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SINGULAR INTEGRALS RELATED TO HOMOGENEOUS MAPPINGS IN TRIEBEL–LIZORKIN SPACES

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Abstract. In this note we establish the boundedness for the singular integral operators related to homogeneous mappings with rough kernels in Triebel-Lizorkin spaces. Some previous results are improved and extended substantially. A main ingredient in the proofs is to establish a criterion of boundedness for the convolution type operator in the above function spaces, which presents a systematic treatment for the related singular integral operators.

1. Introduction

The main purpose of this paper is to establish the bounds of singular integral operators with rough kernels supported by homogeneous mappings in Triebel-Lizorkin spaces. Let us recall some definitions. For $m \geqslant 2$, $\alpha \in \mathbb{R}$ and $0 < p, q \leqslant \infty$ $(p \neq \infty)$, the homogeneous Triebel-Lizorkin spaces $\dot{F}_{\alpha}^{p,q}(\mathbb{R}^m)$ is defined by

$$\dot{F}_{\alpha}^{p,q}(\mathbb{R}^m) := \Big\{ f \in \mathscr{S}'(\mathbb{R}^m) : \|f\|_{\dot{F}_{\alpha}^{p,q}(\mathbb{R}^m)} = \Big\| \Big(\sum_{i\in\mathbb{Z}} 2^{-i\alpha q} |\Psi_i * f|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^m)} < \infty \Big\}, \tag{1}$$

where $\mathscr{S}'(\mathbb{R}^m)$ denotes the tempered distribution class on $\mathbb{R}^m$, $\widehat{\Psi_i}(\xi) = \phi(2^i\xi)$ for $i \in \mathbb{Z}$ and $\phi \in \mathscr{C}_c^{\infty}(\mathbb{R}^m)$ satisfies the conditions: $0 \leqslant \phi(x) \leqslant 1$; $\operatorname{supp}(\phi) \subset \{x \in \mathbb{R}^m : 1/2 \leqslant |x| \leqslant 2\}$; $\phi(x) > c > 0$ if $3/5 \leqslant |x| \leqslant 5/3$. The inhomogeneous versions of Triebel-Lizorkin spaces, which are denoted by $F_{\alpha}^{p,q}(\mathbb{R}^m)$, are obtained by adding the term $\|\Phi * f\|_{L^p(\mathbb{R}^m)}$ to the right hand side of (1) with $\sum_{i\in\mathbb{Z}}$ replaced by $\sum_{i\geqslant 1}$, where $\Phi \in \mathscr{S}(\mathbb{R}^m)$, $\operatorname{supp}(\hat{\Phi}) \subset \{\xi \in \mathbb{R}^m : |\xi| \leqslant 2\}$, $\hat{\Phi}(x) > c > 0$ if $|x| \leqslant 5/3$. It is well known that

$$\dot{F}_0^{p,2}(\mathbb{R}^m) = L^p(\mathbb{R}^m) \ \ \forall 1 < p < \infty; \tag{2}$$

$$F_{\alpha}^{p,q}(\mathbb{R}^m) \sim \dot{F}_{\alpha}^{p,q}(\mathbb{R}^m) \cap L^p(\mathbb{R}^m) \ \text{ and } \ \|f\|_{F_{\alpha}^{p,q}(\mathbb{R}^m)} \sim \|f\|_{\dot{F}_{\alpha}^{p,q}(\mathbb{R}^m)} + \|f\|_{L^p(\mathbb{R}^m)} \ \ \forall \alpha > 0. \tag{3}$$

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See [20, 21, 30] for more properties of $\dot{F}_{\alpha}^{p,q}(\mathbb{R}^m)$.

Let $n \geqslant 2$ and $K(y)$ be a Calderón-Zygmund type kernel of the form

$$K(y) = h(|y|)\frac{\Omega(y)}{|y|^n}, \tag{4}$$

where Ω is homogeneous of degree 0, integrable over S^{n-1} and satisfies

$$\int_{S^{n-1}} \Omega(u)d\sigma(u) = 0, \tag{5}$$

and $h : [0,\infty) \to \mathbb{C}$ is a measurable function. For a suitable mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^m$, we define the singular integral operator $T_{h,\Omega,\Phi}$ associated to Φ by

$$T_{h,\Omega,\Phi}(f)(x) := \text{p.v.} \int_{\mathbb{R}^n} f(x-\Phi(y))K(y)dy, \tag{6}$$

where $x \in \mathbb{R}^m$ and $f \in \mathscr{S}(\mathbb{R}^m)$ (the space of Schwartz functions). If $m = n$ and $\Phi(y) = y$, we denote simply $T_{h,\Omega,\Phi}$ by $T_{h,\Omega}$.

The operator $T_{h,\Omega}$ was initiated by Fefferman [18] and has been studied by many authors (see [1, 14, 16, 17] etc.). For a general mapping Φ, the operator $T_{h,\Omega,\Phi}$ belongs to the class of singular Radon transforms whose L^p mapping properties are relatively well understood when the kernel $K(y)$ is smooth away from the origin. In the case of $\Phi = \mathscr{P}$ being a polynomial mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$, Fan and Pan [16] proved that $T_{h,\Omega,\mathscr{P}}$ is bounded on $L^p(\mathbb{R}^m)$ for p satisfying $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$, provided that $\Omega \in H^1(S^{n-1})$ and $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$, which certainly implies that $T_{h,\Omega}$ has the same L^p-mapping properties. Here $H^1(S^{n-1})$ is the Hardy space on S^{n-1} (see the definition in Section 3), and $\Delta_\gamma(\mathbb{R}^+)\,(\gamma \geqslant 1)$ denotes the set of all measurable functions h defined on $\mathbb{R}^+ := (0,\infty)$ satisfying the condition

$$\|h\|_{\Delta_\gamma(\mathbb{R}^+)} := \sup_{R>0}\left(R^{-1}\int_0^R |h(t)|^\gamma dt\right)^{1/\gamma} < \infty.$$

Clearly, $L^\infty(\mathbb{R}^+) = \Delta_\infty(\mathbb{R}^+) \subsetneq \Delta_{\gamma_2}(\mathbb{R}^+) \subsetneq \Delta_{\gamma_1}(\mathbb{R}^+)$ for $1 \leqslant \gamma_1 < \gamma_2 < \infty$. Also, by imposing a more restrictive condition on h, Al-Qassem [1] showed that $T_{h,\Omega}$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$, provided that $\Omega \in L(\log^+ L)^{1/\gamma'}(S^{n-1})$ and $h \in \mathscr{H}_\gamma(\mathbb{R}^+)$ for some $1 < \gamma \leqslant \infty$ (see also [17] for the generalization in non-isotropic setting). Here $\mathscr{H}_\gamma(\mathbb{R}^+)$, $\gamma > 0$, is the set of all measurable functions h on $\mathbb{R}^+$ satisfying

$$\|h\|_{\mathscr{H}_\gamma(\mathbb{R}^+)} := \left(\int_0^\infty |h(t)|^\gamma \frac{dt}{t}\right)^{1/\gamma} < \infty,$$

and $L(\log^+ L)^\beta(S^{n-1})$ (for $\beta > 0$) denotes the space of all those functions Ω on S^{n-1}, which satisfy

$$\int_{S^{n-1}} |\Omega(\theta)| \log^\beta(2 + |\Omega(\theta)|)d\sigma(\theta) < \infty.$$

Note that

$$L^\infty(\mathbb{R}^+) = \mathscr{H}_\infty(\mathbb{R}^+) \text{ and } \mathscr{H}_\gamma(\mathbb{R}^+) \subsetneq \Delta_\gamma(\mathbb{R}^+), \quad 1 \leqslant \gamma < \infty;$$

$$L(\log^+ L)^{\beta_1}(S^{n-1}) \subsetneq L(\log^+ L)^{\beta_2}(S^{n-1}), \quad 0 < \beta_2 < \beta_1;$$

$$L(\log^+ L)^{\beta}(S^{n-1}) \subsetneq H^1(S^{n-1}), \quad \beta \geqslant 1;$$

$$L(\log^+ L)^{\beta}(S^{n-1}) \not\subseteq H^1(S^{n-1}) \not\subseteq L(\log^+ L)^{\beta}(S^{n-1}), \quad 0 < \beta < 1.$$

On the other hand, the boundedness of $T_{h,\Omega}$ and the general operator $T_{h,\Omega,\mathscr{P}}$ in Triebel-Lizorkin spaces $\dot{F}_\alpha^{p,q}(\mathbb{R}^m)$ have been studied by many authors (see [3, 5, 6, 9, 24, 25, 27] etc.). Recently, Yabuta et al. [13, 28] investigated the boundedness of singular integrals associated to surfaces of revolution on the $\dot{F}_\alpha^{p,q}(\mathbb{R}^m)$-valued L^r function space on $\mathbb{R}$, which is denoted by $L^r(\mathbb{R}, \dot{F}_\alpha^{p,q}(\mathbb{R}^m))$. Other interesting works related to this topic are [23, 29, 31, 33].

The primary focus of our investigation is the singular integral operators $T_{h,\Omega,\Phi}$ with Φ being a homogeneous mapping. Let $d = (d_1, \ldots, d_m) \in \mathbb{R}^m$. We say that $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ is a (non-isotropic) homogeneous mappings of degree d if

$$\Phi(ty) = \delta_t(\Phi(y)), \quad \forall t > 0 \text{ and } y \in \mathbb{R}^n,$$

where $\{\delta_t\}_{t>0}$ is the family of dilations on $\mathbb{R}^m$ by

$$\delta_t(x_1, \ldots, x_m) = (t^{d_1}x_1, t^{d_2}x_2, \ldots, t^{d_m}x_m).$$

The L^p-mapping properties of $T_{h,\Omega,\Phi}$ have been studied by several authors (see [2, 7, 15, 26] etc.). In particular, Cheng [7] established the following result.

THEOREM 1. ([7]) *Let $h(t) = 1$ and $\Phi = (\Phi_1, \ldots, \Phi_m)$ be a homogeneous mapping of degree $d = (d_1, \ldots, d_m)$ with $d_i \neq 0$ for $1 \leqslant i \leqslant m$. Assume that $\Omega \in H^1(S^{n-1})$ satisfying (5) and $\Phi|_{S^{n-1}}$ is real-analytic. Then for $1 < p < \infty$, there exists $C_p > 0$ such that*

$$\|T_{h,\Omega,\Phi}(f)\|_{L^p(\mathbb{R}^m)} \leqslant C_p \|f\|_{L^p(\mathbb{R}^m)}.$$

A question that arises naturally is whether the condition $\Omega \in H^1(S^{n-1})$ is also sufficient for the $\dot{F}_\alpha^{p,q}$-boundedness of $T_{h,\Omega,\Phi}$ with Φ being as in Theorem 1. We will give a positive answer by our next theorem.

THEOREM 2. *Let $\Phi = (\Phi_1, \ldots, \Phi_m)$ be a homogeneous mapping of degree $d = (d_1, \ldots, d_m)$ with $d_i \in \mathbb{N}\backslash\{0\}$ for $1 \leqslant i \leqslant m$ and $\Phi|_{S^{n-1}}$ real-analytic. Assume that Ω satisfies (5) and one of the following conditions holds:*

(a) *$h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$ and $\Omega \in H^1(S^{n-1})$;*

(b) *$h \in \mathscr{H}_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$ and $\Omega \in L(\log^+ L)^{1/\gamma'}(S^{n-1})$.*

Then $T_{h,\Omega,\Phi}$ is bounded on $\dot{F}_\alpha^{p,q}(\mathbb{R}^m)$ for $\alpha \in \mathbb{R}$ and $(1/p, 1/q) \in \mathscr{R}_\gamma$, where $\mathscr{R}_\gamma$ is the interior of the convex hull of three squares $(\frac{1}{2}, \frac{1}{2} + \frac{1}{\max\{2,\gamma'\}})^2$, $(\frac{1}{2} - \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2})^2$ and $(\frac{1}{2\gamma}, 1 - \frac{1}{2\gamma})^2$.

REMARK 1. Theorem 2 essentially generalizes Theorem 1 in the following two-folds: (i) add the roughness of kernels in the radial direction; (ii) extend the boundedness of $T_{h,\Omega,\Phi}$ on Lebesgue spaces to Triebel-Lizorkin spaces. On the other hand, the results of Theorem 2 for h, Ω with satisfying the condition (b) are new even in the special case of that $\alpha = 0$ with $q = 2$, i.e., in Lebesgue spaces. It should be pointed out that Theorem 2 is not true, if replacing $h \in \mathscr{H}_\gamma(\mathbb{R}^+)$ by $h \in \Delta_\gamma(\mathbb{R}^+)$ for $\gamma > 1$, because of that $L^\infty(\mathbb{R}^+) \subset \Delta_\gamma(\mathbb{R}^+)$, $L\log^+ L(S^{n-1}) \subsetneq L(\log^+ L)^\alpha(S^{n-1})$ for any $0 < \alpha < 1$, and Calderón-Zygmund's celebrated result in [8].

See the following Figures 1–3 for $\mathscr{R}_\gamma$. Here $P_1 = (\frac{1}{2} - \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2} - \frac{1}{\max\{2,\gamma'\}})$, $P_2 = (\frac{1}{2}, \frac{1}{2} - \frac{1}{\max\{2,\gamma'\}})$, $P_3 = (\frac{1}{2} + \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2})$, $P_4 = (\frac{1}{2} + \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2} + \frac{1}{\max\{2,\gamma'\}})$, $P_5 = (\frac{1}{2}, \frac{1}{2} + \frac{1}{\max\{2,\gamma'\}})$, $P_6 = (\frac{1}{2} - \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2})$, $R_1 = (1 - \frac{1}{2\gamma}, \frac{1}{2\gamma})$, $R_2 = (\frac{1}{2\gamma}, 1 - \frac{1}{2\gamma})$, $Q_1 = (0,0)$, $Q_2 = (1,0)$, $Q_3 = (1,1)$ and $Q_4 = (0,1)$.

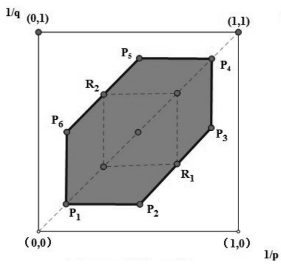

Figure 1: $(1 < \gamma \leqslant 2)$

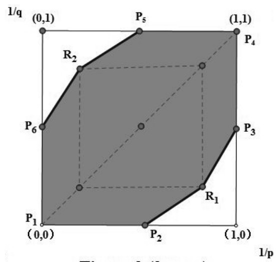

Figure 2: $(2 < \gamma \leqslant \infty)$

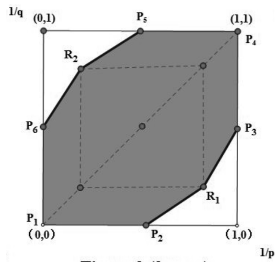

Figure 3: $(\gamma = \infty)$

REMARK 2. We remark that the range of $\mathscr{R}_\gamma$ was first given by Yabuta in [32]. One can easily see that the ranges of p, and q belong to $(1,\infty)$ when $\gamma = \infty$. Thus Theorem 2 generalizes the result of [5] (see Section 5 in [5])

Applying (2)–(3) and Theorem 2, we have the following conclusion immediately.

COROLLARY 1. *Under the same conditions of Theorem* 2 *with* $\alpha > 0$, *the operator* $T_{h,\Omega,\Phi}$ *is bounded on* $F_\alpha^{p,q}(\mathbb{R}^m)$.

The paper is organized as follows. A few lemmas will be recalled or proved in Section 2. The proof of Theorem 2 for the case $\Omega \in H^1(S^{n-1})$ will be given in Section 3. In Section 4, we shall present the proof of Theorem 2 for the case $\Omega \in L(\log^+ L)^{1/\gamma'}(S^{n-1})$. Finally, we end this paper by presenting some more general results in Section 5. We remark that our works and ideas are motivated by [6, 15, 16, 26, 32]. The main ingredient is to present a criterion of boundedness for the operator of convolution type on the Triebel-Lizorkin spaces (see Lemma 5) and a switched technique on the linear transformations in estimating the Fourier transforms of some measures (see Section 3).

We end this section by giving some notations: we denote p' by the conjugate index of p, which satisfies $1/p + 1/p' = 1$; $\delta_{\mathbb{R}^n}$ denotes the Dirac delta function on $\mathbb{R}^n$; J^{-1} denotes the inverse transform of linear transformation J; D^t denotes the transpose of the linear transformation D and π_n^m denotes the projection operator from $\mathbb{R}^m$ to $\mathbb{R}^n$; $\hat{f}$ denotes the Fourier transform of f. Finally, we set $\sum_{j\in\emptyset} a_j = 0$ and $\prod_{j\in\emptyset} a_j = 1$.

2. Preliminary lemmas

In this section, we shall present some necessary lemmas, which will play key roles in the proof of Theorem 2.

LEMMA 1. ([7]) *Let $l \in \mathbb{N}\backslash\{0\}$, $\mu_1,\dots,\mu_l \in \mathbb{R}$ and $d_1,\dots,d_l$ be distinct nonzero real numbers. Let $\psi \in \mathscr{C}^1([0,1])$. Then there exists $C>0$, independent of $\{\mu_j\}_{j=1}^l$, such that*

$$\Big|\int_\delta^\tau \exp(i(\mu_1 t^{d_1}+\ldots+\mu_l t^{d_l}))\psi(t)dt\Big| \leqslant C|\mu_1|^{-1/l}\Big(|\psi(\tau)|+\int_\delta^\tau |\psi'(t)|dt\Big)$$

holds for $1/2 \leqslant \delta < \tau \leqslant 1$.

LEMMA 2. ([26]) *Let $l \in \mathbb{N}\backslash\{0\}$ and $h_1,\dots,h_l$ be distinct nonzero real numbers and*

$$Q(t,u) = t^{h_1}\sum_{|\alpha|\leqslant s} a_\alpha u^\alpha + \sum_{j=2}^{l} t^{h_j} w_j(u),$$

where $t\in\mathbb{R}$, $u=(u_1,\dots,u_{n-1})\in\mathbb{R}^{n-1}$, $\alpha\in\mathbb{N}^{n-1}$, $a_\alpha\in\mathbb{R}$, and $w_j(\cdot)$ are real-valued. Let $r>0$ and $b(\cdot)$ be a measurable function on $[-r,r]^{n-1}$ that satisfies $\|b\|_\infty \leqslant r^{-(n-1)}$. Then there exist positive constants C and γ independent of $\{a_\alpha\}$, $\{w_j(\cdot)\}$, r such that

$$\int_{1/2}^1\Big|\int_{[-r,r]^{n-1}} \exp(iQ(t,u))b(u)du\Big|dt \leqslant C\Big(r^s\sum_{|\alpha|=s}|a_\alpha|\Big)^{-\gamma}.$$

Below are two important vector-valued norm inequalities.

LEMMA 3. ([6]) *Let $\mathscr{P}=(P_1,\dots,P_m)$ with P_j being real-valued polynomials in $\mathbb{R}^n$. For $1<p,q<\infty$, the operator $\mathscr{M}_\mathscr{P}$ given by*

$$\mathscr{M}_\mathscr{P}(f)(x) = \sup_{r>0}\frac{1}{r^n}\int_{|y|\leqslant r}|f(x-\mathscr{P}(y))|dy$$

satisfies the following $L^p(\mathbb{R}^m,\ell^q)$ inequality

$$\Big\|\Big(\sum_{i\in\mathbb{Z}}(\mathscr{M}_\mathscr{P}(f_i))^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)} \leqslant C_{p,q}\Big\|\Big(\sum_{i\in\mathbb{Z}}|f_i|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)},$$

where $C_{p,q}>0$ is independent of the coefficients of P_j for $1\leqslant j\leqslant m$.

LEMMA 4. ([24]) *Let* $0 < M \leqslant N$ *and* $H : \mathbb{R}^M \to \mathbb{R}^M$, $G : \mathbb{R}^N \to \mathbb{R}^N$ *be two nonsingular linear transformations. Let* $\{a_k\}_{k\in\mathbb{Z}}$ *be a lacunary sequence of positive numbers satisfying* $\inf_{k\in\mathbb{Z}} a_{k+1}/a_k \geqslant a > 1$. *Let* $\tilde{\Phi}(\xi) \in \mathscr{S}(\mathbb{R}^M)$ *with* $\widehat{\tilde{\Phi}}(0) = 0$ *and* $\tilde{\Phi}_k(\xi) = a_k^{-M}\tilde{\Phi}(\xi/a_k)$. *Define the transformations* J *and* X_k *by*

$$J(f)(x) = f(G^t(H^t \otimes id_{\mathbb{R}^{N-M}})x) \text{ and } X_k(f)(x) = J^{-1}((\tilde{\Phi}_k \otimes \delta_{\mathbb{R}^{N-M}}) * J(f))(x).$$

Then for any $1 < p, q < \infty$, $\{g_j\}_{j\in\mathbb{Z}} \in L^p(\mathbb{R}^N, \ell^q)$ *and* $\{g_{k,j}\}_{j,k\in\mathbb{Z}} \in L^p(\mathbb{R}^N, \ell^q(\ell^2))$, *there exists a positive constant* $C_{M,a}$ *such that*

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|X_k(g_j)|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^N)} \leqslant C_{M,a}\Big\|\Big(\sum_{j\in\mathbb{Z}}|g_j|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^N)};$$

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|X_k(g_{k,j})|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^N)} \leqslant C_{M,a}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^N)}.$$

To prove Theorem 2, we will establish a criterion on the bounds of the convolution operators in Triebel-Lizorkin spaces.

LEMMA 5. *Let* $\Lambda, v \in \mathbb{N}\backslash\{0\}$ *and* $\{\sigma_{s,k} : 0 \leqslant s \leqslant \Lambda \text{ and } k \in \mathbb{Z}\}$ *be a family of measures on* $\mathbb{R}^m$ *with* $\sigma_{0,k} = 0$ *for every* $k \in \mathbb{Z}$. *For* $1 \leqslant s \leqslant \Lambda$, *let* $\eta_s > 1$, $\delta_s, \beta_s, \gamma_s > 0$, $\ell_s \in \mathbb{N}\backslash\{0\}$ *and* $L_s : \mathbb{R}^m \to \mathbb{R}^{\ell_s}$ *be linear transformations. Suppose that there exist some* $1 < p_0, q_0 < \infty$ *with satisfying* $(p_0, q_0) \neq (2,2)$ *and* $c, A > 0$ *independent of* v *and* $\{L_s\}_{s=1}^{\Lambda}$ *such that the following conditions are satisfied for any* $1 \leqslant s \leqslant \Lambda$, $k \in \mathbb{Z}$, $\xi \in \mathbb{R}^m$ *and* $\{g_{k,j}\}_{k,j\in\mathbb{Z}} \in L^{p_0}(\mathbb{R}^m, \ell^{q_0}(\ell^2))$:

(i) $|\widehat{\sigma_{s,k}}(\xi)| \leqslant cA\min\{1, |\eta_s^{kv\gamma_s}L_s(\xi)|^{-\delta_s/v}\}$;

(ii) $|\widehat{\sigma_{s,k}}(\xi) - \widehat{\sigma_{s-1,k}}(\xi)| \leqslant cA|\eta_s^{kv\gamma_s}L_s(\xi)|^{\beta_s/v}$;

(iii)

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|\sigma_{s,k} * g_{k,j}|^2\Big)^{q_0/2}\Big)^{1/q_0}\Big\|_{L^{p_0}(\mathbb{R}^m)} \leqslant CA\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,j}|^2\Big)^{q_0/2}\Big)^{1/q_0}\Big\|_{L^{p_0}(\mathbb{R}^m)}.$$

Then for $\alpha \in \mathbb{R}$ *and* $(1/p, 1/q) \in A_1A_2\backslash\{(1/p_0, 1/q_0), (1/2, 1/2)\}$, *there exists* $C > 0$ *independent of* v *and* $\{L_s\}_{s=1}^{l}$ *such that*

$$\Big\|\sum_{k\in\mathbb{Z}}\sigma_{\Lambda,k} * f\Big\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^m)} \leqslant CA\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^m)}, \tag{7}$$

where $A_1 = (1/2, 1/2)$, $A_2 = (1/p_0, 1/q_0)$ *and* A_1A_2 *is the line segment from* A_1 *to* A_2.

Proof. For any $1 \leqslant s \leqslant \Lambda$, we set $r(s) = \text{rank}(L_s)$. By [16, Lemma 6.1], there are two nonsingular linear transformations $H_s : \mathbb{R}^{r(s)} \to \mathbb{R}^{r(s)}$ and $G_s : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$|H_s\pi_{r(s)}^m G_s\xi| \leqslant |L_s(\xi)| \leqslant \ell_s|H_s\pi_{r(s)}^m G_s\xi|. \tag{8}$$

Let $\tilde{\phi} \in \mathscr{C}_0^\infty(\mathbb{R})$ such that $\tilde{\phi}(t) \equiv 1$ for $|t| \leqslant 1/2$ and $\tilde{\phi}(t) \equiv 0$ for $|t| \geqslant 1$. Let $\tilde{\psi}(t) = \tilde{\phi}(t^2)$. For any $1 \leqslant s \leqslant l$, define the family of measures $\{\mu_{s,k}\}_{k\in\mathbb{Z}}$ by

$$\widehat{\mu_{s,k}}(\xi) = \widehat{\sigma_{s,k}}(\xi) \prod_{j=s+1}^{\Lambda} \tilde{\psi}(|\eta_j^{kv\gamma_j} H_j \pi_{r(j)}^m G_j \xi|) - \widehat{\sigma_{s-1,k}}(\xi) \prod_{j=s}^{\Lambda} \tilde{\psi}(|\eta_j^{kv\gamma_j} H_j \pi_{r(j)}^m G_j \xi|). \tag{9}$$

One can easily check that

$$\sigma_{\Lambda,k} = \sum_{s=1}^{\Lambda} \mu_{s,k}, \tag{10}$$

$$|\widehat{\mu_{s,k}}(\xi)| \leqslant CA \min\{1, (|\eta_s^{kv\gamma_s} L_s(\xi)|^{\beta_s/v} + |\eta_s^{kv\gamma_s} L_s(\xi)|^{1/v})\}, \tag{11}$$

$$|\widehat{\mu_{s,k}}(\xi)| \leqslant CA |\eta_s^{kv\gamma_s} L_s(\xi)|^{-\delta_s/v}, \quad \text{if } |\eta_s^{kv\gamma_s} H_s \pi_{r(s)}^m G_s \xi| \geqslant 1. \tag{12}$$

From (10) we can write

$$\sum_{k\in\mathbb{Z}} \sigma_{\Lambda,k} * f = \sum_{k\in\mathbb{Z}} \sum_{s=1}^{\Lambda} \mu_{s,k} * f = \sum_{s=1}^{\Lambda} \sum_{k\in\mathbb{Z}} \mu_{s,k} * f := \sum_{s=1}^{\Lambda} \mathscr{A}_s(f). \tag{13}$$

Thus, to prove (7), it suffices to prove that for any $1 \leqslant s \leqslant \Lambda$, there exists $C > 0$ independent of $\{L_s\}_{s=1}^{\Lambda}$ such that

$$\|\mathscr{A}_s(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^m)} \leqslant CA \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^m)} \tag{14}$$

for $\alpha \in \mathbb{R}$ and p, q satisfying the condition in Lemma 5.

Let $\zeta \in \mathscr{S}(\mathbb{R}^+)$ such that

$$\zeta(0) = 0; \;\; 0 \leqslant \zeta(t) \leqslant 1; \; \operatorname{supp}(\zeta) \subset [\eta_s^{-v\gamma_s}, \eta_s^{v\gamma_s}]; \;\; \sum_{k\in\mathbb{Z}} \zeta_k^2(t) = 1,$$

where $\zeta_k(t) = \zeta(\eta_s^{kv\gamma_s} t)$. For any $1 \leqslant s \leqslant \Lambda$, we define the family of operators $\{S_{k,s}\}_{k\in\mathbb{Z}}$ by

$$\widehat{S_{k,s} f}(\xi) := \zeta_k(|H_s \pi_{r(s)}^m G_s \xi|) \hat{f}(\xi). \tag{15}$$

We can write

$$\mathscr{A}_s(f) = \sum_{k\in\mathbb{Z}} \mu_{s,k} * \Big(\sum_{j\in\mathbb{Z}} S_{j+k,s} S_{j+k,s} f \Big) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} S_{j+k,s}(\mu_{s,k} * S_{j+k,s} f) := \sum_{j\in\mathbb{Z}} \mathscr{A}_{s,j}(f). \tag{16}$$

By (11)–(12), Littlewood-Paley theory and Plancherel's theorem,

$$\begin{aligned} &\|\mathscr{A}_{s,j}(f)\|_{L^2(\mathbb{R}^m)} \\ &\leqslant CA \Big(\sum_{k\in\mathbb{Z}} \int_{\{\xi\in\mathbb{R}^m:\ \eta_s^{-(j+k+1)v\gamma_s} \leqslant |H_s \pi_{r(s)}^m G_s \xi| \leqslant \eta_s^{-(j+k-1)v\gamma_s}\}} |\widehat{\mu_{s,k}}(\xi)|^2 |\hat{f}(\xi)|^2 d\xi \Big)^{1/2} \\ &\leqslant CA \eta_s^{-c|j|} \|f\|_{L^2(\mathbb{R}^m)}, \end{aligned} \tag{17}$$

where $c > 0$ is independent of v. Combining (17) with (2) yields

$$\|\mathscr{A}_{s,j}(f)\|_{\dot{F}_0^{2,2}(\mathbb{R}^m)} \leqslant CA \eta_s^{-c|j|} \|f\|_{\dot{F}_0^{2,2}(\mathbb{R}^m)}. \tag{18}$$

Below we estimate $\|\mathscr{A}_{s,j}(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^m)}$. Let $\xi=(\xi^1,\xi^2)$ with $\xi^1=(\xi_1,\dots,\xi_{r(s)})$ and $\xi^2=(\xi_{r(s)+1},\dots,\xi_m)$. We set $\widehat{F_k}(\xi^1)=\hat{F}(\eta_s^{kv\gamma_s}\xi^1)=\zeta_k(|\pi_{r(s)}^m\xi|)$, where ζ_k is as in (15). It is clear that $F\in\mathscr{S}(\mathbb{R}^{r(s)})$ and $\hat{F}(0)=0$. Define the nonsingular linear transformation J on $\mathbb{R}^m$ by $J=G_s^{-1}(H_s^{-1}\otimes\delta_{\mathbb{R}^{m-r(s)}})$. It is easy to verify that

$$S_{k,s}(f)(x)=|J|F_k\otimes\delta_{\mathbb{R}^{m-r(s)}}*f^J(J^tx), \tag{19}$$

where $f^J(x)=|J|^{-1}f((J^t)^{-1}x)$. By change of variables, (19) and Lemma 4 we have that for any $1<p,q<\infty$ and $\{g_i\}_{i\in\mathbb{Z}}\in L^p(\mathbb{R}^m,\ell^q)$, there exists a constant $C>0$ such that

$$\Big\|\Big(\sum_{i\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|S_{k,s}(g_i)|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)}\leqslant C\Big\|\Big(\sum_{i\in\mathbb{Z}}|g_i|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)}. \tag{20}$$

By our assumption (iii), Lemma 4 and the arguments similar to those used in deriving [6, Proposition 2.3], we get

$$\Big\|\Big(\sum_{i\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|\mu_{s,k}*g_{i,k}|^2\Big)^{q_0/2}\Big)^{1/q_0}\Big\|_{L^{p_0}(\mathbb{R}^m)}\leqslant CA\Big\|\Big(\sum_{i\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{i,k}|^2\Big)^{q_0/2}\Big)^{1/q_0}\Big\|_{L^{p_0}(\mathbb{R}^m)}. \tag{21}$$

From the duality and (20)–(21) it follows that there exists $C>0$ such that

$$\begin{aligned}
&\Big\|\Big(\sum_{i\in\mathbb{Z}}|\mathscr{A}_{s,j}(g_i)|^{q_0}\Big)^{1/q_0}\Big\|_{L^{p_0}(\mathbb{R}^m)}\\
&=\sup_{\|\{f_i\}\|_{L^{p_0'}(\mathbb{R}^m,\ell^{q_0'})}\leqslant 1}\Big|\int_{\mathbb{R}^m}\sum_{i\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}S_{j+k,s}(\mu_{s,k}*S_{j+k,s}(g_i))(x)f_i(x)dx\Big|\\
&\leqslant C\sup_{\|\{f_i\}\|_{L^{p_0'}(\mathbb{R}^m,\ell^{q_0'})}\leqslant 1}\Big\|\Big(\sum_{i\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|S_{j+k,s}^*(f_i)|^2\Big)^{q_0'/2}\Big)^{1/q_0'}\Big\|_{L^{p_0'}(\mathbb{R}^m)}\\
&\quad\times\Big\|\Big(\sum_{i\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|\mu_{s,k}*S_{j+k,s}(g_i)|^2\Big)^{q_0/2}\Big)^{1/q_0}\Big\|_{L^{p_0}(\mathbb{R}^m)}\\
&\leqslant CA\Big\|\Big(\sum_{i\in\mathbb{Z}}|g_i|^{q_0}\Big)^{1/q_0}\Big\|_{L^{p_0}(\mathbb{R}^m)},
\end{aligned} \tag{22}$$

which leads to

$$\begin{aligned}
\|\mathscr{A}_{s,j}(f)\|_{\dot{F}_\alpha^{p_0,q_0}(\mathbb{R}^m)}&=\Big\|\Big(\sum_{i\in\mathbb{Z}}2^{-i\alpha q_0}|\Psi_i*\mathscr{A}_{s,j}(f)|^{q_0}\Big)^{1/q_0}\Big\|_{L^{p_0}(\mathbb{R}^m)}\\
&\leqslant\Big\|\Big(\sum_{i\in\mathbb{Z}}|\mathscr{A}_{s,j}(2^{-i\alpha}\Psi_i*f)|^{q_0}\Big)^{1/q_0}\Big\|_{L^{p_0}(\mathbb{R}^m)}\\
&=CA\|f\|_{\dot{F}_\alpha^{p_0,q_0}(\mathbb{R}^m)}
\end{aligned} \tag{23}$$

for any $\alpha\in\mathbb{R}$, where Ψ_i is given as in (1). Interpolation (see [19, 21]) between (18) and (23) implies that for $\alpha\in\mathbb{R}$, $(1/p,1/q)\in A_1A_2\backslash\{(1/p_0,1/q_0),(1/2,1/2)\}$ and $1\leqslant s\leqslant\Lambda$, there exists $\varepsilon>0$ such that

$$\|\mathscr{A}_{s,j}(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^m)}\leqslant CAB_j^{\varepsilon}\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^m)}. \tag{24}$$

Combining (24) with (16) yields (14) and completes the proof of Lemma 5. □

In what follows, we set

$$|||h|||_{\mu,\gamma} = \sup_{k\in\mathbb{Z}} \Big(\int_{2^{(\mu+1)(k-1)}}^{2^{(\mu+1)k}} |h(t)|^\gamma \frac{dt}{t} \Big)^{1/\gamma}, \ \ \gamma > 1.$$

For a suitable mapping $\Gamma : \mathbb{R}^n \to \mathbb{R}^m$ and $\mu \in \mathbb{N}$, define the sequence of measures $\{\sigma_{k,\mu,\Gamma,\Omega}\}_{k\in\mathbb{Z}}$ by

$$\int_{\mathbb{R}^m} f d\sigma_{k,\mu,\Gamma,\Omega} = \int_{D_{\mu,k}} f(\Gamma(x))K(x)dx,$$

where $K(\cdot)$ is as in (4) and $D_{\mu,k} = \{x \in \mathbb{R}^n : 2^{(\mu+1)(k-1)} \leqslant |x| < 2^{(\mu+1)k}\}$.

LEMMA 6. *Let $\Gamma(y) = (P_1(|y|)a_1(\frac{y}{|y|}),\dots,P_m(|y|)a_m(\frac{y}{|y|}))$, where $P_1,\dots,P_m$ are real-valued polynomials on $\mathbb{R}^+$ and $a_1,\dots,a_m$ are arbitrary functions defined on S^{n-1}. Suppose that $\Omega \in L^1(S^{n-1})$ satisfying* (5) *and $|||h|||_{\mu,\gamma} < \infty$ for some $\mu \in \mathbb{N}$ and $\gamma > 1$. If $(1/p,1/q) \in \mathscr{R}_\gamma$ with $\mathscr{R}_\gamma$ being as in Theorem 2. Then for $\{g_{k,j}\}_{k,j\in\mathbb{Z}} \in L^p(\mathbb{R}^m,\ell^q(\ell^2))$, there exists $C > 0$, independent of μ and γ, such that*

$$\begin{aligned} &\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|\sigma_{k,\mu,\Gamma,\Omega} * g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)} \\ &\leqslant C(\mu+1)^{1/\gamma'}\|\Omega\|_{L^1(S^{n-1})}|||h|||_{\mu,\gamma}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)}. \end{aligned} \tag{25}$$

Proof. We consider the following two cases:

Case 1 $(1 < \gamma \leqslant 2)$. Firstly we shall prove (25) for $2 < p,q < 2\gamma/(2-\gamma)$. Given functions $\{f_j\}_{j\in\mathbb{Z}}$ with $\|\{f_j\}\|_{L^{(p/2)'}(\mathbb{R}^m,\ell^{(q/2)'})} \leqslant 1$. By the similar arguments as in getting (7.7) in [16], we have

$$\int_{\mathbb{R}^m} |\sigma_{k,\mu,\Gamma,\Omega} * g_{k,j}(x)|^2 |f_j(x)|dx \leqslant C\|\Omega\|_{L^1(S^{n-1})}|||h|||_{\mu,\gamma}^\gamma \int_{\mathbb{R}^m} |g_{k,j}(x)|^2 \mathscr{M}_\Gamma(f_j)(x)dx, \tag{26}$$

where

$$\mathscr{M}_\Gamma(f)(x) = \sup_{k\in\mathbb{Z}} \int_{2^{(\mu+1)(k-1)}}^{2^{(\mu+1)k}} \int_{S^{n-1}} |f(x+\Gamma(ty'))||\Omega(y')|d\sigma(y')|h(t)|^{2-\gamma}\frac{dt}{t}.$$

Using Hölder's inequality we obtain

$$\begin{aligned} &\mathscr{M}_\Gamma(f)(x) \\ &\leqslant |||h|||_{\mu,\gamma}^{2-\gamma} \int_{S^{n-1}} \Big(\sup_{k\in\mathbb{Z}} \int_{2^{(\mu+1)(k-1)}}^{2^{(\mu+1)k}} |f(x+\Gamma(ty'))|^{\gamma'/2}\frac{dt}{t}\Big)^{2/\gamma'} |\Omega(y')|d\sigma(y') \\ &\leqslant |||h|||_{\mu,\gamma}^{2-\gamma} \int_{S^{n-1}} \Big(\sum_{i=0}^{\mu}\sup_{k\in\mathbb{Z}} \int_{2^{(\mu+1)(k-1)+i}}^{2^{(\mu+1)(k-1)+i+1}} |f(x+\Gamma(ty'))|^{\gamma'/2}\frac{dt}{t}\Big)^{2/\gamma'} |\Omega(y')|d\sigma(y') \\ &\leqslant (\mu+1)^{2/\gamma'}|||h|||_{\mu,\gamma}^{2-\gamma} \int_{S^{n-1}} |\Omega(y')|\Big(\sup_{r>0}\frac{1}{r}\int_{|t|\leqslant r} |f(x+\Gamma(ty'))|^{\gamma'/2}dt\Big)^{2/\gamma'} d\sigma(y'). \end{aligned}$$

Invoking Lemma 4 and Minkowski's inequality we get

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|\mathscr{M}_\Gamma(f_j)|^v\Big)^{1/v}\Big\|_{L^u(\mathbb{R}^m)} \leqslant (\mu+1)^{2/\gamma'}\||h|\|_{\mu,\gamma}^{2-\gamma}\|\Omega\|_{L^1(S^{n-1})}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^v\Big)^{1/v}\Big\|_{L^u(\mathbb{R}^m)} \tag{27}$$

for $\gamma'/2<u,v<\infty$. Then (27) together with (26) yields

$$\begin{aligned}
&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|\sigma_{k,\mu,\Gamma,\Omega}*g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)}^2\\
&=\sup_{\|\{f_j\}\|_{L^{(p/2)'}(\mathbb{R}^m,\ell^{(q/2)'})}\leqslant 1}\int_{\mathbb{R}^m}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|\sigma_{k,\mu,\Gamma,\Omega}*g_{k,j}(x)|^2 f_j(x)dx\\
&\leqslant C\|\Omega\|_{L^1(S^{n-1})}\||h|\|_{\mu,\gamma}^{\gamma}\sup_{\|\{f_j\}\|_{L^{(p/2)'}(\mathbb{R}^m,\ell^{(q/2)'})}\leqslant 1}\int_{\mathbb{R}^m}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|g_{k,j}(x)|^2\mathscr{M}_\Gamma(f_j)(x)dx\\
&\leqslant C\|\Omega\|_{L^1(S^{n-1})}\||h|\|_{\mu,\gamma}^{\gamma}\sup_{\|\{f_j\}\|_{L^{(p/2)'}(\mathbb{R}^m,\ell^{(q/2)'})}\leqslant 1}\Big\|\Big(\sum_{j\in\mathbb{Z}}|\mathscr{M}_\Gamma(f_j)|^v\Big)^{1/v}\Big\|_{L^u(\mathbb{R}^m)}\\
&\quad\times\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)}^2\\
&\leqslant C(\mu+1)^{2/\gamma'}\|\Omega\|_{L^1(S^{n-1})}^2\||h|\|_{\mu,\gamma}^2\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)}^2,
\end{aligned}$$

where we take $u=(p/2)'$ and $v=(q/2)'$. From this we prove (25) for $1<\gamma\leqslant 2$ and $(1/p,1/q)$ belonging to the interior of the square $(\frac12-\frac1{\gamma'},\frac12)^2$. By duality we can obtain (25) for $1<\gamma\leqslant 2$ and $(1/p,1/q)$ belonging to the interior of the square $(\frac12,\frac12+\frac1{\gamma'})^2$. Interpolating these two cases, we get (25) for the case $1<\gamma\leqslant 2$ and $(1/p,1/q)$ belonging to the interior of the convex hull of two squares $(\frac12-\frac1{\gamma'},\frac12)^2$ and $(\frac12,\frac12+\frac1{\gamma'})^2$. Note that in this case the interior of the square $(\frac1{2\gamma},1-\frac1{2\gamma})^2$ contains in the interior of the convex hull of two squares $(\frac12-\frac1{\max\{2,\gamma'\}},\frac12)^2$ and $(\frac12,\frac12+\frac1{\max\{2,\gamma'\}})^2$.

Case 2 $(\gamma>2)$. Since $\||h|\|_{\mu,2}\leqslant(\mu+1)^{1/2-1/\gamma}\||h|\|_{\mu,\gamma}$ for $\gamma>2$. We get (25) for $(1/p,1/q)$ belonging to the interior of the convex hull of two squares $(0,\frac12)^2$ and $(\frac12,1)^2$. Below we shall prove (25) for $(1/p,1/q)$ belonging to the interior of the square $(\frac1{2\gamma},1-\frac1{2\gamma})^2$. For convenience, we define the measure $|\sigma_{k,\mu,\Gamma,\Omega}|$ in the same way as $\sigma_{k,\mu,\Gamma,\Omega}$, but with Ω replaced by $|\Omega|$ and h replaced by $|h|$. For any arbitrary functions $\{g_j\}\in L^p(\mathbb{R}^m,\ell^q)$ with $p,q>\gamma'$. By a change of variable and Hölder's inequality,

$$\begin{aligned}
&|\sigma_{k,\mu,\Gamma,\Omega}|*|g_j|(x)\\
&\leqslant\int_{2^{(\mu+1)(k-1)}}^{2^{(\mu+1)k}}\int_{S^{n-1}}|g_j(x-\Gamma(ty'))||\Omega(y')|d\sigma(y')|h(t)|\frac{dt}{t}\\
&\leqslant\||h|\|_{\mu,\gamma}\|\Omega\|_{L^1(S^{n-1})}^{1/\gamma}\Big(\int_{S^{n-1}}\int_{2^{(\mu+1)(k-1)}}^{2^{(\mu+1)k}}|g_j(x-\Gamma(ty'))|^{\gamma'}\frac{dt}{t}|\Omega(y')|d\sigma(y')\Big)^{1/\gamma'}
\end{aligned}$$

$$\leqslant (\mu+1)^{1/\gamma'}\||h|\|_{\mu,\gamma}\|\Omega\|_{L^1(S^{n-1})}^{1/\gamma}$$
$$\times\Big(\int_{S^{n-1}}\sup_{r>0}\frac{1}{r}\int_{|t|\leqslant r}|g_j(x-\Gamma(ty'))|^{\gamma'}dt|\Omega(y')|d\sigma(y')\Big)^{1/\gamma'},$$

which combining Minkowski's inequality with Lemma 4 implies

$$\begin{aligned}&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sup_{k\in\mathbb{Z}}|\sigma_{k,\mu,\Gamma,\Omega}|*|g_j|\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)}\\ &\leqslant (\mu+1)^{1/\gamma'}\||h|\|_{\mu,\gamma}\|\Omega\|_{L^1(S^{n-1})}^{1/\gamma}\\ &\quad\times\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\int_{S^{n-1}}\sup_{r>0}\frac{1}{r}\int_{|t|\leqslant r}|g_j(\cdot-\Gamma(ty'))|^{\gamma'}dt|\Omega(y')|d\sigma(y')\Big)^{q/\gamma'}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)}\\ &\leqslant C_{p,q}(\mu+1)^{1/\gamma'}\||h|\|_{\mu,\gamma}\|\Omega\|_{L^1(S^{n-1})}\Big\|\Big(\sum_{j\in\mathbb{Z}}|g_j|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)}\end{aligned}\tag{28}$$

for any $\gamma' < p,q < \infty$, It follows from (28) that

$$\begin{aligned}&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sup_{k\in\mathbb{Z}}|\sigma_{k,\mu,\Gamma,\Omega}*g_{k,j}|\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)}\\ &\leqslant\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sup_{k\in\mathbb{Z}}|\sigma_{k,\mu,\Gamma,\Omega}|*\sup_{k\in\mathbb{Z}}|g_{k,j}|\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)}\\ &\leqslant C_{p,q}(\mu+1)^{1/\gamma'}\||h|\|_{\mu,\gamma}\|\Omega\|_{L^1(S^{n-1})}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sup_{k\in\mathbb{Z}}|g_{k,j}|\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)}\end{aligned}\tag{29}$$

for any $\{g_{k,j}\}_{k,j\in\mathbb{Z}}\in L^p(\mathbb{R}^m,\ell^q(\ell^\infty))$ with $\gamma'<p,q<\infty$. On the other hand, for any $1<p,q<\gamma$, then $\gamma <' p',q'<\infty$. By the dual argument, there exists $\{h_j\}_{j\in\mathbb{Z}}\in L^{p'}(\mathbb{R}^m,\ell^{q'})$ with $\|\{h_j\}\|_{L^{p'}(\mathbb{R}^m,\ell^{q'})}=1$ such that

$$\begin{aligned}&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|\sigma_{k,\mu,\Gamma,\Omega}*g_{k,j}|\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)}\\ &=\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^m}\sum_{k\in\mathbb{Z}}|\sigma_{k,\mu,\Gamma,\Omega}*g_{k,j}(x)|h_j(x)dx\\ &\leqslant\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^m}\sum_{k\in\mathbb{Z}}|g_{k,j}(x)||\sigma_{k,\mu,\Gamma,\Omega}|*|\widetilde{h_j}|(-x)dx\\ &\leqslant\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,j}|\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sup_{k\in\mathbb{Z}}|\sigma_{k,\mu,\Gamma,\Omega}|*|\widetilde{h_j}|\Big)^{q'}\Big)^{1/q'}\Big\|_{L^{p'}(\mathbb{R}^m)},\end{aligned}$$

where $\widetilde{h_j}(x)=h_j(-x)$. This together with (29) implies

$$\begin{aligned}&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|\sigma_{k,\mu,\Gamma,\Omega}*g_{k,j}|\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)}\\ &\leqslant C_{p,q}(\mu+1)^{1/\gamma'}\||h|\|_{\mu,\gamma}\|\Omega\|_{L^1(S^{n-1})}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,j}|\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)}.\end{aligned}\tag{30}$$

for any $1 < p, q < \gamma$. Interpolation between (29) and (30) yields (25) for $(1/p, 1/q)$ belonging to the interior of the square $(\frac{1}{2\gamma}, 1 - \frac{1}{2\gamma})^2$. By interpolation we get (25) for the case $\gamma \geqslant 2$ and complete the proof of Lemma 6. □

3. Proof of Theorem 2 for $\Omega \in H^1(S^{n-1})$

Let us begin with recalling Hardy space on S^{n-1} and its atomic decomposition. The Hardy space $H^1(S^{n-1})$ is the set of all functions $\Omega \in L^1(S^{n-1})$ with satisfying

$$\|\Omega\|_{H^1(S^{n-1})} := \Big\| \sup_{0 \leqslant r < 1} \Big| \int_{S^{n-1}} \Omega(\theta) P_{r(\cdot)}(\theta) d\sigma(\theta) \Big| \Big\|_{L^1(S^{n-1})} < \infty,$$

where $P_{rw}(\theta)$ denotes the Poisson kernel on S^{n-1} defined by

$$P_{rw}(\theta) = \frac{1 - r^2}{|rw - \theta|^n}, \quad 0 \leqslant r < 1 \text{ and } \theta,\ w \in S^{n-1}.$$

Now we give the definition of atom and atomic decomposition of $H^1(S^{n-1})$.

DEFINITION 1. A function $a(\cdot)$ on S^{n-1} is a regular atom if there exist $\varepsilon \in S^{n-1}$ and $\rho \in (0, 2]$ such that

$$\mathrm{supp}(a) \subset S^{n-1} \cap B(\varepsilon, \rho), \ \text{ where } B(\varepsilon, \rho) = \{y \in \mathbb{R}^n : \ |y - \varepsilon| < \rho\}; \tag{31}$$

$$\|a\|_{L^2(S^{n-1})} \leqslant \rho^{(1-n)/2}; \tag{32}$$

$$\int_{S^{n-1}} a(y) d\sigma(y) = 0. \tag{33}$$

Following from [10, 11], we have the following atomic decomposition of Hardy space.

LEMMA 7. *If* $\Omega \in H^1(S^{n-1})$ *satisfies* (5)*, then there are complex numbers* $\{c_j\}_{j \in \mathbb{Z}}$ *and regular atoms* $\{\Omega_j\}_{j \in \mathbb{Z}}$ *such that*

$$\Omega = \sum_j c_j \Omega_j \ \text{ and } \ \|\Omega\|_{H^1(S^{n-1})} \approx \sum_j |c_j|.$$

Proof of Theorem 2 for $\Omega \in H^1(S^{n-1})$. In what follows, we denote $\mathscr{V}_{n-1}$ by the set of polynomials in $n-1$ variables with real coefficients and set $[x] := \max\{k \in \mathbb{N} : k \leqslant x\}$ for any $x \in \mathbb{R}$. For $s \in \mathbb{N}$, let $\mathscr{V}_{n-1,s}$ denote the subset of $\mathscr{V}_{n-1}$ which contains homogeneous polynomials of degree s.

By Lemma 7, it suffices to prove Theorem 2 for Ω being an H^1 atom on S^{n-1} satisfying (31)–(33). Without loss of generality we may assume that $0 < \rho < \frac{1}{4}$. Let λ be the number of distinct d_j. We may assume that

$$\Phi = (\Phi_1, \dots, \Phi_m) = (\Phi^1, \dots, \Phi^\lambda),$$

where $\Phi^s = (\Phi_{s,1}, \dots, \Phi_{s,a_s})$ with $\Phi_{s,j}(ty) = t^{d_{r_s}}\Phi_{s,j}(y)$ for any $1 \leqslant s \leqslant \lambda$ and $1 \leqslant j \leqslant s$. Obviously, $\sum_{s=1}^{\lambda} a_s = m$ and $\{r_1, \dots, r_\lambda\} \subset \{1, \dots, m\}$. We also assume that $\{\Phi_{s,1}, \dots, \Phi_{s,o_s}\}$ forms a basis for span$\{\Phi_{s,1}, \dots, \Phi_{s,a_s}\}$ for any $1 \leqslant s \leqslant \lambda$. Thus there exist $\{b_{s,j,k}\}$ such that

$$\Phi_{s,j}(y) = b_{s,j,1}\Phi_{s,1}(y) + \cdots + b_{s,j,o_s}\Phi_{s,o_s}(y)$$

for any $1 \leqslant s \leqslant \lambda$ and $1 \leqslant j \leqslant a_s$. In what follows, let $\xi = (\xi_1, \dots, \xi_m) = (\xi^1, \dots, \xi^\lambda)$ with $\xi^s = (\xi_{s,1}, \dots, \xi_{s,a_s})$ for $1 \leqslant s \leqslant \lambda$. For any $1 \leqslant s \leqslant \lambda$, let $\tilde{\Phi}^s = (\Phi_{s,1}, \dots, \Phi_{s,o_s})$ and $\tilde{\xi}^s = (\xi_{s,1}, \dots, \xi_{s,o_s})$. We define two sequences of linear transformations $\{H_{s,i}\}_{i=1}^{a_s} : \mathbb{R}^{o_s} \to \mathbb{R}$ and $\{R_{s,j}\}_{j=1}^{o_s} : \mathbb{R}^{a_s} \to \mathbb{R}$ as follows:

$$H_{s,i}(x) = b_{s,i,1}x_1 + \cdots + b_{s,i,o_s}x_{o_s}, \quad 1 \leqslant i \leqslant a_s;$$

$$R_{s,j}(y) = b_{s,1,j}y_1 + \cdots + b_{s,a_s,j}y_{a_s}, \quad 1 \leqslant j \leqslant o_s.$$

Define the family of linear transformations $\{H_s\}_{s=1}^{\lambda}$ and $\{R_s\}_{s=1}^{\lambda}$ by

$$H_s = (H_{s,1}, \dots, H_{s,a_s}), \quad R_s = (R_{s,1}, \dots, R_{s,o_s}). \tag{34}$$

It is easy to verify that

$$x \cdot H_s(y) = R_s(x) \cdot y, \quad (x, y) \in \mathbb{R}^{a_s} \times \mathbb{R}^{o_s}. \tag{35}$$

Thus we have

$$\xi^s \cdot \Phi^s = \xi^s \cdot H_s(\tilde{\Phi}^s) = R_s(\xi^s) \cdot \tilde{\Phi}^s. \tag{36}$$

For any $1 \leqslant s \leqslant \lambda$ and $z \in S^{o_s - 1}$, since $\{\Phi_{s,1}, \dots, \Phi_{s,o_s}\}$ is linearly independent, thus $z \cdot \tilde{\Phi}^i(\cdot)$ is a nonzero real-analytic function. By (3.8) in [26], there exists $\delta_s > 0$ such that

$$\iint_{(S^{n-1})^2} \left| z \cdot (\tilde{\Phi}^s(y) - \tilde{\Phi}^s(u)) \right|^{-\delta_s} d\sigma(y) d\sigma(u) < \infty. \tag{37}$$

Let $\varepsilon_s = \min\{1/d_{r_s}, 1/s, \delta_s/2\}$. Follows from (5.30) in [15], for any $1 \leqslant s \leqslant \lambda$, there exists an orthogonal $n \times n$ matrix U such that $\varepsilon U = \mathbf{e} = (0, \dots, 0, 1) \in S^{n-1}$ and a polynomial $P_{s,j} \in \mathscr{V}_{n-1}$ such that $\deg(P_{s,j}) \leqslant [\frac{n-1}{\varepsilon_s}]$ and

$$|\Phi_{s,j}(yU^{-1}) - P_{s,j}(\tilde{y})| \leqslant C\rho^{(n-1)/\varepsilon_s} \tag{38}$$

for every $y \in B(\mathbf{e}, \rho) \cap S^{n-1}$ and $1 \leqslant j \leqslant a_s$, where $\tilde{y} = (\frac{y_1}{|y|}, \dots, \frac{y_{n-1}}{|y|})$. For any $1 \leqslant s \leqslant \lambda$, let $\mathscr{P}_s = (P_{s,1}, \dots, P_{s,a_s})$ and $\deg(\mathscr{P}_s) = \max_{1 \leqslant j \leqslant a_s} \deg(P_{s,j})$. Then there are integers $0 \leqslant \Lambda_{s,1} < \Lambda_{s,2} < \cdots < \Lambda_{s,M_s} \leqslant \deg(\mathscr{P}_s)$ and $Q_{s,j,\Lambda_{s,l}} \in \mathscr{V}_{n-1,\Lambda_{s,l}}$ for $1 \leqslant j \leqslant a_s$ and $1 \leqslant l \leqslant M_s$ such that

$$\mathscr{P}_s = \sum_{l=1}^{M_s} \mathscr{Q}_{s,\Lambda_{s,l}}, \tag{39}$$

where $\mathscr{Q}_{s,\Lambda_{s,l}} = (Q_{s,1,\Lambda_{s,l}}, Q_{s,2,\Lambda_{s,l}}, \dots, Q_{s,a_s,\Lambda_{s,l}})$ and $\mathscr{Q}_{s,\Lambda_{s,l}} \neq (0, \dots, 0)$. For any $1 \leqslant l \leqslant M_s$, let $\tilde{\mathscr{Q}}_{s,\Lambda_{s,l}} = (Q_{s,1,\Lambda_{s,l}}, Q_{s,2,\Lambda_{s,l}}, \dots, Q_{s,o_s,\Lambda_{s,l}})$ and

$$\tilde{\mathscr{P}}_s = \sum_{l=1}^{M_s} \tilde{\mathscr{Q}}_{s,\Lambda_{s,l}}. \tag{40}$$

We get form (38) that

$$|\tilde{\Phi}^s(y) - \tilde{\mathscr{P}}_s(\widetilde{yU})| \leqslant C\rho^{(n-1)/\varepsilon_s} \tag{41}$$

for every $y \in B(\varepsilon,\rho)\cap S^{n-1}$ and $1 \leqslant s \leqslant \lambda$. For $1 \leqslant s \leqslant \lambda$, $1 \leqslant l \leqslant M_s$ and $1 \leqslant j \leqslant a_s$, we set

$$Q_{s,j,\Lambda_{s,l}}(y) = \sum_{|\beta|=\Lambda_{s,l}} b_{s,j,l,\beta} y^\beta. \tag{42}$$

Let $\varpi(u) = \sum_{s=1}^{u}(M_s+1)$ for $1 \leqslant u \leqslant \lambda$, $\varpi(0) = 0$, and define $\Theta_0,\dots,\Theta_{\varpi(\lambda)}$ by $\Theta_0(y) = (0,\dots,0)$ and

$$\Theta_{\varpi(u)+\theta}(y) = \Big(\Phi^1(y),\dots,\Phi^u(y), |y|^{d_{r_{u+1}}} H_{u+1}\Big(\sum_{l=1}^{\theta} \tilde{\mathscr{Q}}_{u+1,\Lambda_{u+1,l}}\Big(\frac{\widetilde{yU}}{|y|}\Big)\Big),0,\dots,0\Big) \tag{43}$$

for $0 \leqslant u \leqslant \lambda-1$, $0 \leqslant \theta < \varpi(u+1)-\varpi(u)$, and

$$\Theta_{\varpi(\lambda)}(y) = \Phi(y). \tag{44}$$

It follows from (43) that

$$\Theta_{\varpi(u)-1}(y) = \Big(\Phi^1(y),\dots,\Phi^{u-1}(y), |y|^{d_{r_u}} H_u\Big(\tilde{\mathscr{P}}_u\Big(\frac{\widetilde{yU}}{|y|}\Big)\Big),0,\dots,0\Big), \ \ 1 \leqslant u \leqslant \lambda. \tag{45}$$

For $0 \leqslant s \leqslant \varpi(\lambda)$, let $\nu_{k,s} = \sigma_{k,0,\Theta_s,\Omega}$. Note that

$$T_{h,\Omega,\Phi}(f) = \sum_{k\in\mathbb{Z}} \nu_{k,\varpi(\lambda)} * f; \tag{46}$$

$$\nu_{k,0}(y) = 0, \ \ \forall k \in \mathbb{Z} \text{ and } y \in \mathbb{R}^m. \tag{47}$$

For any $1 \leqslant s \leqslant \varpi(\lambda)$, by a change of variable, Hölder's inequality and the fact that $\|\Omega\|_{L^1(S^{n-1})} \leqslant C$,

$$\begin{aligned}
|\widehat{\nu_{k,s}}(\xi)| &= \Big|\int_{2^{k-1}}^{2^k}\int_{S^{n-1}} \Omega(y')\exp(-2\pi i\xi\cdot\Theta_s(ty'))d\sigma(y')h(t)\frac{dt}{t}\Big| \\
&\leqslant 2\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\Big(\int_{2^{k-1}}^{2^k}\Big|\int_{S^{n-1}} \Omega(y')\exp(-2\pi i\xi\cdot\Theta_s(ty'))d\sigma(y')\Big|^{\gamma'}\frac{dt}{t}\Big)^{1/\gamma'} \\
&\leqslant C\Big(\int_{2^{k-1}}^{2^k}\Big|\int_{S^{n-1}} \Omega(y')\exp(-2\pi i\xi\cdot\Theta_s(ty'))d\sigma(y')\Big|^{2}\frac{dt}{t}\Big)^{1/\max\{2,\gamma'\}}.
\end{aligned} \tag{48}$$

For any $1 \leqslant s \leqslant \varpi(\lambda)$, let

$$I_{s,k}(\xi) = \int_{2^{k-1}}^{2^k}\Big|\int_{S^{n-1}} \Omega(y')\exp(-2\pi i\xi\cdot\Theta_s(ty'))d\sigma(y')\Big|^{2}\frac{dt}{t}.$$

By a change of variable and Lemma 1 we have

$$
\begin{aligned}
|I_{\varpi(u),k}(\xi)| &= \Big|\int_{2^{k-1}}^{2^k}\iint_{(S^{n-1})^2}\Omega(y')\overline{\Omega(z')} \\
&\quad\times\exp\Big(-2\pi i\sum_{j=1}^{u}\xi^j\cdot(\Phi^j(y')-\Phi^j(z'))t^{d_{r_j}}\Big)d\sigma(y')d\sigma(z')\frac{dt}{t} \\
&\leqslant \iint_{(S^{n-1})^2}|\Omega(y')\overline{\Omega(z')}| \\
&\quad\times\Big|\int_{2^{k-1}}^{2^k}\exp\Big(-2\pi i\sum_{j=1}^{u}R_j(\xi^j)\cdot(\tilde{\Phi}^j(y')-\tilde{\Phi}^j(z'))t^{d_{r_j}}\Big)\frac{dt}{t}\Big|d\sigma(y')d\sigma(z') \\
&\leqslant C\iint_{(S^{n-1})^2}|\Omega(y')\overline{\Omega(z')}||2^{kd_{r_u}}R_u(\xi^u)\cdot(\tilde{\Phi}^u(y')-\tilde{\Phi}^u(z'))|^{-\varepsilon}d\sigma(y')d\sigma(z') \\
&\leqslant C|2^{kd_{r_u}}R_u(\xi^u)|^{-\varepsilon}\|\Omega\|^2_{L^2(S^{n-1})} \\
&\quad\times\Big(\iint_{(S^{n-1})^2}\Big|\frac{R_u(\xi^u)}{|R_u(\xi^u)|}\cdot(\tilde{\Phi}^u(y')-\tilde{\Phi}^u(z'))\Big|^{-2\varepsilon}d\sigma(y')d\sigma(z')\Big)^{1/2}
\end{aligned}
$$

for any $1\leqslant u\leqslant\lambda$ and $0<\varepsilon\leqslant\min\{1/d_{r_u},1/u\}$. This together with (32), (37) and (48) yields that

$$
|\widehat{\nu_{k,\varpi(u)}}(\xi)|\leqslant C|2^{kd_{r_u}}\rho^{(n-1)/\varepsilon_u}R_u(\xi^u)|^{-\varepsilon_u/\max\{2,\gamma'\}},\quad 1\leqslant u\leqslant\lambda. \tag{49}
$$

For $0\leqslant u\leqslant\lambda-1$ and $0<\theta<\varpi(u+1)-\varpi(u)$, by a change of variable and Hölder's inequality again,

$$
\begin{aligned}
&|\widehat{\nu_{k,\varpi(u)+\theta}}(\xi)| \\
&=\Big|\int_{2^{k-1}}^{2^k}\int_{S^{n-1}}\exp(-2\pi i\xi\cdot\Theta_{\varpi(u)+\theta}(ty'))\Omega(y')d\sigma(y')h(t)\frac{dt}{t}\Big| \\
&\leqslant 2\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\Big(\int_{2^{k-1}}^{2^k}\Big|\int_{S^{n-1}}\exp(-2\pi i\xi\cdot\Theta_{\varpi(u)+\theta}(ty'))\Omega(y')d\sigma(y')\Big|^{\gamma'}\frac{dt}{t}\Big)^{1/\gamma'} \\
&\leqslant C\Big(\int_{1/2}^{1}\Big|\int_{S^{n-1}}\exp\Big(-2\pi i\Big(\sum_{j=1}^{u}\xi^j\cdot\Phi^j(y')(2^kt)^{d_{r_j}} \\
&\quad+\xi^{u+1}\cdot H_{u+1}\Big(\sum_{l=1}^{\theta}\tilde{\mathscr{Q}}_{u+1,\Lambda_{u+1,l}}(\widetilde{y'U})\Big)(2^kt)^{d_{r_{u+1}}}\Big)\Big)\Omega(y')d\sigma(y')\Big|dt\Big)^{1/\gamma'} \\
&= C\Big(\int_{1/2}^{1}\Big|\int_{S^{n-1}}\exp\Big(-2\pi i\Big(\sum_{j=1}^{u}\xi^j\cdot\Phi^j(yU^{-1})(2^kt)^{d_{r_j}} \\
&\quad+\xi^{u+1}\cdot H_{u+1}\Big(\sum_{l=1}^{\theta}\tilde{\mathscr{Q}}_{u+1,\Lambda_{u+1,l}}(\tilde{y})\Big)(2^kt)^{d_{r_{u+1}}}\Big)\Big)\Omega(yU^{-1})d\sigma(y)\Big|dt\Big)^{1/\gamma'}.
\end{aligned}
$$

We get from (35) and (45) that

$$\begin{aligned}\xi^{u+1}\cdot H_{u+1}\Big(\sum_{l=1}^{\theta}\tilde{\mathscr{Q}}_{u+1,\Lambda_{u+1,l}}(\tilde{y})\Big) &= R_{u+1}(\xi^{u+1})\cdot\Big(\sum_{l=1}^{\theta}\tilde{\mathscr{Q}}_{u+1,\Lambda_{u+1,l}}(\tilde{y})\Big)\\ &=\sum_{j=1}^{o_{u+1}}R_{u+1,j}(\xi^{u+1})\cdot\Big(\sum_{l=1}^{\theta}Q_{u+1,j,\Lambda_{u+1,l}}(\tilde{y})\Big)\\ &=\sum_{l=1}^{\theta}\sum_{|\beta|=\Lambda_{u+1,l}}\Big(\sum_{j=1}^{o_{u+1}}b_{u+1,j,l,\beta}R_{u+1,j}(\xi^{u+1})\Big)(\tilde{y})^{\beta}.\end{aligned}$$

Invoking Lemma 2, there exists $\gamma_{u,\theta}>0$ such that

$$|\widehat{\nu_{k,\varpi(u)+\theta}}(\xi)|\leqslant C|2^{kd_{r_{u+1}}}\rho^{\Lambda_{u+1,\theta}}L^{(\Lambda_{u+1,\theta})}(\xi)|^{-\gamma_{u,\theta}/\gamma'} \tag{50}$$

for $0\leqslant u\leqslant\lambda-1$ and $0<\theta<\varpi(u+1)-\varpi(u)$, where

$$L^{(\Lambda_{u+1,\theta})}(\xi)=\Big(\sum_{j=1}^{o_{u+1}}b_{u+1,j,l,\beta}R_{u+1,j}(\xi^{u+1})\Big)_{|\beta|=\Lambda_{u+1,\theta}}. \tag{51}$$

Note that $L^{(\Lambda_{u+1,\theta})}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^{\dim(\mathscr{V}_{n-1,\Lambda_{u+1,\theta}})}$. On the other hand, by a change of variable, (35)–(36), (41) and (45) we have

$$\begin{aligned}&|\widehat{\nu_{k,\varpi(u)}}(\xi)-\widehat{\nu_{k,\varpi(u)-1}}(\xi)|\\ &=\Big|\int_{2^{k-1}\leqslant|y|<2^k}(\exp(-2\pi i\xi\cdot\Theta_{\varpi(u)}(y))-\exp(-2\pi i\xi\cdot\Theta_{\varpi(u)-1}(y)))\frac{\Omega(y)h(|y|)}{|y|^n}dy\Big|\\ &\leqslant C2^{kd_{r_u}}\int_{2^{k-1}}^{2^k}|h(t)|\frac{dt}{t}\int_{S^{n-1}}|\Omega(y)||\xi^u\cdot\Phi^u(y)-\xi^u\cdot H_u(\mathscr{P}_u(\widetilde{yU}))|d\sigma(y)\\ &\leqslant C2^{kd_{r_u}}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\int_{S^{n-1}}|\Omega(y)||R_u(\xi^u)\cdot(\tilde{\Phi}^u(y)-\tilde{\mathscr{P}}_u(\widetilde{yU}))|d\sigma(y)\\ &\leqslant C|2^{kd_{r_u}}\rho^{(n-1)/\varepsilon_u}R_u(\xi^u)|\end{aligned} \tag{52}$$

for $1\leqslant u\leqslant\lambda$. By (36), (42)–(43) and a change of variable we have

$$\begin{aligned}&|\widehat{\nu_{k,\varpi(u)+\theta}}(\xi)-\widehat{\nu_{k,\varpi(u)+\theta-1}}(\xi)|\\ &=\Big|\int_{2^{k-1}\leqslant|y|<2^k}(\exp(-2\pi i\xi\cdot\Theta_{\varpi(u)+\theta}(y))-\exp(-2\pi i\xi\cdot\Theta_{\varpi(u)+\theta-1}(y)))\\ &\quad\times\frac{\Omega(y)h(|y|)}{|y|^n}dy\Big|\\ &\leqslant C\int_{2^{k-1}\leqslant|y|<2^k}\Big|2^{kd_{r_{u+1}}}\xi^{u+1}\cdot H_{u+1}\Big(\tilde{\mathscr{Q}}_{u+1,\Lambda_{u+1,\theta}}\Big(\frac{\widetilde{yU}}{|y|}\Big)\Big)\Big|\frac{|\Omega(y)h(|y|)|}{|y|^n}dy\\ &=C\int_{2^{k-1}}^{2^k}|h(t)|\frac{dt}{t}\int_{S^{n-1}}|2^{kd_{r_{u+1}}}\xi^{u+1}\cdot H_{u+1}(\tilde{\mathscr{Q}}_{u+1,\Lambda_{u+1,\theta}}(\tilde{y}))\Omega(yU^{-1})|d\sigma(y)\\ &\leqslant C|2^{kd_{r_{u+1}}}\rho^{\Lambda_{u+1,\theta}}L^{(\Lambda_{u+1,\theta})}(\xi)|\end{aligned} \tag{53}$$

for $1\leqslant u\leqslant\lambda-1$ and $1\leqslant\theta<\varpi(u)-\varpi(u-1)$. Define the linear transformations $\{L_s\}_{s=1}^{\varpi(\lambda)}$ by

$$L_s(\xi)=\begin{cases}\rho^{\Lambda_{u+1,\theta}}L^{(\Lambda_{u+1,\theta})}(\xi), & s=\varpi(u)+\theta,\ 0\leqslant u\leqslant\lambda-1,\ 0<\theta<\varpi(u+1)-\varpi(u);\\ \rho^{(n-1)/\varepsilon_u}R_u(\xi^u), & s=\varpi(u),\ 1\leqslant u\leqslant\lambda.\end{cases}$$

Also, we define $N_1,\dots,N_{\varpi(\lambda)}$ and $\eta_1,\dots,\eta_{\varpi(\lambda)}$ by

$$N_s := \begin{cases} \frac{\gamma_{u,\theta}}{\gamma'}, & s=\varpi(u)+\theta,\ 0\leqslant u\leqslant \lambda-1,\ 0<\theta<\varpi(u+1)-\varpi(u); \\ \frac{\varepsilon_u}{\max\{2,\gamma'\}}, & s=\varpi(u),\ 1\leqslant u\leqslant \lambda. \end{cases}$$

$$\eta_s := \begin{cases} d_{r_{u+1}}, & s=\varpi(u)+\theta, 0\leqslant u\leqslant \lambda-1,\ 0<\theta<\varpi(u+1)-\varpi(u); \\ d_{r_u}, & s=\varpi(u),\ 1\leqslant u\leqslant \lambda. \end{cases}$$

It follows from (49)–(50) and (52)–(53) that for any $1\leqslant s\leqslant \varpi(\lambda)$,

$$|\widehat{\nu_{k,s}}(\xi)-\widehat{\nu_{k,s-1}}(\xi)|\leqslant C|2^{k\eta_s}L_s(\xi)|; \tag{54}$$

$$|\widehat{\nu_{k,s}}(\xi)|\leqslant C|2^{k\eta_s}L_s(\xi)|^{-N_s}. \tag{55}$$

On the other hand, invoking Lemma 6 with $\mu=0$, we have that

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|\nu_{k,s}*g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)}\leqslant C\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)} \tag{56}$$

holds for any $1\leqslant s\leqslant \varpi(\lambda)$, $\{g_{k,j}\}_{k,j\in\mathbb{Z}}\in L^p(\mathbb{R}^m,\ell^q(\ell^2))$ and $(1/p,1/q)\in\mathscr{R}_\gamma$. By (46)–(47), (54)–(56), Lemma 5 and interpolation, we get Theorem 2 for Ω being an H^1 atom on S^{n-1} satisfying (31)–(33). This proves Theorem 2 for $\Omega\in H^1(S^{n-1})$. □

4. Proof of Theorem 2 for $\Omega\in L(\log^+ L)^{1/\gamma'}(S^{n-1})$

Let $\Omega\in L(\log^+ L)^{\alpha}(S^{n-1})$ for $\alpha>0$ and satisfy (5). Employing the notation in [4], let $E_\mu=\{y'\in S^{n-1}:\ 2^\mu<|\Omega(y')|\leqslant 2^{\mu+1}\}$ for $\mu\in\mathbb{N}\backslash\{0\}$ and $E_0=\{y'\in S^{n-1}:\ |\Omega(y')|\leqslant 2\}$. Set $A(\Omega)=\{\mu\in\mathbb{N}:\ \sigma(E_\mu)>2^{-4\mu}\}$ and for $\mu\geqslant 1$,

$$\Omega_\mu(y')=\Omega(y')\chi_{E_\mu}(y')-(\sigma(S^{n-1}))^{-1}\int_{E_\mu}\Omega(y')d\sigma(y'),$$

and $\Omega_0(y')=\Omega(y')-\sum_{\mu\in A(\Omega)}\Omega_\mu(y')$, where $\sigma(E_\mu)=\int_{E_\mu}d\sigma(\theta)$ and $\sigma(S^{n-1})=\int_{S^{n-1}}d\sigma(\theta)$. One can easily check that

$$\int_{S^{n-1}}\Omega_\mu(y')d\sigma(y')=0,\ \text{ for } \mu\in A(\Omega)\cup\{0\}; \tag{57}$$

$$\|\Omega_0\|_{L^1(S^{n-1})}\leqslant C,\ \ \|\Omega_\mu\|_{L^1(S^{n-1})}\leqslant C\|\Omega\|_{L^1(E_\mu)},\ \text{ for } \mu\in A(\Omega); \tag{58}$$

$$\|\Omega_0\|_{L^2(S^{n-1})}\leqslant C,\ \ \|\Omega_\mu\|_{L^2(S^{n-1})}\leqslant C2^{2\mu}\|\Omega\|_{L^1(E_\mu)},\ \text{ for } \mu\in A(\Omega); \tag{59}$$

$$\Omega(y')=\sum_{\mu\in A(\Omega)\cup\{0\}}\Omega_\mu(y'); \tag{60}$$

$$\sum_{\mu\in A(\Omega)\cup\{0\}}(\mu+1)^\alpha\|\Omega\|_{L^1(E_\mu)}\leqslant C\|\Omega\|_{L(\log^+ L)^\alpha(S^{n-1})},\ \text{ for } \alpha>0; \tag{61}$$

$$T_{h,\Omega,\Phi}(f) = \sum_{\mu \in A(\Omega) \cup \{0\}} T_{h,\Omega_\mu,\Phi}(f). \tag{62}$$

We now give the proof of Theorem 2 for $\Omega \in L(\log^+ L)^{1/\gamma'}(S^{n-1})$.

Proof of Theorem 2 for $\Omega \in L(\log^+ L)^{1/\gamma'}(S^{n-1})$. Let λ be the number of distinct d_j. We may assume without loss of generality that

$$\Phi = (\Phi_1, \ldots, \Phi_m) = (\Phi^1, \ldots, \Phi^\lambda), \tag{63}$$

where $\Phi^s = (\Phi_{s,1}, \ldots, \Phi_{s,a_s})$ with $\Phi_{s,j}(ty) = t^{d_{r_s}} \Phi_{s,j}(y)$ for any $1 \leqslant s \leqslant \lambda$ and $1 \leqslant j \leqslant s$. Obviously, $\sum_{s=1}^{\lambda} a_s = m$ and $\{r_1, \ldots, r_\lambda\} \subset \{1, \ldots, m\}$. We also assume that $\{\Phi_{s,1}, \ldots, \Phi_{s,o_s}\}$ forms a basis for $\mathrm{span}\{\Phi_{s,1}, \ldots, \Phi_{s,a_s}\}$ for any $1 \leqslant s \leqslant \lambda$. Let $\tilde{\Phi}^s = (\Phi_{s,1}, \ldots, \Phi_{s,o_s})$ and $\xi = (\xi_1, \ldots, \xi_m) = (\xi^1, \ldots, \xi^\lambda)$ with $\xi^s = (\xi_{s,1}, \xi_{s,2}, \ldots, \xi_{s,a_s})$ for any $1 \leqslant s \leqslant \lambda$. Following from the proof of Theorem 2 for the case $\Omega \in H^1(S^{n-1})$, there exists a sequence of linear transformations $\{R_s\}_{s=1}^{\lambda}$ such that

$$\xi^s \cdot \Phi^s = R_s(\xi^s) \cdot \tilde{\Phi}^s. \tag{64}$$

Let δ_s be given as in (37) and $\varepsilon_s = \min\{1/d_{r_s}, 1/s, \delta_s/2\}$. Define the mappings: $\Gamma_0, \ldots, \Gamma_\lambda$ by

$$\Gamma_0(y) = (0, \ldots, 0);$$

$$\Gamma_s(y) = (\Phi^1, \ldots, \Phi^s, 0, \ldots, 0), \ \ 1 \leqslant s \leqslant \lambda.$$

For $0 \leqslant s \leqslant \lambda$, let $\omega_{s,\mu,k} = \sigma_{k,\mu,\Gamma_s,\Omega_\mu}$. It is obvious that

$$\omega_{0,\mu,k}(y) = 0, \ \forall k \in \mathbb{Z} \text{ and } y \in \mathbb{R}^m; \tag{65}$$

$$T_{h,\Omega_\mu,\Phi}(f) = \sum_{k \in \mathbb{Z}} \omega_{\lambda,\mu,k} * f. \tag{66}$$

For convenience, we set $A_\mu = (\mu+1)^{1/\gamma'} \|\Omega\|_{L^1(E_\mu)} \||h|\|_{\mu,\gamma}$ for $\gamma > 1$. By a change of variable, (58), (64) and Hölder's inequality,

$$\begin{aligned}
&|\widehat{\omega_{s,\mu,k}}(\xi) - \widehat{\omega_{s-1,\mu,k}}(\xi)| \\
&= \Big| \int_{2^{(\mu+1)(k-1)}}^{2^{(\mu+1)k}} \int_{S^{n-1}} \Omega_\mu(y')(\exp(-2\pi i \xi \cdot \Gamma_s(ty')) \\
&\quad - \exp(-2\pi i \xi \cdot \Gamma_{s-1}(ty'))) d\sigma(y') h(t) \frac{dt}{t} \Big| \\
&\leqslant \int_{2^{(\mu+1)(k-1)}}^{2^{(\mu+1)k}} \int_{S^{n-1}} |\Omega_\mu(y')| |R_s(\xi^s) \cdot \tilde{\Phi}^s(y') t^{d_{r_s}}| d\sigma(y') |h(t)| \frac{dt}{t} \\
&\leqslant C |2^{(\mu+1)k d_{r_s}} R_s(\xi^s)| \|\Omega_\mu\|_{L^1(S^{n-1})} \int_{2^{(\mu+1)(k-1)}}^{2^{(\mu+1)k}} |h(r)| \frac{dr}{r} \\
&\leqslant C A_\mu |2^{(\mu+1)k d_{r_s}} R_s(\xi^s)|.
\end{aligned} \tag{67}$$

On the other hand, by a change of variable and Hölder's inequality again,

$$
\begin{aligned}
|\widehat{\omega_{s,\mu,k}}(\xi)| &= \Big| \int_{2^{(\mu+1)(k-1)}}^{2^{(\mu+1)k}} \int_{S^{n-1}} \Omega_\mu(y') \exp(-2\pi i\xi \cdot \Gamma_s(ty')) d\sigma(y') h(t) \frac{dt}{t} \Big| \\
&\leqslant |||h|||_{\mu,\gamma} \Big(\int_{2^{(\mu+1)(k-1)}}^{2^{(\mu+1)k}} \Big| \int_{S^{n-1}} \Omega_\mu(y') \exp(-2\pi i\xi \cdot \Gamma_s(ty')) d\sigma(y') \Big|^{\gamma'} \frac{dt}{t} \Big)^{1/\gamma'} \\
&=: |||h|||_{\mu,\gamma} \tilde{H}_{s,\mu,k}(\xi).
\end{aligned}
\tag{68}
$$

For $1 < \gamma \leqslant 2$, then $\gamma' \geqslant 2$ and

$$
\begin{aligned}
\tilde{H}_{s,\mu,k}(\xi) &= \Big(\int_{2^{(\mu+1)(k-1)}}^{2^{(\mu+1)k}} \Big| \int_{S^{n-1}} \Omega_\mu(y') \exp(-2\pi i\xi \cdot \Gamma_s(ty')) d\sigma(y') \Big|^{\gamma'-2} \\
&\qquad \times \Big| \int_{S^{n-1}} \Omega_\mu(y') \exp(-2\pi i\xi \cdot \Gamma_s(ty')) d\sigma(y') \Big|^{2} \frac{dt}{t} \Big)^{1/\gamma'} \\
&\leqslant C \|\Omega_\mu\|_{L^1(S^{n-1})}^{1-2/\gamma'} \\
&\qquad \times \Big(\int_{2^{(\mu+1)(k-1)}}^{2^{(\mu+1)k}} \Big| \int_{S^{n-1}} \Omega_\mu(y') \exp(-2\pi i\xi \cdot \Gamma_s(ty')) d\sigma(y') \Big|^{2} \frac{dt}{t} \Big)^{1/\gamma'}.
\end{aligned}
\tag{69}
$$

For $\gamma > 2$, then $1 \leqslant \gamma' < 2$, by Hölder's inequality,

$$
\begin{aligned}
\tilde{H}_{s,\mu,k}(\xi) &\leqslant (\mu+1)^{1/\gamma'-1/2} \\
&\qquad \times \Big(\int_{2^{(\mu+1)(k-1)}}^{2^{(\mu+1)k}} \Big| \int_{S^{n-1}} \Omega_\mu(y') \exp(-2\pi i\xi \cdot \Gamma_s(ty')) d\sigma(y') \Big|^{2} \frac{dt}{t} \Big)^{1/2}.
\end{aligned}
\tag{70}
$$

Let

$$
\tilde{I}_{s,\mu,k}(\xi) := \int_{2^{(\mu+1)(k-1)}}^{2^{(\mu+1)k}} \Big| \int_{S^{n-1}} \Omega_\mu(y') \exp(-2\pi i\xi \cdot \Gamma_s(ty')) d\sigma(y') \Big|^{2} \frac{dt}{t}.
$$

We get from (64) that

$$
\begin{aligned}
&\tilde{I}_{s,\mu,k}(\xi) \\
&= \int_{2^{(\mu+1)(k-1)}}^{2^{(\mu+1)k}} \iint_{(S^{n-1})^2} \Omega_\mu(y') \Omega_\mu(x') \exp(-2\pi i\xi \cdot (\Gamma_s(ty') - \Gamma_s(tx'))) d\sigma(y') d\sigma(x') \frac{dt}{t} \\
&= \iint_{(S^{n-1})^2} \Omega_\mu(y') \overline{\Omega_\mu(x')} \int_{2^{(\mu+1)(k-1)}}^{2^{(\mu+1)k}} \exp(-2\pi i\xi \cdot (\Gamma_s(ty') - \Gamma_s(tx'))) \frac{dt}{t} d\sigma(y') d\sigma(x') \\
&= \iint_{(S^{n-1})^2} \Omega_\mu(y') \overline{\Omega_\mu(x')} \\
&\quad \times \int_{2^{(\mu+1)(k-1)}}^{2^{(\mu+1)k}} \exp\Big(-2\pi i \sum_{j=1}^{s} R_j(\xi^j) \cdot (\tilde{\Phi}^j(y') - \tilde{\Phi}^j(x')) t^{d_{r_j}} \Big) \frac{dt}{t} d\sigma(y') d\sigma(x').
\end{aligned}
$$

Invoking Lemma 1 we have

$$\begin{aligned}
&\Big|\int_{2^{(\mu+1)(k-1)}}^{2^{(\mu+1)k}} \exp\Big(-2\pi i\sum_{j=1}^{s} R_j(\xi^j)\cdot(\tilde{\Phi}^j(y')-\tilde{\Phi}^j(x'))t^{d_{r_j}}\Big)\frac{dt}{t}\Big|\\
&=\Big|\sum_{v=0}^{\mu}\int_{2^{(\mu+1)(k-1)+v}}^{2^{(\mu+1)(k-1)+v+1}} \exp\Big(-2\pi i\sum_{j=1}^{s} R_j(\xi^j)\cdot(\tilde{\Phi}^j(y')-\tilde{\Phi}^j(x'))t^{d_{r_j}}\Big)\frac{dt}{t}\Big|\\
&\leqslant \sum_{v=0}^{\mu}\Big|\int_{1/2}^{1} \exp\Big(-2\pi i\sum_{j=1}^{s} R_j(\xi^j)\cdot(\tilde{\Phi}^j(y')-\tilde{\Phi}^j(x'))2^{((\mu+1)(k-1)+v+1)d_{r_j}}t^{d_{r_j}}\Big)\frac{dt}{t}\Big|\\
&\leqslant (\mu+1)|2^{(\mu+1)(k-1)d_{r_s}}R_s(\xi^s)\cdot(\tilde{\Phi}^s(y')-\tilde{\Phi}^s(x'))|^{-\varepsilon}
\end{aligned}$$

for any $0<\varepsilon\leqslant\min\{1/s,1/d_{r_s}\}$. Then by Hölder's inequality we have

$$\begin{aligned}
|\tilde{I}_{s,\mu,k}(\xi)| &\leqslant (\mu+1)\iint_{(S^{n-1})^2}|\Omega_\mu(y')\overline{\Omega_\mu(x')}|\\
&\qquad\times|R_s(\xi^s)\cdot(\tilde{\Phi}^s(y')-\tilde{\Phi}^s(x'))2^{(\mu+1)(k-1)d_{r_s}}|^{-\varepsilon}d\sigma(y')d\sigma(x')\\
&\leqslant (\mu+1)\|\Omega_\mu\|^2_{L^2(S^{n-1})}(2^{(\mu+1)(k-1)d_{r_s}}|R_s(\xi^s)|)^{-\varepsilon}\\
&\qquad\times\Big(\iint_{(S^{n-1})^2}\Big|\frac{R_s(\xi^s)}{|R_s(\xi^s)|}\cdot(\tilde{\Phi}^s(y')-\tilde{\Phi}^s(x'))\Big|^{-2\varepsilon}d\sigma(y')d\sigma(x')\Big)^{1/2}
\end{aligned}$$

for any $0<\varepsilon\leqslant\min\{1/s,1/d_{r_s}\}$. By letting $\varepsilon_s=\min\{1/s,1/r_s,\delta_s/2\}$ and (37) we have

$$|\tilde{I}_{s,\mu,k}(\xi)|\leqslant(\mu+1)\|\Omega_\mu\|^2_{L^2(S^{n-1})}(2^{(\mu+1)(k-1)d_{r_s}}|R_s(\xi^s)|)^{-\varepsilon_s}. \tag{71}$$

This together with (68)–(70) implies

$$|\widehat{\omega_{s,\mu,k}}(\xi)|\leqslant CA_\mu 2^{4\mu/\max\{2,\gamma'\}}(2^{(\mu+1)(k-1)d_{r_s}}|R_s(\xi^s)|)^{-\varepsilon_s/\max\{2,\gamma'\}}. \tag{72}$$

On the other hand, one can easily check that

$$|\widehat{\omega_{s,\mu,k}}(\xi)|\leqslant CA_\mu. \tag{73}$$

Interpolation between (72) and (73) yields

$$|\widehat{\omega_{s,\mu,k}}(\xi)|\leqslant CA_\mu(2^{(\mu+1)kd_{r_s}}|R_s(\xi^s)|)^{-\varepsilon_s/(\max\{2,\gamma'\}(\mu+1))}. \tag{74}$$

It follows from (67) and (73) that

$$|\widehat{\omega_{s,\mu,k}}(\xi)-\widehat{\omega_{s-1,\mu,k}}(\xi)|\leqslant CA_\mu(2^{(\mu+1)kd_{r_s}}|R_s(\xi^s)|)^{1/(\mu+1)}. \tag{75}$$

On the other hand, invoking Lemma 6 and (59),

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|\omega_{s,\mu,k}*g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)}\leqslant CA_\mu\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^m)} \tag{76}$$

holds for any $1\leqslant s\leqslant\lambda$, functions $\{g_{k,j}\}_{k,j\in\mathbb{Z}}\in L^p(\mathbb{R}^m,\ell^q(\ell^2))$ and $(1/p,1/q)\in\mathscr{R}_\gamma$. Here $C>0$ is independent of μ and γ. Then by (61)–(62), (65)–(66), (73)–(76), Lemma 5 and interpolation, we get Theorem 2 for $\Omega\in L(\log^+L)^{1/\gamma'}(S^{n-1})$. □

5. Concluding results

In this section, we will show that our main results can be extended to a class of singular integral operators associated to more general compound mappings. Precisely, using Theorem 2 and a switched method followed from [12], we can obtain the corresponding results for the more general singular integral operators $T_{h,\Omega,\Phi,\varphi}$ defined by

$$T_{h,\Omega,\Phi,\varphi}(f)(x) := \text{p.v.} \int_{\mathbb{R}^n} f(x - \Phi(\varphi(|y|)y'))K(y)dy, \ \ x \in \mathbb{R}^m,$$

where $K(\cdot)$ is as in (4) and $\varphi \in \mathscr{G}$. Here $\mathscr{G}$ is the set of all nonnegative (or non-positive) and monotonic $\mathscr{C}^1(\mathbb{R}^+)$ functions φ such that $\Upsilon_\varphi(t) := \frac{\varphi(t)}{t\varphi'(t)}$ with $|\Upsilon_\varphi(t)| \leqslant C_\varphi$, where C_φ is a positive constant which depends only on φ. Clearly, $T_{h,\Omega,\Phi}$ is the special case of $T_{h,\Omega,\Phi,\varphi}$ for $\varphi(t) = t$. The general result can be formulated as follows.

THEOREM 3. *Let $\varphi \in \mathscr{G}$ and Φ be given as in Theorem 2. Under the same conditions of Theorem 2 (resp., Corollary 1), the operator $T_{h,\Omega,\Phi,\varphi}$ is also bounded on $\dot{F}_\alpha^{p,q}(\mathbb{R}^m)$ (resp., $F_\alpha^{p,q}(\mathbb{R}^m)$).*

REMARK 3. If $\varphi \in \mathscr{G}$, the following facts are obvious (see [12]):

(i) $\lim_{t\to 0} \varphi(t) = 0$ and $\lim_{t\to\infty} |\varphi(t)| = \infty$ if φ is nonnegative and increasing, or non-positive and decreasing;

(ii) $\lim_{t\to 0} |\varphi(t)| = \infty$ and $\lim_{t\to\infty} \varphi(t) = 0$ if φ is nonnegative and decreasing, or non-positive and increasing.

REMARK 4. Theorem 3 implies [32, Theorem 1] even in the special case $m = n$ and $\Phi(y) = y$.

In order to prove Theorem 3, we need the following two lemmas.

LEMMA 8. ([12, 24]) *Let $\varphi \in \mathscr{G}$. Suppose $h \in \Delta_\gamma(\mathbb{R}^+)$, or $\mathscr{H}_\gamma(\mathbb{R}^+)$, for some $\gamma > 1$, then $h(\varphi^{-1})\Upsilon_\varphi(\varphi^{-1}) \in \Delta_\gamma(\mathbb{R}^+)$,* or *$\mathscr{H}_\gamma(\mathbb{R}^+)$. Precisely, we have*

$$\|h(\varphi^{-1})\Upsilon_\varphi(\varphi^{-1})\|_{\Delta_\gamma(\mathbb{R}^+)} \leqslant C\|h\|_{\Delta_\gamma(\mathbb{R}^+)},$$

$$\|h(\varphi^{-1})\Upsilon_\varphi(\varphi^{-1})\|_{\mathscr{H}_\gamma(\mathbb{R}^+)} \leqslant C\|h\|_{\mathscr{H}_\gamma(\mathbb{R}^+)},$$

where the constant $C > 0$ depends only on φ.

LEMMA 9. *Let $\varphi \in \mathscr{G}$. Then*

(i) *if φ is nonnegative and increasing, $T_{h,\Omega,\Phi,\varphi}(f) = T_{h(\varphi^{-1})\Upsilon_\varphi(\varphi^{-1}),\Omega,\Phi}(f)$;*

(ii) *if φ is nonnegative and decreasing, $T_{h,\Omega,\Phi,\varphi}(f) = -T_{h(\varphi^{-1})\Upsilon_\varphi(\varphi^{-1}),\Omega,\Phi}(f)$;*

(iii) *if φ is non-positive and decreasing, $T_{h,\Omega,\Phi,\varphi}(f) = T_{h(\varphi^{-1})\Upsilon_\varphi(\varphi^{-1}),\tilde{\Omega},\Phi}(f)$;*

(iv) *if φ is non-positive and increasing, $T_{h,\Omega,\Phi,\varphi}(f) = -T_{h(\varphi^{-1})\Upsilon_\varphi(\varphi^{-1}),\tilde{\Omega},\Phi}(f)$, where $\tilde{\Omega}(y) = \Omega(-y)$.*

Proof. We can get this lemma by Remark 3 and the similar arguments as in the proof of [12, Lemma 2.3]. The details are omitted. □

Proof of Theorem 3. Theorem 3 directly follows from Lemmas 8 and 9 and Theorem 2. □

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