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INTERPOLATION COEFFICIENTS MIXED FINITE ELEMENT METHODS AND L^∞–ERROR ESTIMATES FOR NONLINEAR OPTIMAL CONTROL PROBLEM

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Abstract. In this paper, we investigate L^∞-error estimates for the convex optimal control problem governed by nonlinear elliptic equations using interpolation coefficients mixed finite element methods. By using the interpolation coefficient thought to process the nonlinear term of equations, we present the mixed finite element approximation with interpolated coefficients for nonlinear optimal control problem. We derive L^∞-error estimates for the interpolation coefficients mixed finite element approximation of nonlinear optimal control problem. Finally some numerical examples are given to confirm our theoretical results.

1. Introduction

We consider the following nonlinear optimal control problem:

$$\min_{u\in K\subset L^\infty(\Omega)} \left\{ \frac{1}{2}\|p-p_d\|^2 + \frac{1}{2}\|y-y_d\|^2 + \frac{1}{2}\|u\|^2 \right\} \tag{1.1}$$

subject to the state equations

$$\mathrm{div} p + \phi(y) = f + u, \quad p = -A\nabla y, \quad x\in\Omega, \tag{1.2}$$

with the boundary condition $y=0$, $x\in\partial\Omega$, where Ω is a bounded open set in $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$, $f\in H^1(\Omega)$. For any $R>0$ the function $\phi(\cdot)\in W^{2,\infty}(-R,R)$, $\phi'(y)\in L^2(\Omega)$ for any $y\in H^1(\Omega)$, and $\phi'(y)\geqslant\gamma_0>0$. We assume that the two given functions satisfy the regularity $p_d\in(W^{2,p}(\Omega))^2$, $y_d\in W^{1,p}(\Omega)$, $p\geqslant 2$. Furthermore, we assume the coefficient matrix $A(x)=(a_{i,j}(x))_{2\times 2}\in(W^{1,\infty}(\Omega))^{2\times 2}$ is a symmetric 2×2-matrix and there is a constant $c>0$ satisfying for any vector $\mathrm{X}\in\mathbb{R}^2$,

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$X'AX \geqslant c\|X\|_{\mathbb{R}^2}^2$. Here, K denotes the admissible set of the control variable, defined by

$$K = \{u(x) \in L^\infty(\Omega) : \alpha(x) \leqslant u(x) \leqslant \beta(x)\}, \tag{1.3}$$

where $\alpha(x)$ and $\beta(x)$ are two real functions.

For $1 \leqslant p < \infty$ and any nonnegative integer m. Let $W^{m,p}(\Omega) = \{v \in L^p(\Omega);\ D^\alpha v \in L^p(\Omega) \text{ if } |\alpha| \leqslant m\}$ denote the Sobolev spaces endowed with the norm $\|v\|_{m,p}^p = \sum_{|\alpha| \leqslant m} \|D^\alpha v\|_{L^p(\Omega)}^p$, and the semi-norm $|v|_{m,p}^p = \sum_{|\alpha| = m} \|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$. Let $\|\cdot\|_{0,\infty}$ denote the maximum norm.

In the recent years, efficient numerical approximation methods play a critical role in solving optimal control problems. Finite element methods have been extensive studies in this aspect. Systematic introduction of the finite element methods for optimal control problems can be found in, for example, [9, 8]. In finite element literature, progress has been made in proving localized bounds by Schatz and Wahlbin in, for example, [26, 27]. In particular, Kwon and Milner [11] have studied L^∞-error estimates for mixed finite element methods for semilinear second-order elliptic equations, which directly relevant to our work. For optimal control problem governed by linear elliptic state equations, there are two early papers on the numerical approximation for linear control constrained problems [8]. Moreover, Meyer and Rösch have studied the superconvergence property for linear quadratic optimal control problem in [24]. Liu and Yan [20, 21] have derived a posteriori error estimates for finite element approximation of convex optimal control problems and boundary control problems.

Interpolated coefficients mixed finite element methods are economic and graceful methods. The interpolated coefficients finite element methods were introduced and analyzed for semilinear parabolic problems in Zlamal [31]. Later Larsson, Tomee and Zhang [12] studied the semi-discrete interpolation coefficients finite element methods for nonlinear heat equations. Chen et al. [4] presented optimal order convergence on piecewise uniform triangular meshes by use of superconvergence techniques. Xiong and Chen derived superconvergence of triangular finite element methods for semilinear elliptic problems in [29, 30].

Recently, in [23], we considered mixed finite element discretization for general semilinear optimal control problems. The state and co-state were approximated by lowest order Raviart-Thomas mixed finite element spaces, and the control was discretized by piecewise constant functions. A posteriori error estimates were derived for both the coupled state and the control solutions. In [22], we discussed L^∞-error estimates of mixed finite element methods for semilinear optimal control problem. In [7], we considered a bilinear constrained optimal control problem and obtained a priori error estimates and superconvergence of mixed finite element methods for the optimal control problem. However, the interpolated coefficients mixed finite element methods have not been studied and applied for optimal control problems. In this paper, we shall study the interpolation coefficients mixed finite element methods for optimal control problems governed by nonlinear elliptic equations and then derive L^∞-error estimates for the coupled state and control variables. The results seem to be new and are an important step towards developing efficient mixed finite element approximation for optimal control problems.

In the paper, we will transform the nonlinear elliptic optimal control problems into the optimality conditions, including the variational inequality, so we must solve the variational inequality carefully. A systematic introduction of the variational inequality can be found in [16, 17]. In [18, 10], the authors discussed a posteriori error estimates for some elliptic variational inequalities. The authors studied the moving mesh finite element approximations for a class of variational inequalities in [13]. By using some techniques to solve the variational inequality in those references, we can solve the nonlinear optimal control problems easily.

The outline of this paper is as follows. In Section 2, we construct the interpolation coefficients mixed finite element approximation for optimal control problem governed by nonlinear elliptic equations. In Section 3, we derive L^∞-error error estimates for the lowest order Raviart-Thomas mixed finite element approximation for the optimal control problem. Numerical examples are presented in Section 4.

2. Interpolation coefficients mixed methods

In this section, by using the interpolation operator I_h to process the nonlinear term $\phi(y_h)$ of equations, we present the interpolated coefficients mixed finite element discretization for optimal control problem governed by nonlinear equations (1.1)–(1.2).

Let $V = H(\mathrm{div};\Omega) = \{v \in (L^2(\Omega))^2, \mathrm{div}\, v \in L^2(\Omega)\}$ endowed with the norm given by $\|v\|_{H(\mathrm{div};\Omega)} = (\|v\|^2_{0,\Omega} + \|\mathrm{div}\, v\|^2_{0,\Omega})^{1/2}$. We denote $W = L^2(\Omega)$, $U = L^\infty(\Omega)$. We recast (1.1)–(1.2) as the following weak form: find $(p,y,u) \in V \times W \times U$ such that

$$\min_{u\in K\subset U} \left\{ \frac{1}{2}\|p-p_d\|^2 + \frac{1}{2}\|y-y_d\|^2 + \frac{1}{2}\|u\|^2 \right\} \tag{2.1}$$

$$(A^{-1}p,v) - (y,\mathrm{div}v) = 0, \qquad \forall v \in V, \tag{2.2}$$

$$(\mathrm{div}p,w) + (\phi(y),w) = (f+u,w), \qquad \forall w \in W. \tag{2.3}$$

It is well known (see e.g., [15]) that the optimal control problem (2.1)–(2.3) has at least a solution (p,y,u), and that if a triplet (p,y,u) is the solution of (2.1)–(2.3), then there is a co-state $(q,z) \in V \times W$ such that (p,y,q,z,u) satisfies the following optimality conditions:

$$(A^{-1}p,v) - (y,\mathrm{div}v) = 0, \qquad \forall v \in V, \tag{2.4}$$

$$(\mathrm{div}p,w) + (\phi(y),w) = (f+u,w), \qquad \forall w \in W, \tag{2.5}$$

$$(A^{-1}q,v) - (z,\mathrm{div}v) = -(p-p_d,v), \qquad \forall v \in V, \tag{2.6}$$

$$(\mathrm{div}q,w) + (\phi'(y)z,w) = (y-y_d,w), \qquad \forall w \in W, \tag{2.7}$$

$$(z+u,\tilde{u}-u)_U \geqslant 0, \qquad \forall \tilde{u} \in K, \tag{2.8}$$

where $(\cdot,\cdot)_U$ is the inner product of U. For simplify, the product $(\cdot,\cdot)_U$ will be denoted as $(\cdot,\cdot)$.

In order to state the control variable succinctly, we introduce the following projection [2]:

$$\mathrm{Proj}_{[\alpha(x),\beta(x)]}\left(g(x)\right)=\max\left(\alpha(x),\min(g(x),\beta(x))\right),\quad a.e.\quad x\in\Omega,\tag{2.9}$$

we can directly express the control from above optimality condition:

$$u(x)=\mathrm{Proj}_{[\alpha(x),\beta(x)]}\left(-z(x)\right).\tag{2.10}$$

Let $\mathscr{T}_h$ be regular triangulation of Ω, with boundary elements only allowed to have one curved side. They are assumed to satisfy the angle condition which means that there is a positive constant C such that for all $T\in\mathscr{T}_h$, $C^{-1}h_T^2\leqslant|T|\leqslant Ch_T^2$, where $|T|$ is the area of T and h_T is the diameter of T. Let $h=\max h_T$. In addition C or c denotes a general positive constant independent of h.

Let $V_h\times W_h\subset V\times W$ denote the k order $(k\geqslant 0)$ Raviart-Thomas space [5] associated with the triangulation $\mathscr{T}_h$ of Ω. We define

$$\begin{aligned}
V_h &:= \{v_h\in V:\forall T\in\mathscr{T}_h,\ v_h|_T\in P_k^2(T)+x\cdot P_k(T)\},\\
W_h &:= \{w_h\in W:\forall T\in\mathscr{T}_h,\ w_h|_T\in P_k(T)\},\\
K_h &:= \{\tilde{u}_h\in K:\forall T\in\mathscr{T}_h,\ \tilde{u}_h|_T\in P_k(T)\},
\end{aligned}$$

where P_k denotes the space of polynomials of total degree at most k. By the definition of finite element subspace, the mixed finite element discretization of (2.1)–(2.3) is as follows: compute $(p_h,y_h,u_h)\in V_h\times W_h\times K_h$ such that

$$\min_{u_h\in K_h}\left\{\frac{1}{2}\|p_h-p_d\|^2+\frac{1}{2}\|y_h-y_d\|^2+\frac{1}{2}\|u_h\|^2\right\}\tag{2.11}$$

$$(A^{-1}p_h,v_h)-(y_h,\mathrm{div}v_h)=0,\qquad\forall v_h\in V_h,\tag{2.12}$$

$$(\mathrm{div}p_h,w_h)+(\phi(y_h),w_h)=(f+u_h,w_h),\qquad\forall w_h\in W_h.\tag{2.13}$$

Define interpolating operator $I_h:C(\bar{\Omega})\to W_h$ by

$$I_hv=\sum_{j=1}^N v_j\varphi_j(x),$$

where $\{\varphi_j\}_{j=1}^N$ be the standard Lagrangian nodal basis of W_h. Since $y_h=\sum\limits_{j=1}^N y_j\varphi_j(x)$, then $\phi(y_h)=\phi\left(\sum\limits_{j=1}^N y_j\varphi_j(x)\right)$. By using the definition of the interpolating operator I_h, we have

$$I_h\phi(y_h)=\sum_{j=1}^N\phi(y_j)\varphi_j(x),\tag{2.14}$$

and the interpolation error estimate [12]: for $0\leqslant m\leqslant r$ and $1\leqslant p\leqslant\infty$ we have

$$\|v-I_hv\|_{m,p}\leqslant Ch^{r-m}\|v\|_{r,p},\tag{2.15}$$

where v belongs to $C(\bar{\Omega})\cap W^{r,p}(T)$ for all $T\in\mathscr{T}_h$. By substituting $I_h\phi(y_h)$ for $\phi(y_h)$ in (2.13), then the optimal control problem (2.11)–(2.13) again has at least a solution (p_h,y_h,u_h), and that if a triplet (p_h,y_h,u_h) is the solution of (2.11)–(2.13), then there is a co-state $(q_h,z_h)\in V_h\times W_h$ such that (p_h,y_h,q_h,z_h,u_h) satisfies the following optimality conditions:

$$(A^{-1}p_h,v_h)-(y_h,\mathrm{div}v_h)=0, \qquad \forall v_h\in V_h, \tag{2.16}$$
$$(\mathrm{div}p_h,w_h)+(I_h\phi(y_h),w_h)=(f+u_h,w_h), \qquad \forall w_h\in W_h, \tag{2.17}$$
$$(A^{-1}q_h,v_h)-(z_h,\mathrm{div}v_h)=-(p_h-p_d,v_h) \qquad \forall v_h\in V_h, \tag{2.18}$$
$$(\mathrm{div}q_h,w_h)+(\phi'(y_h)z_h,w_h)=(y_h-y_d,w_h), \qquad \forall w_h\in W_h, \tag{2.19}$$
$$(z_h+u_h,\tilde{u}_h-u_h)\geqslant 0, \qquad \forall\tilde{u}_h\in K_h. \tag{2.20}$$

For $\varphi\in W_h$, we shall write

$$\phi(\varphi)-\phi(\psi)=-\tilde{\phi}'(\varphi)(\psi-\varphi)=-\phi'(\psi)(\psi-\varphi)+\tilde{\phi}''(\varphi)(\psi-\varphi)^2, \tag{2.21}$$

where

$$\tilde{\phi}'(\varphi)=\int_0^1\phi'(\varphi+t(\psi-\varphi))dt,$$
$$\tilde{\phi}''(\varphi)=\int_0^1(1-t)\phi''(\psi+t(\varphi-\psi))dt \tag{2.22}$$

are bounded functions in $\bar{\Omega}$ [25].

Let $R_h:W\to W_h$ be the orthogonal L^2-projection into W_h define by [1, 6]:

$$(R_hw-w,\chi)=0, \qquad w\in W, \quad \chi\in W_h, \tag{2.23}$$

which satisfies

$$\|R_hw-w\|_{0,q}\leqslant C\|w\|_{t,q}h^t, \qquad 0\leqslant t\leqslant k+1,\ \text{if } w\in W\cap W^{t,q}(\Omega), \tag{2.24}$$
$$\|R_hw-w\|_{-r}\leqslant C\|w\|_th^{r+t}, \quad 0\leqslant r,t\leqslant k+1,\ \text{if } w\in H^t(\Omega), \tag{2.25}$$
$$(\mathrm{div}v,w-R_hw)=0, \qquad w\in W,\ v\in V_h. \tag{2.26}$$

Let $\Pi_h:V\to V_h$ be the Raviart-Thomas projection [25], which satisfies

$$(\mathrm{div}(\Pi_hv-v),w)=0, \qquad v\in V,\ w\in W_h, \tag{2.27}$$
$$\|\Pi_hv-v\|_{0,q}\leqslant C\|v\|_{t,q}h^t, \qquad 1/q<t\leqslant k+1,\ \text{if } v\in V\cap W^{t,q}(\Omega)^2, \tag{2.28}$$
$$\|\mathrm{div}(\Pi_hv-v)\|_{0,\infty}\leqslant C\|\mathrm{div}v\|_th^t, \quad 0\leqslant t\leqslant k+1,\ \text{if } v\in V\cap H^t(\mathrm{div};\Omega). \tag{2.29}$$

We have the commuting diagram property

$$\mathrm{div}\circ\Pi_h=R_h\circ\mathrm{div}:\ V\to W_h \quad\text{and}\quad \mathrm{div}(I-\Pi_h)V\perp W_h. \tag{2.30}$$

3. L^∞-error estimates

In this section, we will present L^∞-error estimates for the control variable and the state, co-state variables.

For any control function $u_h \in K_h$, we first define the state solution $(p(u_h), y(u_h), q(u_h), z(u_h))$ satisfies

$$(A^{-1}p(u_h), v) - (y(u_h), \mathrm{div}v) = 0, \qquad \forall v \in V, \tag{3.1}$$
$$(\mathrm{div}p(u_h), w) + (\phi(y(u_h)), w) = (f + u_h, w), \qquad \forall w \in W, \tag{3.2}$$
$$(A^{-1}q(u_h), v) - (z(u_h), \mathrm{div}v) = -(p(u_h) - p_d, v), \qquad \forall v \in V, \tag{3.3}$$
$$(\mathrm{div}q(u_h), w) + (\phi'(y(u_h))z(u_h), w) = (y(u_h) - y_d, w), \qquad \forall w \in W. \tag{3.4}$$

Let

$$\varepsilon_1 := p(u_h) - p_h, \quad r_1 := y(u_h) - y_h, \tag{3.5}$$
$$\varepsilon_2 := q(u_h) - q_h, \quad r_2 := z(u_h) - z_h. \tag{3.6}$$

From (2.16)–(2.19), (3.1)–(3.4), and (2.21), we have

$$(A^{-1}\varepsilon_1, v_h) - (r_1, \mathrm{div}v_h) = 0, \qquad \forall v_h \in V_h, \tag{3.7}$$
$$(\mathrm{div}\varepsilon_1, w_h) + (\tilde{\phi}'(y(u_h))r_1, w_h) = (\phi(y_h) - I_h\phi(y_h), w_h), \qquad \forall w_h \in W_h, \tag{3.8}$$
$$(A^{-1}\varepsilon_2, v_h) - (r_2, \mathrm{div}v_h) = -(\varepsilon_1, v_h), \qquad \forall v_h \in V_h, \tag{3.9}$$
$$(\mathrm{div}\varepsilon_2, w_h) + (\phi'(y(u_h))r_2, w_h) = (r_1, w_h) - (\tilde{\phi}''(y(u_h))z_h r_1, w_h), \qquad \forall w_h \in W_h. \tag{3.10}$$

By (3.7)–(3.10) and Theorem 3.1 in [25], we can establish the following error estimates.

LEMMA 3.1. *Let u_h be the solution of (2.20) and $(p(u_h), y(u_h), q(u_h), z(u_h))$ be the solution of (3.1)–(3.4), there is a positive constant C independent of h such that*

$$\|p(u_h) - p_h\|_{H(\mathrm{div};\Omega)} + \|y(u_h) - y_h\|_0 \leqslant Ch^{k+1}, \tag{3.11}$$
$$\|q(u_h) - q_h\|_{H(\mathrm{div};\Omega)} + \|z(u_h) - z_h\|_0 \leqslant Ch^{k+1}. \tag{3.12}$$

Similar to Theorem 3.1 in [25], we can establish the following error estimate.

THEOREM 3.1. *Let $(p, y, q, z, u) \in (V \times W)^2 \times K$ and $(p_h, y_h, q_h, z_h, u_h) \in (V_h \times W_h)^2 \times K_h$ be the solutions of (2.4)–(2.8) and (2.16)–(2.20), respectively. We assume that $u + z \in H^{k+1}(\Omega)$. Then, we have*

$$\|u - u_h\|_0 \leqslant Ch^{k+1}. \tag{3.13}$$

Proof. For the proof, the reader can consult Theorem 3.1 of [22]. □

Now, we introduce the weighted L^2-norms which will play a central role in our work to derive L^∞-error estimates. Let $x_0 \in \bar{\Omega}$ and $\rho > 0$. We define the weight function

$$\mu = |x - x_0|^2 + \rho^2, \quad x \in \bar{\Omega}. \tag{3.14}$$

For any $r \in \mathbb{R}$ we define the r-weighted norm by

$$\|v\|_{r,\mu} = \|\mu^{-\frac{r}{2}} v\|_0, \qquad v \in L^2(\Omega) \text{ or } (L^2(\Omega))^2. \tag{3.15}$$

By Lemma 3.1 in [11], we can obtain the following technical results.

LEMMA 3.2. *Let μ be given by (3.14), if $v \in (L^2(\Omega))^2$, then*

$$\|\nabla \mu^{-1} \cdot v\|_0 \leqslant C\rho^{-2}\|v\|_{1,\mu}. \tag{3.16}$$

LEMMA 3.3. *If $v \in (L^\infty(\Omega))^2$, then*

$$\|v\|_0 \leqslant C\|v\|_{1,\mu}. \tag{3.17}$$

Furthermore, we introduce the following relations between weighted L^2-norms and L^∞-norms and super-approximability results [28]:

$$\|v\|_{1,\mu} \leqslant C|\ln h|^{\frac{1}{2}}\|v\|_{0,\infty}, \quad v \in L^\infty(\Omega) \cap W_h, \tag{3.18}$$

$$\|\mu^{-1}\eta - \Pi_h(\mu^{-1}\eta)\|_{-1,\mu} \leqslant Ch^{k+1}\rho^{-1}\|\eta\|_{1,\mu}, \quad \eta \in V_h. \tag{3.19}$$

If $v \in W_h$ is a fixed element and $x_0 \in \bar{\Omega}$ is chosen so that $\|v\|_{0,\infty} = |v(x_0)|$, then

$$\|v\|_{0,\infty} \leqslant C_\kappa h^{-1}\rho\|v\|_{1,\mu}, \quad \text{for} \quad \rho \leqslant \kappa h. \tag{3.20}$$

Now we recall a priori regularity estimate for the following auxiliary problems:

$$-\mathrm{div}(A^*\nabla\xi) + \Upsilon\xi = g_1, \qquad x \in \Omega, \qquad \xi|_{\partial\Omega} = 0, \tag{3.21}$$

$$-\mathrm{div}(A\nabla\zeta) + \phi'(y(u_h))\zeta = g_2, \qquad x \in \Omega, \qquad \zeta|_{\partial\Omega} = 0. \tag{3.22}$$

where

$$\Upsilon = \begin{cases} \dfrac{\phi(y(u_h)) - \phi(y_h)}{y(u_h) - y_h}, & y(u_h) \neq y_h, \qquad (3.23a) \\ \phi'(y_h), & y(u_h) = y_h. \qquad (3.23b) \end{cases}$$

The next lemma gives the desired priori estimates. (See [19], for example.)

LEMMA 3.4. *Let ξ and ζ be the solutions of (3.21) and (3.22), respectively. Assume that Ω is convex, $A \in (W^{1,\infty}(\Omega))^{(2\times 2)}$, $X'AX \geqslant c\|X\|^2_{\mathbb{R}^2}$ for all $X \in \mathbb{R}^2$. Then*

$$\|\xi\|_{k+2} \leqslant C\|g_1\|_0, \tag{3.24}$$

$$\|\zeta\|_{k+2} \leqslant C\|g_2\|_0. \tag{3.25}$$

Now, we will prove two important theorems.

THEOREM 3.2. *Let (p, y, q, z) and $(p(u_h), y(u_h), q(u_h), z(u_h))$ be the solutions of (2.4)–(2.8) and (3.1)–(3.4), respectively. Then, we have*

$$\|R_h y(u_h) - y_h\|_0 + \|R_h z(u_h) - z_h\|_0 \leqslant Ch^{k+2}. \tag{3.26}$$

Proof. We only prove $\|R_h y(u_h) - y_h\|_0 \leqslant Ch^{k+2}$, the other part of (3.26) can be estimated in the same way. We can rewrite (3.7)–(3.8) as

$$(A^{-1}\varepsilon_1, v_h) - (R_h y(u_h) - y_h, \mathrm{div} v_h) = 0, \qquad \forall v_h \in V_h, \tag{3.27}$$
$$(\mathrm{div}\varepsilon_1, w_h) + (\phi(y(u_h)) - I_h\phi(y_h), w_h) = 0, \qquad \forall w_h \in W_h. \tag{3.28}$$

Then we have

$$(A^{-1}\varepsilon_1, v_h) - (R_h y(u_h) - y_h, \mathrm{div} v_h) = 0, \qquad \forall v_h \in V_h, \tag{3.29}$$
$$(\mathrm{div}\varepsilon_1, w_h) + (\tilde{\phi}'(y(u_h)) r_1, w_h) = (\phi(y_h) - I_h\phi(y_h), w_h), \qquad \forall w_h \in W_h. \tag{3.30}$$

Let $\tau = R_h y(u_h) - y_h$ and ξ be the solution of (3.21) with $g_1 = \tau$, then it follows from (2.27), (3.7)–(3.8), (3.21), and (3.27)–(3.28) that

$$\begin{aligned}
\|\tau\|_0^2 &= (\tau, -\mathrm{div}(A^*\nabla\xi) + \Upsilon\xi) \\
&= (\mathrm{div}\varepsilon_1, \xi) + (\tilde{\phi}'(y(u_h)) r_1, \xi) \\
&= (\mathrm{div}\varepsilon_1, \xi - R_h\xi) + (\tilde{\phi}'(y(u_h)) r_1, \xi - R_h\xi) + (\phi(y_h) - I_h\phi(y_h), R_h\xi).
\end{aligned} \tag{3.31}$$

We then estimate the two terms on the right side of (3.31). First, from Lemma 3.1 and (2.24) it follows that

$$(\mathrm{div}\varepsilon_1, \xi - R_h\xi) \leqslant \|\varepsilon_1\|_{H(\mathrm{div};\Omega)} \cdot \|\xi - R_h\xi\|_0 \leqslant Ch^{k+2}\|\tau\|_0. \tag{3.32}$$

Now, we estimate the second term

$$(\tilde{\phi}'(y(u_h)) r_1, \xi - R_h\xi) \leqslant C\|r_1\|_0 \cdot \|\xi - R_h\xi\|_0 \leqslant Ch^{k+2}\|\tau\|_0. \tag{3.33}$$

For the third term, we have

$$(\phi(y_h) - I_h\phi(y_h), R_h\xi) \leqslant C\|\phi(y_h) - I_h\phi(y_h)\|_0 \cdot \|R_h\xi\|_0 \leqslant Ch^{k+2}\|\tau\|_0. \tag{3.34}$$

Inserting (3.32) and (3.34) into (3.31) and we can deduce that $\|\tau\|_0 \leqslant Ch^{k+2}$, from which the theorem follows immediately. □

THEOREM 3.3. *Let (p, y, q, z) and $(p(u_h), y(u_h), q(u_h), z(u_h))$ be the solutions of (2.4)–(2.8) and (3.1)–(3.4), respectively. Then, we have*

$$\|\Pi_h p(u_h) - p_h\|_{0,\infty} + \|\Pi_h q(u_h) - q_h\|_{0,\infty} \leqslant Ch^{k+\frac{1}{2}}|\ln h|^{\frac{1}{2}}. \tag{3.35}$$

Proof. Let us denote $\sigma = \Pi_h p(u_h) - p_h$, we obtain

$$\begin{aligned}
\|\sigma\|_{1,\mu}^2 \leqslant & C(A^{-1}\sigma, \mu^{-1}\sigma) \\
\leqslant & C\{(A^{-1}\sigma, \mu^{-1}\sigma - \Pi_h(\mu^{-1}\sigma)) + (A^{-1}\varepsilon_1, \Pi_h(\mu^{-1}\sigma)) \\
& + (A^{-1}(\Pi_h p(u_h) - p(u_h)), \Pi_h(\mu^{-1}\sigma))\} \\
\leqslant & C\{h^{k+1}\rho^{-1}\|\sigma\|_{1,\mu} + (A^{-1}\varepsilon_1, \Pi_h(\mu^{-1}\sigma)) \\
& + |\ln h|^{\frac{1}{2}}(1 + h^{k+1}\rho^{-1}) \sup_T \|\Pi_h p(u_h) - p(u_h)\|_{0,\infty,T} \cdot \|\sigma\|_{1,\mu}\},
\end{aligned} \tag{3.36}$$

using ε-cauchy inequality, then we have

$$\begin{aligned}\|\sigma\|_{1,\mu}^2 &\leqslant C(A^{-1}\varepsilon_1,\Pi_h(\mu^{-1}\sigma))+C|\ln h|\sup_T\|\Pi_h p(u_h)-p(u_h)\|_{0,\infty,T}^2\\ &\leqslant C(A^{-1}\varepsilon_1,\Pi_h(\mu^{-1}\sigma))+Ch^{2k+2}|\ln h|. \end{aligned}\tag{3.37}$$

For the first term of the right hand of (3.37), integrating in polar coordinates, we obtain $\|\mu^{-1}\|_0\leqslant C\rho^{-1}$, thus using equation (3.7), we obtain

$$\begin{aligned}&(A^{-1}\varepsilon_1,\Pi_h(\mu^{-1}\sigma))=(r_1,\operatorname{div}\circ\Pi_h(\mu^{-1}\sigma))\\ =&(r_1,R_h\circ\operatorname{div}(\mu^{-1}\sigma))=(\tau,\operatorname{div}(\mu^{-1}\sigma))=(\tau,\nabla\mu^{-1}\sigma)+(\tau,\mu^{-1}\operatorname{div}\sigma)\\ \leqslant&\|\tau\|_0\cdot\|\nabla\mu^{-1}\sigma\|_0+\|\tau\|_0\cdot\|\mu^{-1}\|_0\cdot\|\operatorname{div}\sigma\|_{0,\infty}\\ \leqslant&Ch^{k+2}\left(\rho^{-2}\|\sigma\|_{1,\mu}+\rho^{-1}\cdot\|\operatorname{div}\sigma\|_{0,\infty}\right).\end{aligned}\tag{3.38}$$

Using (3.30) and definition of R_h, we can easily see that

$$R_h\circ\operatorname{div}\varepsilon_1=R_h\left[\phi(y_h)-I_h\phi(y_h)\right]-R_h\left[\tilde{\phi}'(y(u_h))r_1\right],\tag{3.39}$$

then, using (2.30), we can see that

$$\operatorname{div}\sigma=\operatorname{div}\circ\Pi_h\varepsilon_1=R_h\circ\operatorname{div}\varepsilon_1=R_h\left[\phi(y_h)-I_h\phi(y_h)\right]-R_h\left[\tilde{\phi}'(y(u_h))r_1\right],\tag{3.40}$$

thus we have

$$\begin{aligned}\|\operatorname{div}\sigma\|_{0,\infty}&\leqslant\left(\|\phi(y_h)-I_h\phi(y_h)\|_{0,\infty}+\|\tilde{\phi}'(y(u_h))r_1\|_{0,\infty}\right)\\ &\leqslant\left(Ch^{k+1}+\|r_1\|_{0,\infty}\right)\leqslant Ch^{k+1},\end{aligned}\tag{3.41}$$

where we used the priori estimate $\|r_1\|_{0,\infty}\leqslant Ch^{k+1}$, which was demonstrated in [25]. Inserting (3.41) to (3.38) yields the bound

$$(A^{-1}\varepsilon_1,\Pi_h(\mu^{-1}\sigma))\leqslant Ch^{k+2}\rho^{-2}\|\sigma\|_{1,\mu}+Ch^{k+3}\rho^{-1}.\tag{3.42}$$

Inserting (3.42) into (3.37), and using ε-Cauchy inequality, we have

$$\|\sigma\|_{1,\mu}^2\leqslant C(\varepsilon)h^{2k+2}|\ln h|+\varepsilon\|\sigma\|_{1,\mu}^2+Ch^2\rho^{-2}.\tag{3.43}$$

Let $h\rho^{-2}=C^{-2}$, that is to say $\rho=Ch^{\frac{1}{2}}$. Combining (3.20) and (3.43), h sufficiently small, then we have

$$\|\sigma\|_{0,\infty}\leqslant Ch^{-\frac{1}{2}}\|\sigma\|_{1,\mu}\leqslant Ch^{k+\frac{1}{2}}|\ln h|^{\frac{1}{2}}.\tag{3.44}$$

The proof of $\|\Pi_h q(u_h)-q_h\|_{0,\infty}\leqslant h^{k+\frac{1}{2}}|\ln h|^{\frac{1}{2}}$ is quite similar with above and we omitted here. □

Finally, we will give the L^∞-error estimates both for the control variable and the state variables.

THEOREM 3.4. *Let (p, y, q, z, u) and $(p_h, y_h, q_h, z_h, u_h)$ be the solutions of (2.4)–(2.8) and (2.16)–(2.20), respectively. Then, we have*

$$\|u-u_h\|_{0,\infty}+\|y-y_h\|_{0,\infty}+\|z-z_h\|_{0,\infty}\leqslant Ch^{k+1}, \tag{3.45}$$

$$\|p-p_h\|_{0,\infty}+\|q-q_h\|_{0,\infty}\leqslant Ch^{k+\frac{1}{2}}|\ln h|^{\frac{1}{2}}. \tag{3.46}$$

Proof. By (2.24)–(2.25), (3.26), (3.35), and the classical imbedding theorem $H^2(\Omega)\subset C(\bar{\Omega})$, we can see that

$$\begin{aligned}
&\|y-y_h\|_{0,\infty}+\|z-z_h\|_{0,\infty}\\
\leqslant&\|y-y(u_h)\|_{0,\infty}+\|y(u_h)-y_h\|_{0,\infty}+\|z-z(u_h)\|_{0,\infty}+\|z(u_h)-z_h\|_{0,\infty}\\
\leqslant&C\|y-y(u_h)\|_{C(\bar{\Omega})}+\|y(u_h)-R_hy(u_h)\|_{0,\infty}+\|R_hy(u_h)-y_h\|_{0,\infty}\\
&+C\|z-z(u_h)\|_{C(\bar{\Omega})}+\|z(u_h)-R_hz(u_h)\|_{0,\infty}+\|R_hz(u_h)-z_h\|_{0,\infty}\\
\leqslant&C\|y-y(u_h)\|_2+C\|z-z(u_h)\|_2+\|R_hy(u_h)-y_h\|_{0,\infty}+\|R_hz(u_h)-z_h\|_{0,\infty}+Ch^{k+1}\\
\leqslant&C\left(\|u-u_h\|_0+h^{-1}\|R_hy(u_h)-y_h\|_0+h^{-1}\|R_hz(u_h)-z_h\|_0+h^{k+1}\right)\\
\leqslant&Ch^{k+1}.
\end{aligned} \tag{3.47}$$

Similar to Theorem 4.1 in [22], we can obtain the following result

$$\|u-u_h\|_{0,\infty}\leqslant C\|z-z_h\|_{0,\infty}. \tag{3.48}$$

Combining (3.47) and (3.48), we have

$$\|u-u_h\|_{0,\infty}+\|y-y_h\|_{0,\infty}+\|z-z_h\|_{0,\infty}\leqslant Ch^{k+1}+\|z-z_h\|_{0,\infty}\leqslant Ch^{k+1}. \tag{3.49}$$

By (2.28), (3.26), (3.35), and the classical imbedding theorem $W^{2,3}(\Omega)\subset W^{1,\infty}(\Omega)$, we can see that

$$\begin{aligned}
&\|p-p_h\|_{0,\infty}+\|q-q_h\|_{0,\infty}\\
\leqslant&\|p-p(u_h)\|_{0,\infty}+\|p(u_h)-p_h\|_{0,\infty}+\|q-q(u_h)\|_{0,\infty}+\|q(u_h)-q_h\|_{0,\infty}\\
\leqslant&C\|\nabla y-\nabla y(u_h)\|_{0,\infty}+\|p(u_h)-\Pi_hp(u_h)\|_{0,\infty}\\
&+\|\Pi_hp(u_h)-p_h\|_{0,\infty}+\|\nabla(z-z(u_h))+p-p(u_h)\|_{0,\infty}\\
&+\|q(u_h)-\Pi_hq(u_h)\|_{0,\infty}+\|\Pi_hq(u_h)-q_h\|_{0,\infty}\\
\leqslant&C\left(\|y-y(u_h)\|_{2,3}+h^{k+1}+h^{k+\frac{1}{2}}|\ln h|^{\frac{1}{2}}\right)\\
\leqslant&C\left(\|u-u_h\|_{0,\infty}+h^{k+1}+h^{k+\frac{1}{2}}|\ln h|^{\frac{1}{2}}\right)\\
\leqslant&Ch^{k+\frac{1}{2}}|\ln h|^{\frac{1}{2}}.
\end{aligned} \tag{3.50}$$

Thus, we completed the proof. □

4. Numerical examples

In this section, we are going to validate the L^∞-error estimates for the errors in the control, state, and co-state numerically. The optimization problems were dealt numerically with codes developed based on AFEPACK [14]. Our numerical examples are the following optimal control problem:

$$\min_{u\in K}\left\{\frac{1}{2}\|p-p_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{1}{2}\|u\|^2\right\} \tag{4.1}$$

$$\mathrm{div}p+y^5=u+f,\qquad p=-\nabla y,\qquad x\in\Omega,\quad y|_{\partial\Omega}=0, \tag{4.2}$$

$$\mathrm{div}q+5y^4z=y-y_d,\qquad q=-\nabla z-p+p_d,\qquad x\in\Omega,\quad z|_{\partial\Omega}=0. \tag{4.3}$$

In our examples, we choose the domain $\Omega=[0,1]\times[0,1]$, $K=\{u\in L^\infty(\Omega):\alpha(x)\leqslant u(x)\leqslant\beta(x)\}$. The state and co-state are approximated by the lowest order Raviart-Thomas mixed finite element spaces and the control is approximated by piecewise constant functions. We present below two examples to illustrate the theoretical results for the nonlinear optimal control problem.

EXAMPLE 1. In first numerical example, we set

$$\alpha(x_1,x_2)=0.03+0.05\frac{|x_1-x_2|}{\sqrt{3}}, \tag{4.4}$$

$$\beta(x_1,x_2)=0.06+0.09\frac{|1-x_1-x_2|}{\sqrt{3}}. \tag{4.5}$$

We define

$$y(x)=\sin(\pi x_1)\sin(4\pi x_2), \tag{4.6}$$

thus the state variable $p(x)$ can be given by

$$p(x)=-\begin{pmatrix}\pi\cos(\pi x_1)\sin(4\pi x_2)\\ 4\pi\sin(\pi x_1)\cos(4\pi x_2)\end{pmatrix}, \tag{4.7}$$

and the source function $f(x)$ is given by

$$f(x)=\begin{cases} f_1(x)+y^5-\alpha(x), & \text{if } u_f(x)<\alpha(x), & (4.8a)\\ f_1(x)+y^5-u_f(x), & \text{if } u_f(x)\in[\alpha(x),\beta(x)], & (4.8b)\\ f_1(x)+y^5-\beta(x), & \text{if } u_f(x)>\beta(x), & (4.8c)\end{cases}$$

with $f_1(x_1,x_2)=17\sin(\pi x_1)\sin(4\pi x_2)$ and $u_f(x_1,x_2)=\sin(\pi x_1)\sin(4\pi x_2)$. Due to the state equation (4.2), we obtain for the exact control function u as follows:

$$u(x)=\begin{cases} \alpha(x), & \text{if } u_f(x)<\alpha(x), & (4.9a)\\ u_f(x), & \text{if } u_f(x)\in[\alpha(x),\beta(x)], & (4.9b)\\ \beta(x), & \text{if } u_f(x)>\beta(x). & (4.9c)\end{cases}$$

For the optimal co-state function z, we find

$$z(x) = -\sin(\pi x_1)\sin(4\pi x_2), \tag{4.10}$$

then the desired state variables can be given by

$$p_d(x) = 2\begin{pmatrix}\pi\cos(\pi x_1)\sin(4\pi x_2)\\ 4\pi\sin(\pi x_1)\cos(4\pi x_2)\end{pmatrix},$$
$$y_d(x) = y + 68\sin(\pi x_1)\sin(4\pi x_2) - 5y^4 z. \tag{4.11}$$

Table 1: *The numerical errors on uniformly triangle mesh grid.*

resolution	$\|u-u_h\|_{0,\infty}$	$\|y-y_h\|_{0,\infty}$	$\|z-z_h\|_{0,\infty}$	$\|p-p_h\|_{0,\infty}$	$\|q-q_h\|_{0,\infty}$
16×16	5.35785×10^{-2}	2.38136×10^{-1}	2.46124×10^{-1}	1.72316×10^{0}	1.72328×10^{0}
32×32	2.68118×10^{-2}	1.19067×10^{-1}	1.23062×10^{-1}	1.21766×10^{0}	1.21771×10^{0}
64×64	1.32934×10^{-2}	5.95342×10^{-2}	6.09488×10^{-2}	8.60948×10^{-1}	8.60952×10^{-1}
128×128	6.64141×10^{-3}	2.97656×10^{-2}	3.03198×10^{-2}	6.09081×10^{-1}	6.09083×10^{-1}

The profile of the numerical solution u is plotted in Figure 1. In this numerical implementation, the errors $\|u-u_h\|_{0,\infty}$, $\|y-y_h\|_{0,\infty}$, $\|z-z_h\|_{0,\infty}$, $\|p-p_h\|_{0,\infty}$ and $\|q-q_h\|_{0,\infty}$ obtained on a sequence of uniformly refined triangle meshes are presented in Table 1. We show the convergence orders by slopes in Figure 2. The theoretical results can be observed clearly from the data.

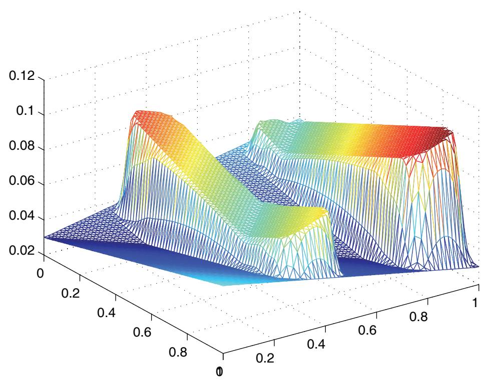

Figure 1: *The profile of the numerical control solution* u *on* 64×64 *mesh grids.*

To show the efficiency of interpolated coefficients mixed finite element methods, we give a numerical comparison with classical mixed finite element methods in Table 2. It is clear that the interpolated coefficients mixed finite element methods are able to save substantial computational time, in comparison with classical mixed methods.

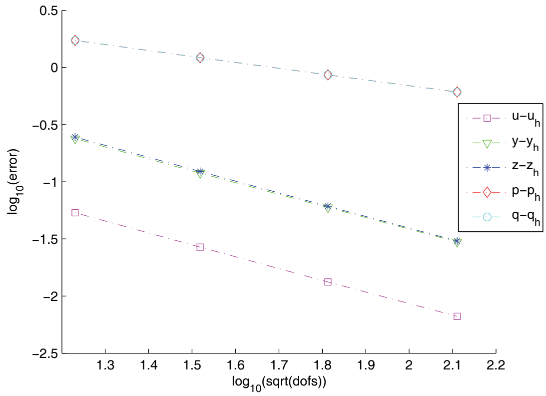

Figure 2: *Convergence orders of* $u-u_h$, $p-p_h$, $y-y_h$, $q-q_h$, *and* $z-z_h$ *on triangle meshes.*

Table 2: *CPU times on classical mixed methods and interpolation coefficients mixed methods.*

resolution	CPU times	
	Classical mixed methods	Interpolation coefficients mixed methods
16×16	11.4s	6.8s
32×32	54.6s	28.4s
64×64	352.9s	148.7s
128×128	2294.9s	802.2s

EXAMPLE 2. In the second example, we set

$$\alpha(x_1,x_2)=0.03+0.05\frac{|x_1-x_2|}{\sqrt{2}}, \tag{4.12}$$

$$\beta(x_1,x_2)=0.05+0.07\frac{|1-x_1-x_2|}{\sqrt{2}}. \tag{4.13}$$

We define

$$y(x)=x_1x_2(1-x_1)(1-x_2), \tag{4.14}$$

thus the state variable p can be given by

$$p(x)=-\begin{pmatrix}(1-2x_1)x_2(1-x_2)\\(1-2x_2)x_1(1-x_1)\end{pmatrix}, \tag{4.15}$$

and

$$f(x)=\begin{cases} f_1(x)+y^5-\alpha(x), & \text{if } u_f(x)<\alpha(x), \quad (4.16a)\\ f_1(x)+y^5-u_f(x), & \text{if } u_f(x)\in[\alpha(x),\beta(x)], \quad (4.16b)\\ f_1(x)+y^5-\beta(x), & \text{if } u_f(x)>\beta(x), \quad (4.16c)\end{cases}$$

with $f_1(x_1,x_2) = 2x_1(1-x_1)+2x_2(1-x_2)$ and $u_f(x_1,x_2) = -2x_1x_2(1-x_1)(1-x_2)$. Due to the state equation (4.2), we obtain for the exact control function u as follows:

$$u(x) = \begin{cases} \alpha(x), & \text{if } u_f(x) < \alpha(x), & \text{(4.17a)} \\ u_f(x), & \text{if } u_f(x) \in [\alpha(x),\beta(x)], & \text{(4.17b)} \\ \beta(x), & \text{if } u_f(x) > \beta(x). & \text{(4.17c)} \end{cases}$$

For the optimal co-state function $z(x)$, we find

$$z(x) = 2x_1x_2(1-x_1)(1-x_2), \tag{4.18}$$

then the desired state variables can be given by

$$\begin{aligned} p_d(x) &= 3\begin{pmatrix} (1-2x_1)x_2(1-x_2) \\ (1-2x_2)x_1(1-x_1) \end{pmatrix}, \\ y_d(x) &= y+4x_1(1-x_1)+4x_2(1-x_2)-5y^4z. \end{aligned} \tag{4.19}$$

Table 3: *The numerical errors on uniformly triangle mesh grid.*

resolution	$\|u-u_h\|_{0,\infty}$	$\|y-y_h\|_{0,\infty}$	$\|z-z_h\|_{0,\infty}$	$\|p-p_h\|_{0,\infty}$	$\|q-q_h\|_{0,\infty}$
16×16	3.26518×10^{-3}	4.94943×10^{-3}	4.94122×10^{-3}	1.41383×10^{-1}	1.41374×10^{-1}
32×32	1.67748×10^{-3}	2.54454×10^{-3}	2.53685×10^{-3}	1.01172×10^{-1}	1.01171×10^{-1}
64×64	8.49715×10^{-4}	1.28784×10^{-3}	1.28519×10^{-3}	7.18876×10^{-2}	7.18874×10^{-2}
128×128	4.19911×10^{-4}	6.47606×10^{-4}	6.46848×10^{-4}	5.09402×10^{-2}	5.09403×10^{-2}

In this numerical example, the profile of the numerical solution is presented in Figure 3. From the error data on the uniform refined triangle meshes, as listed in Table 3, it can be seen that the L^∞-error estimates remain in our data. A numerical comparison with classical mixed finite element methods has been given in Table 4. It is shown from Table 4 that the CPU times have reduced obviously. Furthermore we also show the convergence orders by slopes in Figure 4, the convergence order for the coupled state and control variables can be observed clearly.

Table 4: *CPU times on classical mixed methods and interpolation coefficients mixed methods.*

resolution	CPU times	
	Classical mixed methods	Interpolation coefficients mixed methods
16×16	4.8s	2.1s
32×32	12.1s	3.9s
64×64	49.4s	10.6s
128×128	255.1s	42.3s

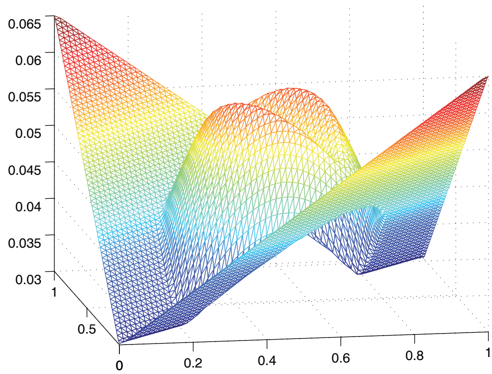

Figure 3: *The profile of the numerical control solution* u *on* 64×64 *mesh grids.*

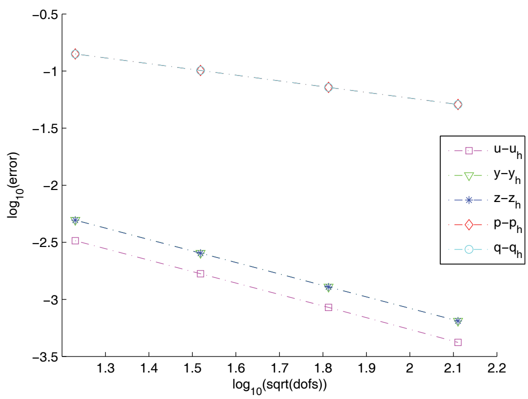

Figure 4: *Convergence orders of* $u-u_h$, $p-p_h$, $y-y_h$, $q-q_h$, *and* $z-z_h$ *on triangle meshes.*

From the two above numerical examples, we can find that the numerical results demonstrate our theoretical results.

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