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MEAN CENTRAL DISTANCE—CENTRAL DISTANCE INEQUALITIES

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Abstract. By means of the analysis, convex geometry, computer and majorization theories, in the centered 2-surround system $S^{(2)}\{P,\Gamma,l\}$, we establish the following mean central distance–central distance inequalities:

$$\frac{\exp\left(\frac{1}{|\Gamma|}\oint_\Gamma \log \overline{r}_P\right)}{\exp\left(\frac{1}{|\Gamma|}\oint_\Gamma \log r_P\right)} \geqslant \frac{1}{2}\left[\sec\frac{l\pi}{|\Gamma|}+\cot\frac{l\pi}{|\Gamma|}\log\left(\tan\frac{l\pi}{|\Gamma|}+\sec\frac{l\pi}{|\Gamma|}\right)\right]$$

and

$$\frac{\left(\frac{1}{|\Gamma|}\oint_\Gamma \overline{r}_P^2\right)^{1/2}}{\frac{1}{|\Gamma|}\oint_\Gamma r_P} \geqslant \frac{1}{2}\left[\sec\frac{l\pi}{|\Gamma|}+\cot\frac{l\pi}{|\Gamma|}\log\left(\tan\frac{l\pi}{|\Gamma|}+\sec\frac{l\pi}{|\Gamma|}\right)\right] \text{ when } 0<\angle APA_+\leqslant\tau,$$

where $\tau = 2.49342812654089\ldots$, and $\tau/2$ is the unique real root of the following equation:

$$\frac{\mathrm{d}^2[\sec\theta+\cot\theta\log(\tan\theta+\sec\theta)]}{\mathrm{d}\theta^2}=0,\ \theta\in\left(0,\frac{\pi}{2}\right).$$

We also demonstrate the applications of our results, and obtain the N-mean central distance – central distance inequality and the mean central distance–central distance–limit inequality.

1. Introduction

We begin by recalling some of the basic concepts as follows [1, 2, 3, 4].

Let $\gamma: I \to \mathbb{R}^2$ be a continuous function, where $I \subset \mathbb{R}$ is an interval, and let the image

$$\Gamma \triangleq \gamma(I) = \left\{\gamma(t)\in\mathbb{R}^2 | \gamma(t) = x(t)\mathbf{i}+y(t)\mathbf{j},\ t\in I\right\}$$

of γ be a smooth curve, that is, the derivatives $x'(t)$ and $y'(t)$ are continuous, and the derivative of the vector $\gamma(t)$ satisfies the condition

$$\gamma'(t) \triangleq x'(t)\mathbf{i}+y'(t)\mathbf{j}\neq\mathbf{0} \Leftrightarrow \|\gamma'(t)\| \triangleq \sqrt{[x'(t)]^2+[y'(t)]^2}>0,\ \forall t\in I,$$

here

$$\mathbf{0}\triangleq(0,0),\ \mathbf{i}\triangleq(1,0),\ \mathbf{j}\triangleq(0,1),\ \mathbb{R}\triangleq(-\infty,\infty),\ \mathbb{R}^2\triangleq\mathbb{R}\times\mathbb{R}.$$

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Then the length $|\Gamma|$ of the curve Γ exists and

$$0<|\Gamma|\triangleq\int_I\|\gamma'(t)\|\mathrm{d}t=\int_I\sqrt{[x'(t)]^2+[y'(t)]^2}\mathrm{d}t\leqslant\infty,$$

and $0<|\Gamma|<\infty$ if $0<|I|<\infty$, where $|I|$ is the measure of the interval I.

In this paper, we assume that Γ is a smooth and convex Jordan closed curve in $\mathbb{R}^2$ [1, 2, 3, 4]. Then

$$\Gamma\triangleq\gamma(\mathbb{R})=\left\{\gamma(t)\in\mathbb{R}^2|\gamma(t)=x(t)\mathbf{i}+y(t)\mathbf{j},\ t\in\mathbb{R}\right\}\text{ and }\gamma(t)\equiv\gamma(t+|\Gamma|),\ \forall t\in\mathbb{R},$$

that is, $\gamma(t)$ is a periodic function with the period $|\Gamma|$, where the parameter t is the natural parameter, that is,

$$0<l\leqslant|\Gamma|\Rightarrow|\gamma([t,t+l])|\triangleq\int_t^{t+l}\sqrt{[x'(t)]^2+[y'(t)]^2}\mathrm{d}t=l,\ \forall t\in\mathbb{R}.$$

We denote by $D(\Gamma)$ the convex region enclosed by the Jordan closed curve Γ and, we also define

$$A_-\triangleq\gamma(t_A-l),\ A\triangleq\gamma(t_A)\text{ and }A_+\triangleq\gamma(t_A+l),\ t_A\in\mathbb{R}. \tag{1}$$

If l is a fixed real number and $0<l<|\Gamma|/2$, then we say that the plane point set

$$D(\Gamma,l)\triangleq\bigcap_{A\in\Gamma}\widehat{A_-AA_+}\subset D(\Gamma)\subset\mathbb{R}^2$$

is an *l-central region* of the curve Γ, where the *angular region*

$$\widehat{A_-AA_+}\triangleq\{(1-\lambda)\gamma(t_A)+\lambda\gamma(t)|0<\lambda<\infty,\ t_A+l<t<t_A-l+|\Gamma|\}.$$

Let the l-central region $D(\Gamma,l)$ be non-empty and the fixed point $P\in D(\Gamma,l)$. We say that the set

$$S^{(2)}\{P,\Gamma,l\}\triangleq\{P,\Gamma,l\}$$

is a *centered 2-surround system* or *centered 2-satellite system*, P is a *center* and $A,A_+\in\Gamma$ are two *satellites* of the system.

For the centered 2-surround system $S^{(2)}\{P,\Gamma,l\}$, we may think of the point P as the center of the earth, Γ as the orbit of two satellites A,A_+. In order to avoid hitting, the satellites A,A_+ must move by same curve velocity, that is,

$$l\triangleq|\gamma([t_A,t_A+l])|\in\left(0,\frac{|\Gamma|}{2}\right)$$

is invariable. This is the significance of the centered 2-surround system $S^{(2)}\{P,\Gamma,l\}$ in the theory of satellite.

For centered 2-surround system $S^{(2)}\{P,\Gamma,l\}$, we let P' denote the projection of the point P in the line AA_+, and we say that the distance

$$r_P\triangleq\mathrm{Distance}(P,AA_+)=\|P'-P\|$$

from the point P to line AA_+ is a *central distance* of the system, and the positive real number

$$\overline{r}_P \triangleq \frac{1}{\|A_+ - A\|} \int_{M\in[AA_+]} \|M-P\|$$

is a *mean central distance* of the system, which is the mean of the distance between the point P and the point M in the straight line segment $[AA_+]$, see Figure 1.

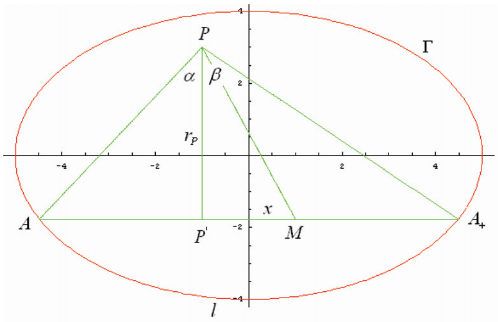

Figure 1: *The graph of the centered* 2*-surround system* $S^{(2)}\{P,\Gamma,l\}$.

Let $f:\Gamma\to(0,\infty)$ be a continuous function [5, 6] defined on the smooth Jordan closed curve Γ. Then the functional

$$\mathrm{M}_{\Gamma}^{[p]}(f) \triangleq \begin{cases} \left(\dfrac{1}{|\Gamma|}\displaystyle\oint_{\Gamma} f^p\right)^{1/p}, & p\in\mathbb{R},\ p\neq 0 \\ \exp\left(\dfrac{1}{|\Gamma|}\displaystyle\oint_{\Gamma}\log f\right), & p=0 \end{cases}$$

is called the p*-mean* (or p*-power mean*) of the function f [7, 8, 9, 10, 11, 12], where $\oint_\Gamma$ is the *curve integral* and,

$$\mathrm{M}_{\Gamma}(f) \triangleq \mathrm{M}_{\Gamma}^{[1]}(f) = \frac{1}{|\Gamma|}\oint_{\Gamma} f \text{ and } \mathrm{G}_{\Gamma}(f) \triangleq \mathrm{M}_{\Gamma}^{[0]}(f) = \exp\left(\frac{1}{|\Gamma|}\oint_{\Gamma}\log f\right)$$

are the *mean* and the *geometric mean* of the function f, respectively.

The theory of satellite is important in space science. In [1, 2, 3, 4, 13], the authors systematically studied the theory of satellite and obtained some useful results.

In the convex geometry, a well-known isoperimetric inequality can be expressed as: If Γ is a smooth Jordan closed curve, then we have

$$\mathrm{Area}D(\Gamma) \leqslant \frac{|\Gamma|^2}{4\pi}, \tag{2}$$

where $\mathrm{Area}D(\Gamma)$ denote the area of the region $D(\Gamma)$. Equality in (2) holds if and only if Γ is a circle.

In the convex geometry, a large number of isoperimetric inequalities similar to (2) had been obtained [1, 2, 3, 4, 14, 15, 16].

In this paper, by means of the *theory of majorization* [17], we will study the sharp lower bounds of $\mathrm{G}_\Gamma(\overline{r}_P)/\mathrm{G}_\Gamma(r_P)$ and $\mathrm{M}_\Gamma^{[2]}(\overline{r}_P)/\mathrm{M}_\Gamma(r_P)$, and establish two new isoperimetric inequalities in the centered surround system $S^{(2)}\{P,\Gamma,l\}$.

Our main results are as follows.

THEOREM 1.1. (Mean central distance–central distance inequality) *Let $S^{(2)}\{P,\Gamma,l\}$ be a centered 2-surround system. Then we have the following isoperimetric inequality:*

$$\frac{\mathrm{G}_\Gamma(\overline{r}_P)}{\mathrm{G}_\Gamma(r_P)} \geqslant \frac{1}{2}\left[\sec\frac{l\pi}{|\Gamma|}+\cot\frac{l\pi}{|\Gamma|}\log\left(\tan\frac{l\pi}{|\Gamma|}+\sec\frac{l\pi}{|\Gamma|}\right)\right]. \tag{3}$$

Equality in (3) holds if and only if Γ is a circle and P is the center of the circle.

THEOREM 1.2. (Mean central distance–central distance inequality) *Let $S^{(2)}\{P,\Gamma,l\}$ be a centered 2-surround system. If*

$$A\in\Gamma \Rightarrow 0<\angle APA_+ \leqslant \tau = 2.49342812654089\ldots,$$

where $\tau/2$ is the unique real root of the equation:

$$\frac{\mathrm{d}^2[\sec\theta+\cot\theta\log(\tan\theta+\sec\theta)]}{\mathrm{d}\theta^2}=0,\ \theta\in\left(0,\frac{\pi}{2}\right), \tag{4}$$

then we have the following isoperimetric inequality:

$$\frac{\mathrm{M}_\Gamma^{[2]}(\overline{r}_P)}{\mathrm{M}_\Gamma(r_P)} \geqslant \frac{1}{2}\left[\sec\frac{l\pi}{|\Gamma|}+\cot\frac{l\pi}{|\Gamma|}\log\left(\tan\frac{l\pi}{|\Gamma|}+\sec\frac{l\pi}{|\Gamma|}\right)\right]. \tag{5}$$

Equality in (5) holds if and only if Γ is a circle and P is the center of the circle.

In Section 5, we will demonstrate the applications of Theorems 1.1 and 1.2.

We remark here that, the relevant calculations in this paper are dependent on the Mathematica software since these calculations are very complex.

2. Preliminaries

In order to prove Theorems 1.1 and 1.2, we need to establish several identities and inequalities involving the centered 2-surround system as follows.

LEMMA 2.1. (See Lemma 4 in [4]) *Let $S^{(2)}\{P,\Gamma,l\}$ be a centered 2-surround system. Then we have the following identity:*

$$\oint_\Gamma \angle A_-PA_+ = 2l\pi. \tag{6}$$

LEMMA 2.2. (Hadmard inequality [18]) *Let the function* $\varphi : [a,b] \to \mathbb{R}$ *be a continuous convex function. Then we have*

$$\frac{\varphi(a)+\varphi(b)}{2} \geqslant \frac{1}{b-a}\int_a^b \varphi(x)\mathrm{d}x \geqslant \varphi\left(\frac{a+b}{2}\right). \tag{7}$$

LEMMA 2.3. *Let* $\alpha,\beta \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ *and* $\alpha+\beta \in (0,\pi)$. *Then we have*

$$\frac{\int_{-\tan\alpha}^{\tan\beta}\sqrt{x^2+1}\mathrm{d}x}{\tan\alpha+\tan\beta} \geqslant \tan\frac{\alpha+\beta}{2}\int_0^1\sqrt{x^2+\cot^2\frac{\alpha+\beta}{2}}\mathrm{d}x. \tag{8}$$

Equality in (8) holds if and only if $\alpha=\beta$.

Proof. This proof is based on the *theory of majorization* [17, 19, 20]. Let

$$F(\alpha,\beta) \triangleq \frac{\int_{-\tan\alpha}^{\tan\beta}\sqrt{x^2+1}\mathrm{d}x}{\tan\alpha+\tan\beta} \text{ and } E \triangleq \left\{(\alpha,\beta)\,|\,\alpha,\beta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right),\ \alpha+\beta\in(0,\pi)\right\}.$$

Then

$$F(\alpha,\beta)=F(\beta,\alpha),\ \forall\,(\alpha,\beta)\in E. \tag{9}$$

Let $x=-t$. Then we have

$$\begin{aligned} F(\alpha,\beta) &= \frac{\int_{-\tan\alpha}^{\tan\beta}\sqrt{x^2+1}\mathrm{d}x}{\tan\alpha+\tan\beta} = \frac{\int_{\tan\alpha}^{-\tan\beta}\sqrt{(-t)^2+1}\mathrm{d}(-t)}{\tan\alpha+\tan\beta} \\ &= -\frac{\int_{\tan\alpha}^{-\tan\beta}\sqrt{t^2+1}\mathrm{d}t}{\tan\alpha+\tan\beta} = \frac{\int_{-\tan\beta}^{\tan\alpha}\sqrt{t^2+1}\mathrm{d}t}{\tan\alpha+\tan\beta} \\ &= \frac{\int_{-\tan\beta}^{\tan\alpha}\sqrt{x^2+1}\mathrm{d}x}{\tan\beta+\tan\alpha} = F(\beta,\alpha). \end{aligned}$$

That is, (9) holds.

Now we prove that the symmetric function $F(\alpha,\beta)$ is a Schur-convex function [17, 19] on the symmetric convex set E. By the theory of majorization, we just need to prove that

$$(\alpha-\beta)\left(\frac{\partial F}{\partial\alpha}-\frac{\partial F}{\partial\beta}\right) \geqslant 0,\ \forall(\alpha,\beta)\in E. \tag{10}$$

Indeed, since

$$\frac{\mathrm{d}\tan\alpha}{\mathrm{d}\alpha}=\sec^2\alpha,\ \frac{\mathrm{d}\tan\beta}{\mathrm{d}\beta}=\sec^2\beta,$$

and

$$(\alpha,\beta)\in E \Rightarrow \sqrt{\tan^2\alpha+1}=\sec\alpha \text{ with } \sqrt{\tan^2\beta+1}=\sec\beta,$$

we have

$$\frac{\partial}{\partial\alpha}\int_{-\tan\alpha}^{\tan\beta}\sqrt{x^2+1}\mathrm{d}x=\sec^3\alpha \text{ and } \frac{\partial}{\partial\beta}\int_{-\tan\alpha}^{\tan\beta}\sqrt{x^2+1}\mathrm{d}x=\sec^3\beta.$$

Hence

$$\begin{aligned}\frac{\partial F}{\partial \alpha} &= \frac{\partial}{\partial \alpha}\frac{\int_{-\tan\alpha}^{\tan\beta}\sqrt{x^2+1}\mathrm{d}x}{\tan\alpha+\tan\beta}\\ &= \frac{(\tan\alpha+\tan\beta)\frac{\partial}{\partial\alpha}\int_{-\tan\alpha}^{\tan\beta}\sqrt{x^2+1}\mathrm{d}x-\frac{\partial}{\partial\alpha}(\tan\alpha+\tan\beta)\int_{-\tan\alpha}^{\tan\beta}\sqrt{x^2+1}\mathrm{d}x}{(\tan\alpha+\tan\beta)^2}\\ &= \frac{(\tan\alpha+\tan\beta)\sec^3\alpha-\sec^2\alpha\int_{-\tan\alpha}^{\tan\beta}\sqrt{x^2+1}\mathrm{d}x}{(\tan\alpha+\tan\beta)^2},\end{aligned}$$

that is,

$$\frac{\partial F}{\partial \alpha} = \frac{(\tan\alpha+\tan\beta)\sec^3\alpha-\sec^2\alpha\int_{-\tan\alpha}^{\tan\beta}\sqrt{x^2+1}\mathrm{d}x}{(\tan\alpha+\tan\beta)^2}. \tag{11}$$

Similarly, from (9), we have

$$\frac{\partial F}{\partial \beta} = \frac{(\tan\alpha+\tan\beta)\sec^3\beta-\sec^2\beta\int_{-\tan\alpha}^{\tan\beta}\sqrt{x^2+1}\mathrm{d}x}{(\tan\alpha+\tan\beta)^2}. \tag{12}$$

Set $\tan\alpha = u,\ \tan\beta = v$. Then, by (11), (12) and

$$\sec^2\alpha = \tan^2\alpha+1 = u^2+1,\ \sec^2\beta = \tan^2\beta+1 = v^2+1,\ u+v>0,$$

we get

$$\begin{aligned}&\frac{\partial F}{\partial \alpha}-\frac{\partial F}{\partial \beta}\\ &= \frac{\left(\sec^3\alpha-\sec^3\beta\right)(\tan\alpha+\tan\beta)-\left(\sec^2\alpha-\sec^2\beta\right)\int_{-\tan\alpha}^{\tan\beta}\sqrt{x^2+1}\mathrm{d}x}{(\tan\alpha+\tan\beta)^2}\\ &= \left(\sec^2\alpha-\sec^2\beta\right)\frac{\frac{\sec^3\alpha-\sec^3\beta}{\sec^2\alpha-\sec^2\beta}(\tan\alpha+\tan\beta)-\int_{-\tan\alpha}^{\tan\beta}\sqrt{x^2+1}\mathrm{d}x}{(\tan\alpha+\tan\beta)^2}\\ &= \frac{\tan^2\alpha-\tan^2\beta}{(\tan\alpha+\tan\beta)^2}\left[\frac{\sec^2\alpha+\sec^2\beta+\sec\alpha\sec\beta}{\sec\alpha+\sec\beta}(\tan\alpha+\tan\beta)-\int_{-\tan\alpha}^{\tan\beta}\sqrt{x^2+1}\mathrm{d}x\right]\\ &= \frac{\tan\alpha-\tan\beta}{\tan\alpha+\tan\beta}\left[\frac{\sec^2\alpha+\sec^2\beta+\sec\alpha\sec\beta}{\sec\alpha+\sec\beta}(\tan\alpha+\tan\beta)-\int_{-\tan\alpha}^{\tan\beta}\sqrt{x^2+1}\mathrm{d}x\right]\\ &= \frac{u-v}{u+v}\left[\frac{u^2+v^2+2+\sqrt{(u^2+1)(v^2+1)}}{\sqrt{u^2+1}+\sqrt{v^2+1}}(u+v)-\int_{-u}^{v}\sqrt{x^2+1}\mathrm{d}x\right]\\ &= (u-v)\left[\frac{u^2+v^2+2+\sqrt{(u^2+1)(v^2+1)}}{\sqrt{u^2+1}+\sqrt{v^2+1}}-\frac{\int_{-u}^{v}\sqrt{x^2+1}\mathrm{d}x}{u+v}\right],\end{aligned}$$

that is,

$$(\alpha-\beta)\left(\frac{\partial F}{\partial \alpha}-\frac{\partial F}{\partial \beta}\right) = (\alpha-\beta)(u-v)G(u,v), \tag{13}$$

where

$$G(u,v) \triangleq \frac{u^2+v^2+2+\sqrt{(u^2+1)(v^2+1)}}{\sqrt{u^2+1}+\sqrt{v^2+1}} - \frac{\int_{-u}^{v}\sqrt{x^2+1}\mathrm{d}x}{u+v}.$$

Since

$$(\alpha-\beta)(u-v) = \frac{(\alpha-\beta)\sin(\alpha-\beta)}{\cos\alpha\cos\beta} \geqslant 0,\ \forall(\alpha,\beta)\in E,$$

inequality (13) is equivalent to the inequality

$$G(u,v) \geqslant 0,\ \forall u,v\in\mathbb{R},\ u+v>0. \tag{14}$$

Let

$$\varphi:\mathbb{R}\to\mathbb{R},\ \varphi(x)\triangleq\sqrt{x^2+1}. \tag{15}$$

Then

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x} = \frac{x}{\sqrt{x^2+1}} \text{ and } \frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = \frac{1}{(x^2+1)^{3/2}} > 0.$$

Hence the function φ is a continuous convex function on $\mathbb{R}$. By Lemma 2.2, we have

$$\frac{\int_{-u}^{v}\sqrt{x^2+1}\mathrm{d}x}{u+v} \leqslant \frac{\varphi(-u)+\varphi(v)}{2} = \frac{1}{2}\left(\sqrt{u^2+1}+\sqrt{v^2+1}\right). \tag{16}$$

By (16), we get

$$\begin{aligned}
G(u,v) &= \frac{u^2+v^2+2+\sqrt{(u^2+1)(v^2+1)}}{\sqrt{u^2+1}+\sqrt{v^2+1}} - \frac{\int_{-u}^{v}\sqrt{x^2+1}dx}{u+v}\\
&\geqslant \frac{u^2+v^2+2+\sqrt{(u^2+1)(v^2+1)}}{\sqrt{u^2+1}+\sqrt{v^2+1}} - \frac{1}{2}\left(\sqrt{u^2+1}+\sqrt{v^2+1}\right)\\
&= \frac{u^2+v^2+2}{2(\sqrt{u^2+1}+\sqrt{v^2+1})}\\
&> 0.
\end{aligned}$$

Hence (13) and (14) are proved.

Since the symmetric function $F(\alpha,\beta)$ is a Schur-convex function on the symmetric convex set E, and

$$(\alpha,\beta) \succ \left(\frac{\alpha+\beta}{2},\frac{\alpha+\beta}{2}\right),$$

by the definition of the Schur-convex function [17, 19], we have

$$\begin{aligned}
F(u,v) \geqslant F\left(\frac{\alpha+\beta}{2},\frac{\alpha+\beta}{2}\right) &= \frac{1}{2\tan\frac{\alpha+\beta}{2}}\int_{-\tan\frac{\alpha+\beta}{2}}^{\tan\frac{\alpha+\beta}{2}}\sqrt{x^2+1}\mathrm{d}x\\
&= \frac{1}{\tan\frac{\alpha+\beta}{2}}\int_{0}^{\tan\frac{\alpha+\beta}{2}}\sqrt{x^2+1}\mathrm{d}x
\end{aligned}$$

$$= \tan\frac{\alpha+\beta}{2}\int_0^{\tan\frac{\alpha+\beta}{2}}\sqrt{\left(x\cot\frac{\alpha+\beta}{2}\right)^2+\cot^2\frac{\alpha+\beta}{2}}\mathrm{d}\left(x\cot\frac{\alpha+\beta}{2}\right)$$
$$= \tan\frac{\alpha+\beta}{2}\int_0^1\sqrt{t^2+\cot^2\frac{\alpha+\beta}{2}}\mathrm{d}t.$$

That is, inequality (8) is proved. This ends the proof of Lemma 2.3. □

We remark here that, in [19], the authors extended the theory of majorization and established the theory of weak monotonic function, and they show that a Schur-convex function is a weak increasing function under the proper hypotheses, that is, they obtained the following result: Let $\Omega \subset \mathbb{R}^n_{++}$ be a symmetrical and convex domain, and let $f:\Omega\to\mathbb{R}$ be a homogeneous with degree $\gamma \geqslant 0$, symmetrical and differentiable function. If $f:\Omega\to\mathbb{R}$ is a Schur-convex function and $f(\mathbf{e})\geqslant 0$, then $f:\Omega\to\mathbb{R}$ is a weak increasing function.

LEMMA 2.4. *Let $S^{(2)}\{P,\Gamma,l\}$ be centered 2-surround system. Then*

$$\overline{r}_P \geqslant r_P\tan\frac{\angle APA_+}{2}\int_0^1\sqrt{t^2+\cot^2\frac{\angle APA_+}{2}}\mathrm{d}t. \tag{17}$$

Equality in (17) holds if and only if P' is the midpoint of line segment $[AA_+]$.

Proof. Let

$$A \triangleq x_A\mathbf{i},\ A_+ \triangleq x_{A_+}\mathbf{i},\ M \triangleq x\mathbf{i},\ x_A \leqslant x \leqslant x_{A_+},\ P \triangleq r_P\mathbf{j},\ P' \triangleq 0\mathbf{i}+0\mathbf{j},$$

and let

$$\alpha \triangleq \angle APP' = -\arctan\frac{x_A}{r_P},\ \beta \triangleq \angle A_+PP' = \arctan\frac{x_{A_+}}{r_P}.$$

Then

$$\angle APA_+ = \alpha+\beta,\ x_A = -r_P\tan\alpha,\ x_{A_+} = r_P\tan\beta, \tag{18}$$
$$\alpha,\beta \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right),\ \alpha+\beta\in(0,\pi), \tag{19}$$

and

$$\|M-P\| = \sqrt{x^2+r_P^2},\ \|A_+-A\| = x_{A_+}-x_A,\ r_P = \frac{\|A_+-A\|}{\tan\alpha+\tan\beta}. \tag{20}$$

See Figure 1.

Notice that

$$\overline{r}_P = \frac{1}{\|A_+-A\|}\int_{[AA_+]}\|P-M\| = \frac{1}{x_{A_+}-x_A}\int_{x_A}^{x_{A_+}}\sqrt{x^2+r_P^2}\mathrm{d}x$$
$$= \frac{1}{x_{A_+}-x_A}\int_{-r_P\tan\alpha}^{r_P\tan\beta}\sqrt{x^2+r_P^2}\mathrm{d}x = \frac{1}{r_P(\tan\alpha+\tan\beta)}\int_{-r_P\tan\alpha}^{r_P\tan\beta}\sqrt{x^2+r_P^2}\mathrm{d}x$$
$$= \frac{r_P}{\tan\alpha+\tan\beta}\int_{-r_P\tan\alpha}^{r_P\tan\beta}\sqrt{\left(\frac{x}{r_P}\right)^2+1}\mathrm{d}\left(\frac{x}{r_P}\right) = \frac{r_P}{\tan\alpha+\tan\beta}\int_{-\tan\alpha}^{\tan\beta}\sqrt{x^2+1}\mathrm{d}x,$$

that is,

$$\overline{r}_P = r_P \frac{\int_{-\tan\alpha}^{\tan\beta} \sqrt{x^2+1}\,\mathrm{d}x}{\tan\alpha + \tan\beta}. \tag{21}$$

By (21) and Lemma 2.3, we get

$$\begin{aligned}\overline{r}_P &= r_P \frac{\int_{-\tan\alpha}^{\tan\beta} \sqrt{x^2+1}\,\mathrm{d}x}{\tan\alpha + \tan\beta} \geqslant r_P \tan\frac{\alpha+\beta}{2} \int_0^1 \sqrt{x^2 + \cot^2\frac{\alpha+\beta}{2}}\,\mathrm{d}x \\ &= r_P \tan\frac{\angle APA_+}{2} \int_0^1 \sqrt{t^2 + \cot^2\frac{\angle APA_+}{2}}\,\mathrm{d}t.\end{aligned}$$

That is, inequality (17) is proved.

Based on the above proof, we know that equality in (17) holds if and only if P' is the midpoint of line segment $[AA_+]$. The proof of Lemma 2.4 is completed. $\square$

LEMMA 2.5. (Jensen's inequality, see [21, 22]) *Let $E \subset \mathbb{R}^m$ be a bounded and closed region (or curve), and let the functions $f : E \to \mathbb{R}$ and $\phi : f(E) \to \mathbb{R}$ be Riemann integrable, where $f(E)$ is an interval. If $\phi : f(E) \to \mathbb{R}$ is a convex function, then we have the following Jensen' inequality:*

$$\frac{\int_E \phi(f)}{\int_E} \geqslant \phi\left(\frac{\int_E f}{\int_E}\right). \tag{22}$$

A well-known Jensen's inequality [21, 22, 23, 24, 20] can be stated as:

$$f(A(x)) \triangleq f\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right) \leqslant \frac{1}{n}\sum_{j=1}^{n} f(x_j) \triangleq A(f(x)), \tag{23}$$

where $f : I \to \mathbb{R}$ is a convex function, and $x_j \in I$, $j = 1, 2, \ldots, n$. Inequality (23) is reversed if $f : I \to \mathbb{R}$ is a concave function.

We remark here that, in [22], the authors generalized the inequalities (22) and (23) by means of the theory of majorization, and obtained the following result: Let two functions $f : [a,b] \to (0,\infty)$ and $g : [a,b] \to (0,\infty)$ satisfy the condition

$$\sup_{t\in[a,b]}\left\{\left|\frac{g''(t)}{f''(t)}\right|\right\} < \inf_{t\in[a,b]}\left\{\frac{g(t)}{f(t)}\right\}.$$

If $f''(t) > 0$, $\forall t \in [a,b]$, then for any $x \in [a,b]^n$, we have the following *J-P-S-F type inequalities*:

$$\frac{f(A(x))}{g(A(x))} \leqslant \cdots \leqslant \frac{f_{i+1,n}(x)}{g_{i+1,n}(x)} \leqslant \frac{f_{i,n}(x)}{g_{i,n}(x)} \leqslant \cdots \leqslant \frac{A(f(x))}{A(g(x))}, \tag{24}$$

where

$$f_{k,n}(x) \triangleq \frac{1}{\binom{n}{k}} \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right),\ 1 \leqslant k \leqslant n,\ n \geqslant 2.$$

This inequalities are reversed if $f''(t) < 0,\ \forall t \in [a,b]$. Further, let $E \subset \mathbb{R}^m$ be a bounded closed domain, where the measure $|E| = 1$, and let $\phi : E \to [a,b]$ be a Riemann integrable function. If $f''(t) > 0,\ \forall t \in [a,b]$, then we have the following *Jensen-type inequality*:

$$\frac{f\left(\int_E \phi\right)}{g\left(\int_E \phi\right)} \leqslant \frac{\int_E f(\phi)}{\int_E g(\phi)}, \tag{25}$$

which is also an extension of inequality (22). Inequality (25) is reversed if $f''(t) < 0,\ \forall t \in [a,b]$.

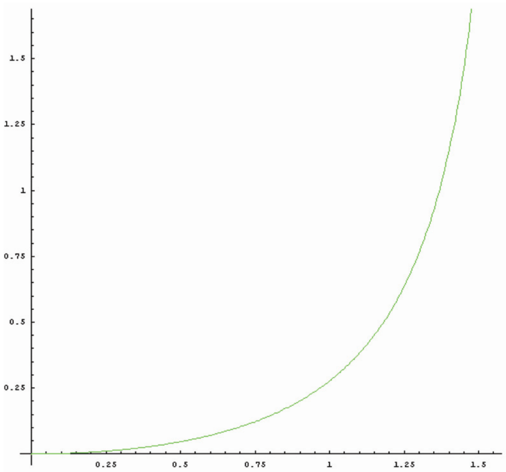

Figure 2: *The graph of the function* $\omega(\theta)$ *where* $\theta \in \left(0, \frac{\pi}{2}\right)$.

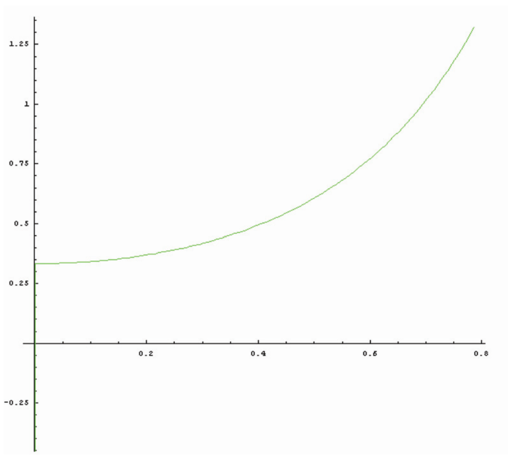

Figure 3: *The graph of the function* $\frac{d^2\omega}{d\theta^2}$ *where* $\theta \in \left(0, \frac{\pi}{4}\right]$.

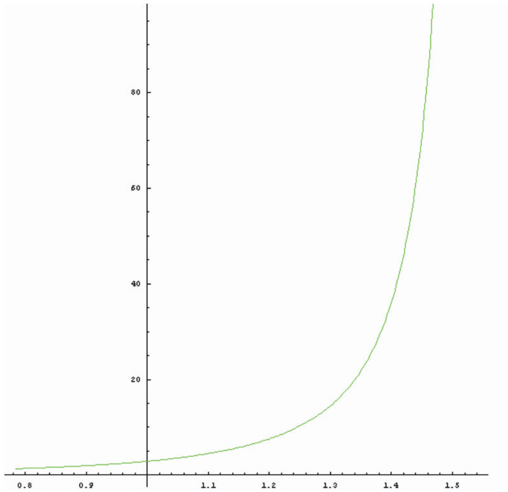

Figure 4: *The graph of the function* $\frac{d^2\omega}{d\theta^2}$ *where* $\theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$.

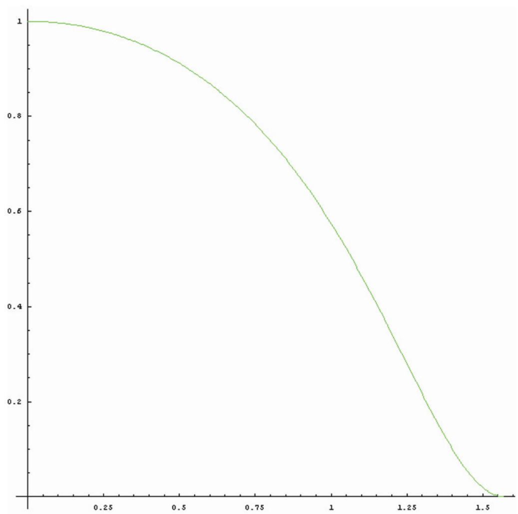

Figure 5: *The graph of the function* $\chi(\theta)$ *where* $\theta \in \left(0, \frac{\pi}{2}\right)$.

3. Proof of Theorem 1.1

Proof. We first prove inequality (3). Since [4]

$$\int_0^1 \sqrt{t^2+\cot^2\theta}\,\mathrm{d}t = \frac{1}{2}\left[\csc\theta + \cot^2\theta \log\left(\tan\theta + \sec\theta\right)\right], \tag{26}$$

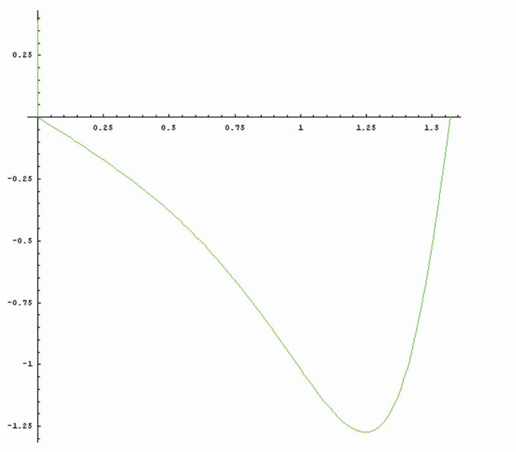

Figure 6: *The graph of the function* $\frac{d\chi(\theta)}{d\theta}$ *where* $\theta \in \left(0, \frac{\pi}{2}\right)$.

we have

$$\tan\theta \int_0^1 \sqrt{t^2 + \cot^2\theta}\,\mathrm{d}t = \frac{1}{2}[\sec\theta + \cot\theta\log(\tan\theta + \sec\theta)]. \tag{27}$$

Consider the following auxiliary function

$$\omega : \left(0, \frac{\pi}{2}\right) \to \mathbb{R},\ \omega(\theta) \triangleq \log\left\{\frac{1}{2}[\sec\theta + \cot\theta\log(\tan\theta + \sec\theta)]\right\}. \tag{28}$$

By means of the command D[ ] of the Mathematica software, we get

$$\frac{\mathrm{d}^2\omega}{\mathrm{d}\theta^2} = -\frac{\left[-\csc^2\theta\log(\sec\theta + \tan\theta) + \sec\theta\tan\theta + \frac{\cot\theta\left(\sec^2\theta + \sec\theta\tan\theta\right)}{\sec\theta + \tan\theta}\right]^2}{[\cot\theta\log(\sec\theta + \tan\theta) + \sec\theta]^2} + \frac{\varpi(\theta)}{\cot\theta\log(\sec\theta + \tan\theta) + \sec\theta},$$

where

$$\begin{aligned}\varpi(\theta) \triangleq{}& 2\cot\theta\csc^2\theta\log(\sec\theta + \tan\theta) + \sec^3\theta + \sec\theta\tan^2\theta \\ &- \frac{2\csc^2\theta\left(\sec^2\theta + \sec\theta\tan\theta\right)}{\sec\theta + \tan\theta} - \frac{\cot\theta\left(\sec^2\theta + \sec\theta\tan\theta\right)^2}{(\sec\theta + \tan\theta)^2} \\ &+ \frac{\cot\theta\left(\sec^3\theta + 2\sec^2\theta\tan\theta + \sec\theta\tan^2\theta\right)}{\sec\theta + \tan\theta}.\end{aligned}$$

By means of the command Plot[ ], we know that the graph of the function $\omega(\theta)$ is depicted in Figures 2, and the graph of the function $\mathrm{d}^2\omega/\mathrm{d}\theta^2$ is depicted in Figures 3 and 4, and by means of the commands Solve[ ] and Limit[ ], we know that the equation $\mathrm{d}^2\omega/\mathrm{d}\theta^2 = 0$ has no any real root in the interval $(0,\pi/2)$ and

$$\frac{\mathrm{d}^2\omega}{\mathrm{d}\theta^2} > \lim_{\theta\to 0}\frac{\mathrm{d}^2\omega}{\mathrm{d}\theta^2} = \frac{1}{3} > 0.$$

So the ω is a convex function with respect to the variable $\theta \in (0,\pi/2)$.

By Lemma 2.4, (27) and (28), we get

$$\log\frac{\overline{r}_P}{r_P} \geqslant \log\left(\tan\frac{\angle APA_+}{2}\int_0^1\sqrt{t^2+\cot^2\frac{\angle APA_+}{2}}\mathrm{d}t\right) = \omega\left(\frac{\angle APA_+}{2}\right). \tag{29}$$

Since the function ω is a convex function, according to (28), (29) and Lemma 2.5, we get

$$\begin{aligned}\frac{1}{|\Gamma|}\oint_\Gamma \log\frac{\overline{r}_P}{r_P} &\geqslant \frac{1}{|\Gamma|}\oint_\Gamma \omega\left(\frac{\angle APA_+}{2}\right) \geqslant \omega\left(\frac{1}{|\Gamma|}\oint_\Gamma\frac{\angle APA_+}{2}\right) = \omega\left(\frac{l\pi}{|\Gamma|}\right)\\ &= \log\left\{\frac{1}{2}\left[\sec\frac{l\pi}{|\Gamma|} + \cot\frac{l\pi}{|\Gamma|}\log\left(\tan\frac{l\pi}{|\Gamma|}+\sec\frac{l\pi}{|\Gamma|}\right)\right]\right\},\end{aligned}$$

that is,

$$\frac{\exp\left(\frac{1}{|\Gamma|}\oint_\Gamma \log\overline{r}_P\right)}{\exp\left(\frac{1}{|\Gamma|}\oint_\Gamma \log r_P\right)} \geqslant \frac{1}{2}\left[\sec\frac{l\pi}{|\Gamma|} + \cot\frac{l\pi}{|\Gamma|}\log\left(\tan\frac{l\pi}{|\Gamma|}+\sec\frac{l\pi}{|\Gamma|}\right)\right]. \tag{30}$$

Hence inequality (3) is proved. The proof of Theorem 1.1 is completed. □

4. Proof of Theorem 1.2

Proof. Consider a new auxiliary function:

$$\chi : \left(0,\frac{\pi}{2}\right) \to \mathbb{R},\ \chi(\theta) \triangleq \left\{\frac{1}{2}[\sec\theta + \cot\theta\log(\tan\theta+\sec\theta)]\right\}^{-2}. \tag{31}$$

By means of the command D[ ] of the Mathematica software, we get

$$\frac{\mathrm{d}\chi(\theta)}{\mathrm{d}\theta} = -\frac{8\left[-\csc^2\theta\log(\sec\theta+\tan\theta)+\sec\theta\tan\theta+\frac{\cot\theta\left(\sec^2\theta+\sec\theta\tan\theta\right)}{\sec\theta+\tan\theta}\right]}{[\cot\theta\log(\sec\theta+\tan\theta)+\sec\theta]^3},$$

and by means of the command FindMinimum[ ], we get

$$\min_{0<\theta<\pi/2}\left\{\frac{\mathrm{d}\chi(\theta)}{\mathrm{d}\theta}\right\} = \left(\frac{\mathrm{d}\chi(\theta)}{\mathrm{d}\theta}\right)_{\theta=\theta_0} = -1.2737520635993627\ldots,$$

where $\theta_0 = 1.246714063270445\ldots$, which is also the unique real root of equation (4), and the strictly decreasing interval of $\mathrm{d}\chi(\theta)/\mathrm{d}\theta$ is $(0,\theta_0]$, which is also strictly concave interval of the function $\chi(\theta)$.

By means of the command Plot[ ] of the Mathematica software, we know that the graph of the function $\chi(\theta)$ is depicted in Figure 5, and the graph of the function $\mathrm{d}\chi(\theta)/\mathrm{d}\theta$ is depicted in Figure 6.

Since

$$A\in\Gamma \Rightarrow 0<\angle APA_+\leqslant\tau\Rightarrow 0<\frac{\angle APA_+}{2}\leqslant\frac{\tau}{2}=\theta_0,$$

according to (28), (31), Cauchy inequality [2]

$$\frac{1}{|\Gamma|}\oint_\Gamma f\times g\leqslant\sqrt{\frac{1}{|\Gamma|}\oint_\Gamma f^2\times\frac{1}{|\Gamma|}\oint_\Gamma g^2},$$

Lemmas 2.1, 2.4 and 2.5, we get

$$\begin{aligned}
\mathrm{M}_\Gamma(r_P) &\triangleq \frac{1}{|\Gamma|}\oint_\Gamma r_P\\
&\leqslant \frac{1}{|\Gamma|}\oint_\Gamma \overline{r}_P\left(\tan\frac{\angle APA_+}{2}\int_0^1\sqrt{t^2+\cot^2\frac{\angle APA_+}{2}}\mathrm{d}t\right)^{-1}\\
&= \frac{1}{|\Gamma|}\oint_\Gamma \overline{r}_P\times\sqrt{\chi\left(\frac{\angle APA_+}{2}\right)}\\
&\leqslant \sqrt{\frac{1}{|\Gamma|}\oint_\Gamma \overline{r}_P^2\times\frac{1}{|\Gamma|}\oint_\Gamma\chi\left(\frac{\angle APA_+}{2}\right)}\\
&= \mathrm{M}_\Gamma^{[2]}(\overline{r}_P)\sqrt{\frac{1}{|\Gamma|}\oint_\Gamma\chi\left(\frac{\angle APA_+}{2}\right)}\\
&\leqslant \mathrm{M}_\Gamma^{[2]}(\overline{r}_P)\sqrt{\chi\left(\frac{1}{|\Gamma|}\oint_\Gamma\frac{\angle APA_+}{2}\right)}\\
&= \mathrm{M}_\Gamma^{[2]}(\overline{r}_P)\sqrt{\chi\left(\frac{1}{2|\Gamma|}\oint_\Gamma\angle APA_+\right)}\\
&= \mathrm{M}_\Gamma^{[2]}(\overline{r}_P)\sqrt{\chi\left(\frac{1}{2|\Gamma|}\times 2l\pi\right)}\\
&= \mathrm{M}_\Gamma^{[2]}(\overline{r}_P)\sqrt{\chi\left(\frac{l\pi}{|\Gamma|}\right)}\\
&= \mathrm{M}_\Gamma^{[2]}(\overline{r}_P)\left\{\frac{1}{2}\left[\sec\frac{l\pi}{|\Gamma|}+\cot\frac{l\pi}{|\Gamma|}\log\left(\tan\frac{l\pi}{|\Gamma|}+\sec\frac{l\pi}{|\Gamma|}\right)\right]\right\}^{-1}.
\end{aligned}$$

That is, inequality (5) holds.

Based on the above proof, we know that equality in (5) holds if and only if Γ is a circle and P is the center of the circle. This completes the proof of Theorem 1.2. □

5. Applications

Let $S^{(2)}\left\{P,\Gamma,l_j\right\}$ be a centered 2-surround system, where $j=1,2,\ldots,N$, $N\geqslant 3$. Then we say the set

$$S^{(2)}\left\{P,\Gamma,\mathbf{l}\right\}\triangleq\left\{P,\Gamma,\mathbf{l}\right\}$$

is a *centered N-surround system* and P is a *center* of the system, where

$$\mathbf{l}\triangleq(l_1,l_2,\ldots,l_N)\in\mathbb{R}^N,\ 0<l_j<\frac{|\Gamma|}{2},\ \forall\, j:1\leqslant j\leqslant N,\ \sum_{j=1}^{N}l_j=|\Gamma|.$$

If we define

$$A_j\triangleq\gamma\left(t_A+\sum_{k=1}^{j}l_k\right),\ j=1,2,\ldots,N,$$

then we say that the points $A_1,A_2,\ldots,A_N$ are N *satellites* of the system [4].

Suppose that $S^{(2)}\left\{P,\Gamma,\mathbf{l}\right\}$ is a centered N-surround system. Then we define

$$r_P^{(j)}\triangleq\text{Distance}(P,A_{j-1}A_j)\text{ and }\overline{r}_P^{(j)}\triangleq\frac{1}{\|A_j-A_{j-1}\|}\int_{M\in[A_{j-1}A_j]}\|M-P\|,$$

where $1\leqslant j\leqslant N$, $N\geqslant 3$.

Theorem 1.1 implies the following result.

THEOREM 5.1. (N–mean central distance–central distance inequality) *Let $S^{(2)}\left\{P,\Gamma,\boldsymbol{l}\right\}$ be a centered N-surround system. Then we have the following inequality:*

$$\frac{\mathrm{G}_\Gamma\left(\sqrt[N]{\prod_{j=1}^{N}\overline{r}_P^{(j)}}\right)}{\mathrm{G}_\Gamma\left(\sqrt[N]{\prod_{j=1}^{N}r_P^{(j)}}\right)}\geqslant\frac{1}{2}\left[\sec\frac{\pi}{N}+\cot\frac{\pi}{N}\log\left(\tan\frac{\pi}{N}+\sec\frac{\pi}{N}\right)\right].\tag{32}$$

Equality in (32) holds if and only if Γ is a circle, P is the center of the circle and

$$l_1=l_2=\cdots=l_N=\frac{|\Gamma|}{N}.\tag{33}$$

Proof. According to Theorem 1.1, we have

$$\frac{\mathrm{G}_\Gamma(\overline{r}_P^{(j)})}{\mathrm{G}_\Gamma(r_P^{(j)})}\geqslant\frac{1}{2}\left[\sec\frac{l_j\pi}{|\Gamma|}+\cot\frac{l_j\pi}{|\Gamma|}\log\left(\tan\frac{l_j\pi}{|\Gamma|}+\sec\frac{l_j\pi}{|\Gamma|}\right)\right],\ j=1,2,\ldots,N,$$

that is,

$$\log\left[\frac{\mathrm{G}_\Gamma(\overline{r}_P^{(j)})}{\mathrm{G}_\Gamma(r_P^{(j)})}\right] \geqslant \omega\left(\frac{l_j\pi}{|\Gamma|}\right),\ j=1,2,\ldots,N, \tag{34}$$

where the function ω is defined by (28).

Since the function ω is a convex function, according to (34) and the Jensen's inequality (23), we have

$$\begin{aligned}\log\left[\frac{\mathrm{G}_\Gamma\left(\sqrt[N]{\prod_{j=1}^N \overline{r}_P^{(j)}}\right)}{\mathrm{G}_\Gamma\left(\sqrt[N]{\prod_{j=1}^N r_P^{(j)}}\right)}\right] &= \frac{1}{N}\sum_{j=1}^N \log\left[\frac{\mathrm{G}_\Gamma(\overline{r}_P^{(j)})}{\mathrm{G}_\Gamma(r_P^{(j)})}\right] \geqslant \frac{1}{N}\sum_{j=1}^N \omega\left(\frac{l_j\pi}{|\Gamma|}\right)\\ &\geqslant \omega\left(\frac{1}{N}\sum_{j=1}^N \frac{l_j\pi}{|\Gamma|}\right) = \omega\left(\frac{\pi}{N}\right)\\ &= \log\left\{\frac{1}{2}\left[\sec\frac{\pi}{N} + \cot\frac{\pi}{N}\log\left(\tan\frac{\pi}{N} + \sec\frac{\pi}{N}\right)\right]\right\}.\end{aligned}$$

That is, inequality (32) holds.

Based on the above proof, we know that equality in (32) holds if and only if Γ is a circle, P is the center of the circle and equations (33) hold. This completes the proof of Theorem 5.1. □

Theorem 1.2 implies the following result.

THEOREM 5.2. (Mean central distance–central distance–limit inequality) *Let $S^{(2)}\{P,\Gamma,l\}$ be a centered 2-surround system. then we have the following isoperimetric inequality:*

$$\lim_{l\to 0}\frac{\mathrm{M}_\Gamma^{[2]}(\overline{r}_P) - \mathrm{M}_\Gamma(r_P)}{l^2} \geqslant \frac{\pi^2}{3}\times\frac{\mathrm{Area}D(\Gamma)}{|\Gamma|^3}, \tag{35}$$

here we assume that the above limit exists.

Proof. Since

$$0 < \|A_+ - A\| \leqslant l,\ \forall A\in\Gamma,$$

we have

$$\lim_{l\to 0}\angle APA_+ = 0.$$

Consequently, there exists a $\delta > 0$ such that

$$0 < \angle APA_+ \leqslant \tau,\ \forall l\in(0,\delta),$$

where τ is defined by Theorem 1.2.

Since Γ is a smooth curve, so, for any point $A \in \Gamma$, there exists a line AT, such that AT is tangent to Γ at the point A. Let $\rho \triangleq \text{Distance}(P, AT)$. Then ρ is a *support function* of the curve Γ, and we have [25]

$$\frac{1}{2}\oint_{\Gamma}\rho = \text{Area}D(\Gamma) \text{ with } \lim_{l\to 0} r_P = \rho. \tag{36}$$

By means of the command Limit[ ] of the Mathematica software, we get

$$\lim_{\theta\to 0}\frac{\frac{1}{2}[\sec\theta + \cot\theta\log(\tan\theta + \sec\theta)] - 1}{\theta^2} = \frac{1}{6}. \tag{37}$$

Set

$$\theta = \frac{l\pi}{|\Gamma|} \Leftrightarrow l = \left(\frac{\pi}{|\Gamma|}\right)^{-1}\theta.$$

According to Theorem 1.2, (36) and (37), when $l \in (0, \delta)$, we have

$$\frac{\mathrm{M}_{\Gamma}^{[2]}(\bar{r}_P) - \mathrm{M}_{\Gamma}(r_P)}{l^2} \geqslant \mathrm{M}_{\Gamma}(r_P)\frac{\frac{1}{2}[\sec\theta + \cot\theta\log(\tan\theta + \sec\theta)] - 1}{l^2},$$

and

$$\begin{aligned}
\lim_{l\to 0}\frac{\mathrm{M}_{\Gamma}^{[2]}(\bar{r}_P) - \mathrm{M}_{\Gamma}(r_P)}{l^2} &\geqslant \lim_{l\to 0}\mathrm{M}_{\Gamma}(r_P)\frac{\frac{1}{2}[\sec\theta + \cot\theta\log(\tan\theta + \sec\theta)] - 1}{l^2} \\
&= \lim_{l\to 0}\mathrm{M}_{\Gamma}(r_P)\lim_{l\to 0}\left(\frac{\pi}{|\Gamma|}\right)^2\frac{\frac{1}{2}[\sec\theta + \cot\theta\log(\tan\theta + \sec\theta)] - 1}{\theta^2} \\
&= \left(\frac{\pi}{|\Gamma|}\right)^2\mathrm{M}_{\Gamma}(\lim_{l\to 0} r_P)\lim_{\theta\to 0}\frac{\frac{1}{2}[\sec\theta + \cot\theta\log(\tan\theta + \sec\theta)] - 1}{\theta^2} \\
&= \frac{1}{6}\left(\frac{\pi}{|\Gamma|}\right)^2\mathrm{M}_{\Gamma}(\rho) \\
&= \frac{1}{6}\left(\frac{\pi}{|\Gamma|}\right)^2\frac{1}{|\Gamma|}\oint_{\Gamma}\rho \\
&= \frac{1}{6}\left(\frac{\pi}{|\Gamma|}\right)^2\frac{1}{|\Gamma|}\times 2\text{Area}D(\Gamma) \\
&= \frac{\pi^2}{3}\times\frac{\text{Area}D(\Gamma)}{|\Gamma|^3}.
\end{aligned}$$

That is, inequality (35) holds. The proof of Theorem 5.2 is completed. □

We remark here that, by (35), we know that: for any $\varepsilon > 0$, there exists a real $\delta > 0$ such that

$$0 < l < \delta \Rightarrow \frac{\mathrm{M}_{\Gamma}^{[2]}(\bar{r}_P) - \mathrm{M}_{\Gamma}(r_P)}{l^2} > \frac{\pi^2}{3}\times\frac{\text{Area}D(\Gamma)}{|\Gamma|^3} - \varepsilon. \tag{38}$$

6. Conclusions

In this paper, we first establish several identities and inequalities involving the centered 2-surround system. Next, we prove two isoperimetric inequalities, which are called the *mean central distance–central distance inequalities*. Finally, we demonstrate the applications of our results, and obtain the *N–mean central distance–central distance inequality* and *mean central distance–central distance–limit inequality*. The proofs of our results are perfect coordination of the mathematical proof techniques and computer.

One of the theoretical significance of this paper is to use the computer to deal with some complex inequality problems, and another is to establish the geometric theory on satellite motion. Large pieces of functional analysis, convex geometry, computer and inequality theories are used in this paper, especially the theory of majorization.

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