{k+2}d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - (\overline{x}-a)^{k+2}\right\}\right. \\ &\quad\quad \left. - f^{(k+1)}(b)\left\{\frac{\int_\alpha^\beta (x(t)-b)^{k+2}d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - (\overline{x}-b)^{k+2}\right\}\right]. \end{aligned} \tag{20}$$

(ii) If the inequality (20) holds and the function F defined as in (16) is convex, then the right hand side of (20) is non-negative and

$$f(\overline{x}) \leqslant \frac{\int_\alpha^\beta f(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}. \tag{21}$$

Proof.

(i) Let γ_0 be the largest number in $[\alpha,\beta]$ such that $x(\gamma_0) \geqslant \overline{x}$. But x is decreasing function so we have

$$x(\gamma) \geqslant \overline{x} \text{ for all } \gamma \in [\alpha,\gamma_0] \text{ and } x(\gamma) \leqslant \overline{x} \text{ for all } \gamma \in [\gamma_0,\beta].$$

Case 1. If $x(\gamma) \geqslant \overline{x}$ for all $\gamma \in [\alpha,\gamma_0]$, then we may write

$$x(t) \geqslant \overline{x} \text{ for all } t \in [\alpha,\gamma], \gamma \in [\alpha,\gamma_0].$$

As λ is increasing so by integrating both sides with respect to λ over $[\alpha,\gamma]$, we get

$$\int_\alpha^\gamma x(t)d\lambda(t) \geqslant \int_\alpha^\gamma \overline{x}d\lambda(t),\ \gamma \in [\alpha,\gamma_0]. \tag{22}$$

Case 2. If $x(\gamma) \leqslant \bar{x}$ for all $\gamma \in [\gamma_0, \beta]$, then we may write

$$x(t) \leqslant \bar{x} \text{ for all } t \in [\gamma, \beta], \gamma \in [\gamma_0, \beta].$$

But λ is increasing so by integrating both sides with respect to λ over $[\gamma, \beta]$, we get

$$\int_\gamma^\beta x(t) d\lambda(t) \leqslant \int_\gamma^\beta \bar{x} d\lambda(t).$$

Therefore we have

$$\begin{aligned} \int_\alpha^\gamma x(t) d\lambda(t) &= \int_\alpha^\beta x(t) d\lambda(t) - \int_\gamma^\beta x(t) d\lambda(t) \\ &\geqslant \int_\alpha^\beta \bar{x} d\lambda(t) - \int_\gamma^\beta \bar{x} d\lambda(t) = \int_\alpha^\gamma \bar{x} d\lambda(t) \end{aligned}$$

i.e.

$$\int_\alpha^\gamma x(t) d\lambda(t) \geqslant \int_\alpha^\gamma \bar{x} d\lambda(t), \ \gamma \in [\gamma_0, \beta]. \tag{23}$$

From (22) and (23) we have

$$\int_\alpha^\gamma x(t) d\lambda(t) \geqslant \int_\alpha^\gamma \bar{x} d\lambda(t), \ \gamma \in [\alpha, \beta].$$

Also the equality

$$\int_\alpha^\beta x(t) d\lambda(t) = \int_\alpha^\beta \bar{x} d\lambda(t) \quad \text{holds.}$$

Since the conditions (6) and (7) are satisfied, therefore using Corollary 2 for $y(t) = x(t)$ and $x(t) = \bar{x}$, we get the inequality (20).

(ii) We may write the right hand side of (20) as

$$\frac{\int_\alpha^\beta F(x(t)) d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - F(\bar{x}).$$

Since F is convex so by Jensen's inequality, we have

$$\frac{\int_\alpha^\beta F(x(t)) d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - F(\bar{x}) \geqslant 0.$$

Hence (21) holds. □

REMARK 1. If we take $x(t) = t$, $\lambda(t) = t$, in the inequality (20), then we obtain generalization of Hermite-Hadamard inequality.

3. Generalization of Jensen-Steffensen's inequality

THEOREM 11. *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous, $\mathbf{x} = (x_1,\dots,x_m) \in [a,b]^m$ be decreasing m-tuple. Let $\mathbf{w} = (w_1,\dots,w_m)$ be real m-tuple such that $0 \leqslant W_k \leqslant W_m$ $(k = 1,2,\dots,m)$, $W_m > 0$ where $W_k = \sum_{i=1}^{k} w_i$ and $\overline{x} = \frac{1}{W_m}\sum_{i=1}^{m} w_i x_i$.*

(i) Then for any $2n$-convex function $f : [a,b] \to \mathbb{R}$, the inequality (15) holds.

(ii) If the inequality (15) holds and the function F defined as in (16) is convex, then the right hand side of (15) is non-negative and (17) holds.

Proof. (i) Let k be the largest number $\{1,2,\dots,m\}$ such that $x_k \geqslant \overline{x}$ then $x_l \geqslant \overline{x}$ for $l = 1,\dots,k$, and we have

$$\sum_{i=1}^{l} w_i x_i - W_l x_l = \sum_{i=1}^{l-1} (x_i - x_{i+1}) W_i \geqslant 0$$

and so we obtain

$$\sum_{i=1}^{l} w_i \overline{x} = W_l \overline{x} \leqslant W_l x_l \leqslant \sum_{i=1}^{l} x_i w_i. \tag{24}$$

Also for $l = k+1,\dots,m$ we have $x_{k+1} < \overline{x}$, therefore

$$x_l (W_m - W_l) - \sum_{i=l+1}^{m} w_i x_i = \sum_{i=l+1}^{m} (x_{i-1} - x_i)(W_m - W_{i-1}) \geqslant 0.$$

Hence, we conclude that

$$\sum_{i=l+1}^{m} w_i \overline{x} = (W_m - W_l)\overline{x} > (W_m - W_l) x_l \geqslant \sum_{i=l+1}^{m} w_i x_i. \tag{25}$$

From (24) and (25), we get

$$\sum_{i=1}^{l} w_i \overline{x} \leqslant \sum_{i=1}^{l} x_i w_i \quad \text{for all} \quad l = 1,2,\dots m-1.$$

Obviously the equality

$$\sum_{i=1}^{m} w_i \overline{x} = \sum_{i=1}^{m} x_i w_i$$

holds. Since the conditions (4) and (5) are satisfied, therefore using Corollary 1 for $\mathbf{y} = (x_1,\dots x_m)$ and $\mathbf{x} = (\overline{x},\dots,\overline{x})$, we get (15).

(ii) The proof is similar to the proof of Theorem 9(ii). □

THEOREM 12. *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous, $x : [\alpha,\beta] \to \mathbb{R}$ be continuous decreasing function such that $x([\alpha,\beta]) \subseteq [a,b]$, $\lambda : [\alpha,\beta] \to \mathbb{R}$ is either continuous or of bounded variation with $\lambda(\alpha) \leqslant \lambda(t) \leqslant \lambda(\beta)$ for all $t \in [\alpha,\beta]$ and $\overline{x} = \frac{\int_\alpha^\beta x(t)d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}$.*

(i) Then for any $2n$-convex function f, the inequality (20) holds.

(ii) If the inequality (20) holds and the function F defined as in (16) is convex, then the right hand side of (20) is non-negative and (21) holds.

Proof. (i) Let γ_0 be the largest number in $[\alpha,\beta]$ such that $x(\gamma_0) \geqslant \overline{x}$. But x is decreasing function so we have

$$x(\gamma) \geqslant \overline{x} \text{ for all } \gamma \in [\alpha,\gamma_0] \text{ and } x(\gamma) \leqslant \overline{x} \text{ for all } \gamma \in [\gamma_0,\beta].$$

(a) If $x(\gamma) \geqslant \overline{x}$ for all $\gamma \in [\alpha,\gamma_0]$, then we may write

$$x(t) \geqslant \overline{x} \text{ for all } t \in [\alpha,\gamma], \gamma \in [\alpha,\gamma_0].$$

Therefore we have

$$\overline{x}\int_\alpha^\gamma d\lambda(t) \leqslant x(\gamma)\int_\alpha^\gamma d\lambda(t),\ \gamma \in [\alpha,\gamma_0]. \tag{26}$$

But

$$\int_\alpha^\gamma x(t)d\lambda(t) - x(\gamma)\int_\alpha^\gamma d\lambda(t) = -\int_\alpha^\gamma x'(t)\left(\int_\alpha^t d\lambda(x)\right)dt \geqslant 0. \tag{27}$$

From (26) and (27), we get

$$\overline{x}\int_\alpha^\gamma d\lambda(t) \leqslant \int_\alpha^\gamma x(t)d\lambda(t),\ \gamma \in [\alpha,\gamma_0]. \tag{28}$$

(b) If $x(\gamma) \leqslant \overline{x}$ for all $\gamma \in [\gamma_0,\beta]$, then we may write

$$x(t) \leqslant \overline{x} \text{ for all } t \in [\gamma,\beta], \gamma \in [\gamma_0,\beta],$$

therefore we have

$$\overline{x}\int_\gamma^\beta d\lambda(t) \geqslant x(\gamma)\int_\gamma^\beta d\lambda(t). \tag{29}$$

But

$$x(\gamma)\int_\gamma^\beta d\lambda(t) - \int_\gamma^\beta x(t)d\lambda(t) = -\int_\gamma^\beta x'(t)\left(\int_t^\beta d\lambda(x)\right)dt \geqslant 0. \tag{30}$$

From (29) and (30), we get

$$\overline{x}\int_\gamma^\beta d\lambda(t) \geqslant \int_\gamma^\beta x(t)d\lambda(t) \text{ for all } \gamma \in [\gamma_0,\beta]. \tag{31}$$

From (28) and (31), we get

$$\bar{x}\int_{\alpha}^{\beta} d\lambda(t) \geqslant \int_{\alpha}^{\beta} x(t)d\lambda(t).$$

The equality

$$\bar{x}\int_{\alpha}^{\beta} d\lambda(t) = \int_{\alpha}^{\beta} x(t)d\lambda(t),$$

obviously holds for all $\gamma \in [\alpha,\beta]$. Since the conditions (6) and (7) are satisfied, therefore using Corollary 2 for $y(t) = x(t)$ and $x(t) = \bar{x}$, we get (20).

(ii) The proof is similar to the proof of Theorem 10 (ii). □

4. Generalization of converse of Jensen's inequality

THEOREM 13. *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous. Let $\mathbf{x} = (x_1,\ldots,x_r)$ be real r-tuple with $x_i \in [m,M] \subseteq [a,b]$, $i = 1,2,\ldots,r$, $\mathbf{w} = (w_1,\ldots w_r)$ be positive r-tuple, $W_r = \sum_{i=1}^{r} w_i$ and $\bar{x} = \frac{1}{W_r}\sum_{i=1}^{r} w_i x_i$.*

(i) Then for any $2n$-convex function $f : [a,b] \to \mathbb{R}$, the following inequality holds

$$\begin{aligned}\frac{1}{W_r}\sum_{i=1}^{r} w_i f(x_i) &\leqslant \frac{\bar{x}-m}{M-m}f(M) + \frac{M-\bar{x}}{M-m}f(m) - \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{1}{k!(k+2)}\\ &\times\Bigg[f^{(k+1)}(a)\left\{\frac{\bar{x}-m}{M-m}(M-a)^{k+2} + \frac{M-\bar{x}}{M-m}(m-a)^{k+2} - \frac{1}{W_r}\sum_{i=1}^{r} w_i(x_i-a)^{k+2}\right\}\\ &- f^{(k+1)}(b)\left\{\frac{\bar{x}-m}{M-m}(M-b)^{k+2} + \frac{M-\bar{x}}{M-m}(m-b)^{k+2} - \frac{1}{W_r}\sum_{i=1}^{r} w_i(x_i-b)^{k+2}\right\}\Bigg].\end{aligned} \tag{32}$$

(ii) If the inequality (32) holds and the function F defined as in (16) is convex, then

$$\frac{1}{W_r}\sum_{i=1}^{r} w_i f(x_i) \leqslant \frac{\bar{x}-m}{M-m}f(M) + \frac{M-\bar{x}}{M-m}f(m).$$

Proof. (i) Putting $m = 2$, $x_1 = M$, $x_2 = m$, $w_1 = \frac{x_i-m}{M-m}$ and $w_2 = \frac{M-x_i}{M-m}$ in (15), we have

$$\begin{aligned}f(x_i) &\leqslant \frac{x_i-m}{M-m}f(M) + \frac{M-x_i}{M-m}f(m) - \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{1}{k!(k+2)}\\ &\times\Bigg[f^{(k+1)}(a)\left\{\frac{x_i-m}{M-m}(M-a)^{k+2} + \frac{M-x_i}{M-m}(m-a)^{k+2} - (x_i-a)^{k+2}\right\}\\ &- f^{(k+1)}(b)\left\{\frac{x_i-m}{M-m}(M-b)^{k+2} + \frac{M-x_i}{M-m}(m-b)^{k+2} - (x_i-b)^{k+2}\right\}\Bigg].\end{aligned} \tag{33}$$

Multiplying (33) with w_i, dividing by W_r and taking the summation from $i=1$ to r, we get (32).

(ii) Using similar arguments as in the proof of Theorem 9(ii), we get the required result. □

REMARK 2. In Theorem 13, assume that $x_0, \sum_{i=1}^{r} w_i x_i \in [m,M]$ with $x_0 \neq \sum_{i=1}^{r} w_i x_i$ and $(x_i - x_0)\left(\sum_{i=1}^{r} w_i x_i - x_i\right) \geqslant 0, i=1,2,\ldots,r$. If $x_0 < \sum_{i=1}^{r} w_i x_i$, then by taking $m = x_0$ and $M = \sum_{i=1}^{r} w_i x_i$, in inequality (32) we obtain the generalization of Giaccardi inequality. Similarly if $x_0 > \sum_{i=1}^{r} w_i x_i$, then by taking $M = x_0$ and $m = \sum_{i=1}^{r} w_i x_i$, in inequality (32) we obtain the generalization of Giaccardi inequality.

Moreover, if we take $m = x_0 = 0$ in the generalized Giaccardi inequality we obtain generalization of Jensen-Petrović's inequality.

The integral version of the above theorem can be stated as:

THEOREM 14. *Let $n \in \mathbb{N}$, $f:[a,b] \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous, $x:[\alpha,\beta] \to \mathbb{R}$ be continuous function such that $x([\alpha,\beta]) \subseteq [m,M] \subseteq [a,b]$, $\lambda:[\alpha,\beta] \to \mathbb{R}$ increasing, bounded function with $\lambda(\alpha) \neq \lambda(\beta)$ and $\bar{x} = \frac{\int_\alpha^\beta x(t)d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}$.*

(i) Then for any $2n$-convex function $f:[a,b] \to \mathbb{R}$, the following inequality holds

$$\frac{\int_\alpha^\beta f(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} \leqslant \frac{\bar{x}-m}{M-m} f(M) + \frac{M-\bar{x}}{M-m} f(m) - \frac{1}{b-a} \sum_{k=0}^{2n-2} \frac{1}{k!(k+2)}$$
$$\times \left[f^{(k+1)}(a) \left\{ \frac{\bar{x}-m}{M-m}(M-a)^{k+2} + \frac{M-\bar{x}}{M-m}(m-a)^{k+2} - \frac{\int_\alpha^\beta (x(t)-a)^{k+2} d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} \right\} \right.$$
$$\left. - f^{(k+1)}(b) \left\{ \frac{\bar{x}-m}{M-m}(M-b)^{k+2} + \frac{M-\bar{x}}{M-m}(m-b)^{k+2} - \frac{\int_\alpha^\beta (x(t)-b)^{k+2} d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} \right\} \right]. \tag{34}$$

(ii) If the inequality (34) holds and the function F defined as in (16) is convex, then

$$\frac{\int_\alpha^\beta f(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} \leqslant \frac{\bar{x}-m}{M-m} f(M) + \frac{M-\bar{x}}{M-m} f(m).$$

COROLLARY 3. *Let $n \in \mathbb{N}$, $\mathbf{x} = (x_1,\ldots,x_r)$ be real r-tuple with $x_i \in [m,M]$, $\mathbf{w} = (w_1,\ldots w_r)$ be positive r-tuple, $W_r = \sum_{i=1}^{r} w_i$ and $\bar{x} = \frac{1}{W_r}\sum_{i=1}^{r} w_i x_i$. Then for $2n$-convex* *function* $f:[m,M]\to\mathbb{R}$, *the following inequality holds*

$$\frac{1}{W_r}\sum_{i=1}^{r} w_i f(x_i) \leqslant \frac{\bar{x}-m}{M-m}f(M)+\frac{M-\bar{x}}{M-m}f(m)-\frac{1}{M-m}\sum_{k=0}^{2n-2}\frac{1}{k!(k+2)}$$
$$\times\Bigg[f^{(k+1)}(m)\left\{\frac{\bar{x}-m}{M-m}(M-m)^{k+2}-\frac{1}{W_r}\sum_{i=1}^{r}w_i(x_i-m)^{k+2}\right\}$$
$$-f^{(k+1)}(M)\left\{\frac{M-\bar{x}}{M-m}(m-M)^{k+2}-\frac{1}{W_r}\sum_{i=1}^{r}w_i(x_i-M)^{k+2}\right\}\Bigg].$$

Proof. Use the inequality (32) for $a=m$ and $b=M$. □

REMARK 3. Similarly we can give integral version of Corollary 3.

5. Bounds for identities related to the generalization of Jensen's inequality

For two Lebesgue integrable functions $\phi,\psi:[a,b]\to\mathbb{R}$, we consider Čebyšev functional

$$T(\phi,\psi)=\frac{1}{b-a}\int_a^b\phi(t)\psi(t)dt-\frac{1}{b-a}\int_a^b\phi(t)dt\frac{1}{b-a}\int_a^b\psi(t)dt. \tag{35}$$

The following results can be found in [12].

THEOREM 15. *Let* $\phi:[a,b]\to\mathbb{R}$ *be a Lebesgue integrable function and* $\psi:[a,b]\to\mathbb{R}$ *be an absolutely continuous function with* $(.-a)(b-.)[\psi']^2\in L[a,b]$. *Then the inequality*

$$|\,T(\phi,\psi)\,|\leqslant\frac{1}{\sqrt{2}}[T(\phi,\phi)]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left(\int_a^b(x-a)(b-x)[\psi'(x)]^2dx\right)^{\frac{1}{2}} \tag{36}$$

holds. The constant $\frac{1}{\sqrt{2}}$ *in (36) is the best possible.*

THEOREM 16. *Suppose that* $\phi:[a,b]\to\mathbb{R}$ *is absolutely continuous with* $\phi'\in L_\infty[a,b]$ *and* $\psi:[a,b]\to\mathbb{R}$ *is monotonic nondecreasing on* $[a,b]$. *Then the inequality*

$$|\,T(\phi,\psi)\,|\leqslant\frac{1}{2(b-a)}\parallel\phi'\parallel_\infty\int_a^b(x-a)(b-x)d\psi(x) \tag{37}$$

holds. The constant $\frac{1}{2}$ *in (37) is the best possible.*

Let $\mathbf{w}=(w_1,\dots,w_m)$ and $\mathbf{x}=(x_1,\dots,x_m)$ be m-tuples with $x_i\in[a,b]$, $w_i\in\mathbb{R}$ $i=1,\dots,m$, $\bar{x}=\frac{1}{W_m}\sum_{i=1}^{m}w_ix_i\in[a,b]$, $W_m\neq0$ and the function T_n be defined as in (10), we denote

$$\delta(s)=\frac{1}{W_m}\sum_{i=1}^{m}w_iT_n(x_i,s)-T_n(\bar{x},s),\ \ s\in[a,b]. \tag{38}$$

Let $x : [\alpha, \beta] \to [a,b]$ be continuous function and $\lambda : [\alpha, \beta] \to \mathbb{R}$ be as in Theorem 10 or in Theorem 12 and let $\overline{x} = \frac{\int_\alpha^\beta x(t)d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}$, we denote

$$\triangle(s) = \frac{\int_\alpha^\beta (T_n(x(t),s)d\lambda(t)}{\int_\alpha^\beta \lambda(t)dt} - T_n(\overline{x},s),\ s \in [a,b]. \tag{39}$$

From Čebyšev functional we may write

$$T(\delta,\delta) = \frac{1}{b-a}\int_a^b \delta^2(s)ds - \left(\frac{1}{b-a}\int_a^b \delta(s)ds\right)^2,$$

$$T(\triangle,\triangle) = \frac{1}{b-a}\int_a^b \triangle^2(s)ds - \left(\frac{1}{b-a}\int_a^b \triangle(s)ds\right)^2.$$

Now, we are in the position to state the main results of this section:

THEOREM 17. *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous with $(.-a)(b-.)[f^{(n+1)}]^2 \in L[a,b]$. Let $x_i \in [a,b]$, $w_i \in \mathbb{R}$, $i = 1,2,\ldots,m$, $W_m = \sum_{i=1}^m w_i \neq 0$ and $\overline{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i \in [a,b]$. Let the functions T_n, T and δ be as defined in (10), (35) and (38) respectively. Then we have*

$$\begin{aligned}\frac{1}{W_m}\sum_{i=1}^m w_i f(x_i) - f(\overline{x}) = \frac{1}{b-a}\sum_{k=0}^{n-2}\frac{1}{k!(k+2)}\Bigg[f^{(k+1)}(a)\left\{\frac{1}{W_m}\sum_{i=1}^m w_i(x_i-a)^{k+2} - (\overline{x}-a)^{k+2}\right\}\\ - f^{(k+1)}(b)\left\{\frac{1}{W_m}\sum_{i=1}^m w_i(x_i-b)^{k+2} - (\overline{x}-b)^{k+2}\right\}\Bigg]\\ + \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(n-1)!(b-a)}\int_a^b \delta(s)ds + H_n^1(f;a,b),\end{aligned} \tag{40}$$

where the remainder $H_n^1(f;a,b)$ satisfies the estimation

$$\left|H_n^1(f;a,b)\right| \leqslant \frac{1}{(n-1)!}\left(\frac{b-a}{2}\left|T(\delta,\delta)\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds\right|\right)^{\frac{1}{2}}.$$

Proof. Using Theorem 7 for $y_i \to x_i$ and $x_i \to \overline{x}$, we get

$$\begin{aligned}\frac{1}{W_m}\sum_{i=1}^m w_i f(x_i) - f(\overline{x}) = \frac{1}{b-a}\sum_{k=0}^{n-2}\frac{1}{k!(k+2)}\Bigg[f^{(k+1)}(a)\left\{\frac{1}{W_m}\sum_{i=1}^m w_i(x_i-a)^{k+2} - (\overline{x}-a)^{k+2}\right\}\\ - f^{(k+1)}(b)\left\{\frac{1}{W_m}\sum_{i=1}^m w_i(x_i-b)^{k+2} - (\overline{x}-b)^{k+2}\right\}\Bigg]\\ + \frac{1}{(n-1)!}\int_a^b \delta(s)f^{(n)}(s)ds.\end{aligned} \tag{41}$$

Now if we apply Theorem 15 for $\phi \to \delta$ and $\psi \to f^{(n)}$, we obtain

$$\left|\frac{1}{b-a}\int_a^b \delta(s)f^{(n)}(s)ds - \left(\frac{1}{b-a}\int_a^b \delta(s)ds\right)\left(\frac{1}{b-a}\int_a^b f^{(n)}(s)ds\right)\right| \leqslant \frac{1}{\sqrt{2}}[T(\delta,\delta)]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left(\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2dx\right)^{\frac{1}{2}}.$$

Therefore we have

$$\frac{1}{(n-1)!}\int_a^b \delta(s)f^{(n)}(s)ds = \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(n-1)!(b-a)}\int_a^b \delta(s)ds + H_n^1(f;a,b). \tag{42}$$

From (41) and (42), we obtain (40). □

The integral version of the above theorem can be stated as follows:

THEOREM 18. *Let $n\in\mathbb{N}$, $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous with $(.-a)(b-.)[f^{(n+1)}]^2 \in L[a,b]$. Let $x:[\alpha,\beta]\to[a,b]$ be continuous function such that $x([\alpha,\beta])\subseteq[a,b]$, $\lambda:[\alpha,\beta]\to\mathbb{R}$ be as defined in Theorem 10 or Theorem 12 and $\overline{x} = \frac{\int_\alpha^\beta x(t)d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}$. Let the functions T_n, T and $\triangle$ be defined in (10), (35) and (39) respectively. Then we have*

$$\begin{aligned}
&\frac{\int_\alpha^\beta f(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - f(\overline{x}) \\
&\quad = \frac{1}{b-a}\sum_{k=0}^{n-2}\frac{1}{k!(k+2)}\left[f^{(k+1)}(a)\left\{\frac{1}{\int_\alpha^\beta d\lambda(t)}\int_\alpha^\beta (x(t)-a)^{k+2}d\lambda(t) - (\overline{x}-a)^{k+2}\right\}\right. \\
&\qquad \left. - f^{(k+1)}(b)\left\{\frac{\int_\alpha^\beta (x(t)-b)^{k+2}d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - (\overline{x}-b)^{k+2}\right\}\right] \\
&\qquad + \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(n-1)!(b-a)}\int_a^b \triangle(s)ds + H_n^2(f;a,b),
\end{aligned} \tag{43}$$

where the remainder $H_n^2(f;a,b)$ satisfies the estimation

$$\left|H_n^2(f;a,b)\right| \leqslant \frac{1}{(n-1)!}\left(\frac{b-a}{2}\left|T(\delta,\delta)\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2ds\right|\right)^{\frac{1}{2}}.$$

In the next theorem we obtain Grüss type inequality.

THEOREM 19. *Let $n\in\mathbb{N}$, $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous with $f^{(n+1)} \geqslant 0$ on $[a,b]$ and let the functions T and δ be defined in (35) and*

(38) respectively. Then we have the representation (40) and the remainder $H_n^1(f;a,b)$ satisfies

$$\left|H_n^1(f;a,b)\right| \leqslant \frac{1}{(n-1)!} \parallel \delta' \parallel_\infty \left[\frac{b-a}{2}\left[f^{(n-1)}(b)+f^{(n-1)}(a)\right]-\left[f^{(n-2)}(b)-f^{(n-2)}(a)\right]\right]. \tag{44}$$

Proof. The proof is similar to the proof of Theorem 7 in [7]. □

The integral version of the above theorem can be given as:

THEOREM 20. *Let $n \in \mathbb{N}$, $f:[a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous with $f^{(n+1)} \geqslant 0$ on $[a,b]$ and let the functions T and $\triangle$ be defined as in (36) and (39) respectively. Then we have the representation (43) and the remainder $H_n^2(f;a,b)$ satisfies*

$$\left|H_n^2(f;a,b)\right| \leqslant \frac{1}{(n-1)!} \parallel \triangle' \parallel_\infty \left[\frac{b-a}{2}\left[f^{(n-1)}(b)+f^{(n-1)}(a)\right]-\left[f^{(n-2)}(b)-f^{(n-2)}(a)\right]\right].$$

Here, the symbol $L_p[a,b]$ $(1 \leqslant p < \infty)$ denotes the space of p-power integrable functions on the interval $[a,b]$ equipped with the norm

$$\parallel f \parallel_p = \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} \text{ for all } f \in L_p[a,b],$$

and space of essentially bounded functions on $[a,b]$, denoted by $L_\infty[a,b]$, with the norm

$$\parallel f \parallel_\infty = ess \sup_{t \in [a,b]} |f(t)|.$$

We present the Ostrowsky type inequalities related to the generalized of Jensen's inequality.

THEOREM 21. *Let $n \in \mathbb{N}$, $f:[a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and $f^{(n)} \in L_p[a,b]$, $\mathbf{x} = (x_1,\ldots,x_m) \in [a,b]^m$, $\mathbf{w} = (w_1,\ldots,w_m)$ be real m-tuple, $W_m = \sum_{i=1}^m w_i \neq 0$ and $\overline{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i \in [a,b]$. Let (p,q) be a pair of conjugate exponents, that is, $1 \leqslant p,q \leqslant \infty$, $\frac{1}{p}+\frac{1}{q} = 1$. Then we have*

$$\begin{aligned}
&\Bigg|\frac{1}{W_m}\sum_{i=1}^m w_i f(x_i) - f(\overline{x}) - \frac{1}{b-a}\sum_{k=0}^{n-2}\frac{1}{k!(k+2)}\Bigg[f^{(k+1)}(a)\left\{\frac{\sum_{i=1}^m w_i(x_i-a)^{k+2}}{W_m} - (\overline{x}-a)^{k+2}\right\} \\
&\qquad - f^{(k+1)}(b)\left\{\frac{\sum_{i=1}^m w_i(x_i-b)^{k+2}}{W_m} - (\overline{x}-b)^{k+2}\right\}\Bigg]\Bigg| \\
&\qquad \leqslant \frac{1}{(n-1)!} \parallel f^{(n)} \parallel_p \left\|\frac{\sum_{i=1}^m w_i T_n(x_i,.)}{W_m} - T_n(\overline{x},.)\right\|_q.
\end{aligned} \tag{45}$$

The constant on the right of (45) is sharp for $1 < p \leqslant \infty$ and the best possible for $p = 1$.

Proof. The arguments of the proof is similar to the proof of Theorem 9 in [7]. □

The integral version of the above theorem given as follows

THEOREM 22. *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and $f^{(n)} \in L_p[a,b]$. Let $x : [\alpha,\beta] \to \mathbb{R}$ be continuous function such that $x([\alpha,\beta]) \subset [a,b]$, $\lambda : [\alpha,\beta] \to \mathbb{R}$ be as defined in Theorem 10 or Theorem 12 and $\overline{x} = \frac{\int_\alpha^\beta x(t)d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}$. Let (p,q) be a pair of conjugate exponents, that is, $1 \leqslant p,q \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then we have*

$$\begin{aligned} &\left| \frac{\int_\alpha^\beta f(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - f(\overline{x}) - \frac{1}{b-a} \sum_{k=0}^{n-2} \frac{1}{k!(k+2)} \right. \\ &\quad \times \left[f^{(k+1)}(a) \left\{ \frac{\int_\alpha^\beta (x(t)-a)^{k+2} d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - (\overline{x}-a)^{k+2} \right\} \right. \\ &\quad \left. \left. - f^{(k+1)}(b) \left\{ \frac{\int_\alpha^\beta (x(t)-b)^{k+2} d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - (\overline{x}-b)^{k+2} \right\} \right] \right| \\ &\leqslant \| f^{(n)} \|_p \left\| \frac{\int_\alpha^\beta (T_n(x(t),.)d\lambda(t)}{\int_\alpha^\beta \lambda(t)dt} - T_n(\overline{x},.) \right\|_q . \end{aligned} \tag{46}$$

The constant on the right of (46) is sharp for $1 < p \leqslant \infty$ and the best possible for $p = 1$.

6. Mean value theorems and n-exponentail convexity

Motivated by the inequalities (15), (20), (32) and (34) we define the functionals $\Upsilon_1(f)$, $\Upsilon_2(f)$, $\Upsilon_3(f)$ and $\Upsilon_4(f)$ respectively by

$$\begin{aligned} \Upsilon_1(f) = &\frac{1}{W_m} \sum_{i=1}^m w_i f(x_i) - f(\overline{x}) \\ &- \frac{1}{b-a} \sum_{k=0}^{2n-2} \frac{1}{k!(k+2)} \left[f^{(k+1)}(a) \left\{ \frac{1}{W_m} \sum_{i=1}^m w_i (x_i - a)^{k+2} - (\overline{x}-a)^{k+2} \right\} \right. \\ &\left. - f^{(k+1)}(b) \left\{ \frac{1}{W_m} \sum_{i=1}^m w_i (x_i - b)^{k+2} - (\overline{x}-b)^{k+2} \right\} \right]. \end{aligned} \tag{47}$$

$$\Upsilon_2(f) = \frac{\int_\alpha^\beta f(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - f(\overline{x}) - \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{1}{k!(k+2)}\left[f^{(k+1)}(a)\left\{\frac{\int_\alpha^\beta (x(t)-a)^{k+2}d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - (\overline{x}-a)^{k+2}\right\} - f^{(k+1)}(b)\left\{\frac{\int_\alpha^\beta (x(t)-b)^{k+2}d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - (\overline{x}-b)^{k+2}\right\}\right]. \tag{48}$$

$$\Upsilon_3(f) = \frac{1}{W_r}\sum_{i=1}^r w_i f(x_i) - \frac{\overline{x}-m}{M-m}f(M) - \frac{M-\overline{x}}{M-m}f(m) + \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{1}{k!(k+2)} \times \left[f^{(k+1)}(a)\left\{\frac{\overline{x}-m}{M-m}(M-a)^{k+2} + \frac{M-\overline{x}}{M-m}(m-a)^{k+2} - \frac{1}{W_r}\sum_{i=1}^r w_i(x_i-a)^{k+2}\right\} - f^{(k+1)}(b)\left\{\frac{\overline{x}-m}{M-m}(M-b)^{k+2} + \frac{M-\overline{x}}{M-m}(m-b)^{k+2} - \frac{1}{W_r}\sum_{i=1}^r w_i(x_i-b)^{k+2}\right\}\right]. \tag{49}$$

$$\Upsilon_4(f) = \frac{\int_\alpha^\beta f(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - \frac{\overline{x}-m}{M-m}f(M) - \frac{M-\overline{x}}{M-m}f(m) + \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{1}{k!(k+2)} \times \left[f^{(k+1)}(a)\left\{\frac{\overline{x}-m}{M-m}(M-a)^{k+2} + \frac{M-\overline{x}}{M-m}(m-a)^{k+2} - \frac{\int_\alpha^\beta (x(t)-a)^{k+2}d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}\right\} - f^{(k+1)}(b)\left\{\frac{\overline{x}-m}{M-m}(M-b)^{k+2} + \frac{M+\overline{x}}{M-m}(m-b)^{k+2} - \frac{\int_\alpha^\beta (x(t)-b)^{k+2}d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}\right\}\right]. \tag{50}$$

THEOREM 23. *Let $f:[a,b]\to\mathbb{R}$ be such that $f\in C^{(2n)}[a,b]$. If the inequalities (15), (20) and the reverse inequalities in (32) and (34) hold, then there exist $\xi_k\in[a,b]$ for $k\in\{1,2,3,4\}$ such that*

$$\Upsilon_k(f) = f^{(2n)}(\xi_k)\Upsilon_k(f_0), \quad k\in\{1,2,3,4\}, \tag{51}$$

where $f_0(x) = \frac{x^{2n}}{(2n)!}$.

Proof. The proof is similar to the proof of Theorem 11 in [7]. □

THEOREM 24. *Let $f,g:[a,b]\to\mathbb{R}$ be such that $f,g\in C^{(2n)}[a,b]$. If the inequality (15) and (20) and the reverse inequality (32) and (34) hold, then there exist $\xi_k\in[a,b]$ for $k\in\{1,2,3,4\}$ such that*

$$\frac{\Upsilon_k(f)}{\Upsilon_k(g)} = \frac{f^{(2n)}(\xi_k)}{g^{(2n)}(\xi_k)},$$

provided that the denominators are non-zero.

Proof. The proof is similar to the proof of Theorem 12 in [7]. □

REMARK 4. If the inverse of $\frac{f^{(2n)}}{g^{(2n)}}$ exists, then from the above mean value theorem we can give the generalized means,

$$\xi_k = \left(\frac{f^{(2n)}}{g^{(2n)}}\right)^{-1}\left(\frac{\Upsilon_k(f)}{\Upsilon_k(g)}\right), \quad k \in \{1,2,3,4\}.$$

Now, we recall definitions and facts about exponentially convex functions.(see for example [15, 16, 18]):

DEFINITION 3. A function $f : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} p_i p_j f\left(\frac{x_i + x_j}{2}\right) \geqslant 0$$

holds for all choices $p_i \in \mathbb{R}$ and $x_i \in I$, $i = 1,\ldots,n$.

REMARK 5. From Definition 3 it follows that 1-exponentially convex functions in the Jensen sense are exactly nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leqslant n$.

DEFINITION 4. A function $f : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

DEFINITION 5. A function $f : I \to \mathbb{R}$ is exponentially convex if it is n-exponentially convex in the Jensen sense and continuous.

PROPOSITION 2. *If $f : I \to \mathbb{R}$ is an exponentially convex, then the matrix*

$$\left[f\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k}$$

is positive semi-definite. Particularly,

$$\det\left[f\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k} \geqslant 0$$

holds for all $k \in \mathbb{N}$ and $x_i \in I$, $i = 1,\ldots,k$.

DEFINITION 6. A function $f : I \to (0,\infty)$ is said to be log-convex if

$$f((1-\lambda)s + \lambda t) \leqslant f(s)^{1-\lambda} f(t)^{\lambda}$$

holds for all $s,t \in I$, $\lambda \in [0,1]$.

DEFINITION 7. A function $f: I \to (0,\infty)$ is said to be log-convex in the Jensen sense if

$$f\left(\frac{x+y}{2}\right) \leqslant \sqrt{f(x)f(y)}$$

holds for all $x,y \in I$.

REMARK 6. If a function is continuous and log-convex in the Jensen sense then it is also log-convex. We can also easily see that for positive functions exponential convexity implies log-convexity (consider the Definition 3 for $n=2$).

REMARK 7. A function $f: I \to (0,\infty)$ is log-convex in Jensens sense if and only if the inequality

$$v_1^2 f(t_1) + 2v_1 v_2 f\left(\frac{t_1+t_2}{2}\right) + v_2^2 f(t_2) \geqslant 0$$

holds for each $t_1, t_2 \in I$ and $v_1, v_2 \in \mathbb{R}$. It follows that a positive function is log-convex in the J-sense if and only if it is 2-exponentially convex in the J-sense. Also, using basic convexity theory it follows that a positive function is log-convex if and only if it is 2-exponentially convex

THEOREM 25. *Let $H_1 = \{f_t : t \in I\}$, where I an interval in $\mathbb{R}$, be a family of functions defined on $[a,b]$ such that the function $t \to f_t[z_0, z_1, \ldots, z_{2l}]$ is n-exponentially convex in the Jensen sense on I for any $2l+1$ mutually distinct points $z_0, z_1, \ldots, z_{2l} \in [a,b]$. Let $\Upsilon_k(f)$ be the linear functionals for $k \in \{1,2,3,4\}$ as defined in (47), (48), (49) and (50). Then the following statements are valid:*

(i) The function $t \to \Upsilon_k(f_t)$ is n-exponentially convex in the Jensen sense on I.

(ii) If the function $t \to \Upsilon(f_t)$ is continuous on I, then it is n-exponentially convex on I.

Proof. The proof is similar to the proof of Theorem 13 in [7]. □

As a consequence of the above theorem we give the following corollaries.

COROLLARY 4. *Let $H_2 = \{f_t : t \in I\}$, where I an interval in $\mathbb{R}$, be a family of functions defined on the interval $[a,b]$ such that the function $t \to f_t[z_0, z_1, \ldots, z_{2l}]$ is exponentially convex in the Jensen sense on I for any $(2l+1)$ mutually distinct points $z_0, z_1, \ldots, z_{2l} \in [a,b]$. Let $\Upsilon_k(f_t)$ be linear functionals for $k \in \{1,2,3,4\}$ as defined in (47), (48), (49) and (50). Then the following statements are valid:*

(i) The function $t \to \Upsilon_k(f_t)$ is exponentially convex in the Jensen sense on I.

(ii) If the function $t \to \Upsilon_k(f_t)$ is continuous on I, then it is exponentially convex on I.

Proof. The proof follows directly from Theorem 25 by using the definition of exponential convexity. □

COROLLARY 5. *Let $H_3 = \{f_t : t \in I\}$, where I an interval in $\mathbb{R}$, be a family of functions defined on $[a,b]$ such that the function $t \to f_t[z_0, z_1, \ldots, z_{2l}]$ is 2-exponentially convex in the Jensen sense on I for any $2l+1$ mutually distinct points $z_0, z_1, \ldots, z_{2l} \in [a,b]$. Let Υ_k be linear functionals for $k \in \{1,2,3,4\}$ as defined in (47), (48), (49) and (50). Then the following statements are valid:*

(i) If the function $t \to \Upsilon_k(f_t)$ is continuous on I, then it is 2-exponentially convex on I. If $t \to \Upsilon_k(f_t)$ is additionally positive, then it is also log-convex on I. Furthermore, for every choice $r,s,t \in I$, such that $r < s < t$, it holds

$$[\Upsilon_k(f_s)]^{t-r} \leqslant [\Upsilon_k(f_r)]^{t-s} [\Upsilon_k(f_t)]^{s-r}.$$

(ii) If the function $t \to \Upsilon_k(f_t)$ is positive and differentiable on I, then for all $r,s,u,v \in I$ such that $r \leqslant u$, $s \leqslant v$, we have

$$\mu_{r,s}(\Upsilon_k, H_3) \leqslant \mu_{u,v}(\Upsilon_k, H_3), \tag{52}$$

where

$$\mu_{r,s}(\Upsilon_k, H_3) = \begin{cases} \left(\frac{\Upsilon_k(f_r)}{\Upsilon_k(f_s)}\right)^{\frac{1}{r-s}}, & r \neq s, \\ \exp\left(\frac{\frac{d}{dr}(\Upsilon_k(f_r))}{\Upsilon_k(f_r)}\right), & r = s. \end{cases} \tag{53}$$

Proof. The arguments of the proof is similar to the proof of Corollary 6 in [7]. □

REMARK 8. Note that the results from Theorem 25, Corollary 4 and Corollary 5 still hold when any two(all) points $z_0, \ldots, z_{2l} \in [a,b]$ coincide for a family of differentiable ($2l$ times differentiable) functions f_t such that the function $t \to f_t[z_0, \ldots, z_{2l}]$ is an n-exponentially convex, exponentially convex and 2-exponentially convex in the Jensen sense, respectively.

7. Examples

Throughout this section we denote

$$A_k = \frac{1}{W_m}\sum_{i=1}^{m} w_i(x_i - a)^{k+2} - (\bar{x} - a)^{k+2}, B_k = \frac{1}{W_m}\sum_{i=1}^{m} w_i(x_i - b)^{k+2} - (\bar{x} - b)^{k+2},$$

$$C_k = \frac{\bar{x} - m}{M - m}(M - a)^{k+2} + \frac{M - \bar{x}}{M - m}(m - a)^{k+2} - \frac{1}{W_r}\sum_{i=1}^{r} w_i(x_i - a)^{k+2},$$

$$D_k = \frac{\bar{x} - m}{M - m}(M - b)^{k+2} + \frac{M - \bar{x}}{M - m}(m - b)^{k+2} - \frac{1}{W_r}\sum_{i=1}^{r} w_i(x_i - b)^{k+2}.$$

where $x_i, w_i, \bar{x}$ are as defined in Theorem 9.

EXAMPLE 1. Let us consider a family of functions

$$\Omega_1 = \{f_t : \mathbb{R} \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$f_t(x) = \begin{cases} \frac{e^{tx}}{t^{2n}}, & t \neq 0, \\ \frac{x^{2n}}{(2n)!}, & t = 0. \end{cases}$$

Since $\frac{d^{2n} f_t}{dx^{2n}}(x) = e^{tx} > 0$, the function f_t is $2n$-convex on $\mathbb{R}$ for every $t \in \mathbb{R}$ and $t \to \frac{d^{2n} f_t}{dx^{2n}}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 25 we also have that $t \to f_t[z_0, \ldots, z_{2n}]$ is exponentially convex (and so exponentially convex in the Jensen sense). Now, using Corollary 4 we conclude that $t \to \Upsilon_k(f_t)$, $k \in \{1,2,3,4\}$ are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous so they are exponentially convex. For this family of functions, $\mu_{s,q}(\Upsilon_k, \Omega_1)$, from (53), $k = 1$, becomes

$$\mu_{s,q}(\Upsilon_1, \Omega_1) = \left(\frac{\Upsilon_1(f_s)}{\Upsilon_1(f_q)} \right)^{\frac{1}{s-q}}, \quad q \neq s,$$

$$\mu_{s,q}(\Upsilon_1, \Omega_1) = \left(\left(\frac{q}{s}\right)^{2n} \frac{\frac{1}{W_m} \sum_{i=1}^{m} w_i e^{s x_i} - e^{s\bar{x}} - K_1}{\frac{1}{W_m} \sum_{i=1}^{m} w_i e^{q x_i} - e^{q\bar{x}} - K_2} \right)^{\frac{1}{s-q}}, \quad s \neq q, s, q \neq 0$$

$$\mu_{s,s}(\Upsilon_1, \Omega_1) = \exp\left(\frac{\frac{1}{W_m} \sum_{i=1}^{m} w_i x_i e^{s x_i} - \bar{x} e^{s\bar{x}} - K_3}{\frac{1}{W_m} \sum_{i=1}^{m} w_i e^{s x_i} - e^{s\bar{x}} - K_1} - \frac{2n}{s} \right), \quad s \neq 0.$$

$$\mu_{0,0}(\Upsilon_1, \Omega_1) = \exp\left(\frac{1}{2n+1} \frac{\frac{1}{W_m} \sum_{i=1}^{m} w_i x_i^{2n+1} - \bar{x}^{2n+1} - K_4}{\frac{1}{W_m} \sum_{i=1}^{m} w_i x_i^{2n} - \bar{x}^{n} - K_5} \right),$$

where

$$K_1 = \frac{1}{b-a} \sum_{k=0}^{2n-2} \frac{s^{k+1}}{k!(k+2)} \left[e^{as} A_k - e^{bs} B_k \right],$$

$$K_2 = \frac{1}{b-a} \sum_{k=0}^{2n-2} \frac{q^{k+1}}{k!(k+2)} \left[e^{aq} A_k - e^{bq} B_k \right],$$

$$K_3 = \frac{1}{b-a} \sum_{k=0}^{2n-2} \frac{s^{k}}{k!(k+2)} \left[(as+k+1) e^{sa} A_k - (bs+k+1) e^{sb} B_k \right],$$

$$K_4 = \frac{1}{b-a} \sum_{k=0}^{2n-2} \frac{2n(2n-1)\ldots(2n-k)}{k!(k+2)} \left[a^{2n-k} A_k - b^{2n-k} B_k \right],$$

$$K_5 = \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{2n(2n-1)\dots(2n-k)}{k!(k+2)}\left[a^{2n-k-1}A_k - b^{2n-k-1}B_k\right].$$

Similarly we can give $\mu_{s,q}(\Upsilon_k,\Omega_k)$ for $k=2,3,4$.

Now, using (52) $\mu_{s,q}(\Upsilon_k,\Omega_k)$ is monotonic function in parameters s and q. Using Corollary 5 and Theorem 24 it follows that:

$$M_{s,q}(\Upsilon_k,\Omega_1) = \ln\mu_{s,q}(\Upsilon_k,\Omega_1),\ k=1,2,3,4$$

satisfy

$$a \leqslant M_{s,q}(\Upsilon_k,\Omega_1) \leqslant b, \quad k=1,2,3,4.$$

This shows that $M_{s,q}(\Upsilon_k,\Omega_1)$ is a mean for $k=1,2,3,4$.

EXAMPLE 2. Let

$$\Omega_2 = \{g_t : (0,\infty) \to (0,\infty) : t \in (0,\infty)\}$$

be a family of functions defined by

$$g_t(x) = \begin{cases} \frac{t^{-x}}{(-\ln t)^{2n}}, & \mathrm{t} \neq 1; \\ \frac{x^{2n}}{(2n)!}, & \mathrm{t} = 1. \end{cases}$$

Since $\frac{d^{2n}g_t}{dx^{2n}}(x) = t^{-x}$ is the Laplace transform of a non-negative function (see [22]) it is exponentially convex. Obviously g_t is $2n$-convex function for every $t>0$.

For this family of functions, $\mu_{s,q}(\Upsilon_1,\Omega_2)$, from (53), becomes

$$\mu_{s,q}(\Upsilon_1,\Omega_2) = \left(\left(\frac{\ln q}{\ln s}\right)^{2n}\frac{\frac{1}{W_m}\sum_{i=1}^m w_i s^{-x_i} - s^{-\overline{x}} - L_1}{\frac{1}{W_m}\sum_{i=1}^m w_i q^{-x_i} - q^{-\overline{x}} - L_2}\right)^{\frac{1}{s-q}}, \ s \neq q;$$

$$\mu_{s,s}(\Upsilon_1,\Omega_2) = \exp\left(\frac{\overline{x}s^{-\overline{x}-1} - \frac{1}{W_m}\sum_{i=1}^m w_i x_i s^{-x_i-1} - L_3}{\frac{1}{W_m}\sum_{i=1}^m w_i s^{-x_i} - s^{-\overline{x}} - L_1} - \frac{2n}{s\ln s}\right), \ s \neq 1.$$

$$\mu_{1,1}(\Upsilon_1,\Omega_2) = \exp\left(-\frac{1}{2n+1}\frac{\frac{1}{W_m}\sum_{i=1}^m w_i x_i^{2n+1} - \overline{x}^{2n+1} - L_4}{\frac{1}{W_m}\sum_{i=1}^m w_i x_i^{2n} - \overline{x}^{2n} - L_5}\right), \ s=1.$$

where

$$L_1 = \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{(-\ln s)^{k+1}}{k!(k+2)}\left[s^{-a}A_k - s^{-b}B_k\right],$$

$$L_2 = \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{(-\ln q)^{k+1}}{k!(k+2)}\left[q^{-a}A_k - q^{-b}B_k\right],$$

$$L_3 = \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{(-\ln s)^{k}}{k!(k+2)}\left[(a\ln s-k-1)s^{-a-1}A_k - (b\ln s-k-1)s^{-b-1}B_k\right],$$

$$L_4 = \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{2n(2n-1)\dots(2n-k)}{k!(k+2)}\left[a^{2n-k}A_k - b^{2n-k}B_k\right],$$

$$L_5 = \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{2n(2n-1)\dots(2n-k)}{k!(k+2)}\left[a^{2n-k-1}A_k - b^{2n-k-1}B_k\right].$$

Similarly we can give $\mu_{s,q}(\Upsilon_k,\Omega_2)$ for $k=2,3,4$.

Now, using (52) it is monotonic function in parameters s and q. Using Corollary 5 and Theorem 24 it follows that:

$$M_{s,q}(\Upsilon_k,\Omega_2) = \ln \mu_{s,q}(\Upsilon_k,\Omega_2),\ k=1,2,3,4$$

satisfy

$$a \leqslant M_{s,q}(\Upsilon_k,\Omega_2) \leqslant b, \quad k=1,2,3,4.$$

This shows that $M_{s,q}(\Upsilon_k,\Omega_2)$ is a mean for $k=1,2,3,4$. Because of the inequality (52), this mean is also monotonic.

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