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# ON TWO PROBLEMS FOR GAUSS COMPOUND MEAN

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*Dedicated to my father*

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*Abstract.* In this paper, we give answers to two open problems concerning Gauss Compound mean posed in the paper: [ZH. H. YANG, *A New Proof of Inequalities for Gauss Compound Mean*, Int. Journal of Math. Analysis, Vol. 4, (2010), no. 21, 1013–1018.] One of the problems is equivalent to the Vamanamurthy problem.

## 1. Introduction

Within the past years, means have been the subject of intensive research. Many interesting results and inequalities for means can be found in the literature [1]–[9]. The well known Gauss compound mean of positive numbers $a,\ b$ with $a \neq b$ is the limit

$$AGM(a,b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

where

$$a_1 = a, \quad b_1 = b, \quad a_{n+1} = \frac{a_n + b_n}{2} = A(a_n, b_n), \quad b_{n+1} = \sqrt{a_n b_n} = G(a_n, b_n). \tag{1}$$

The power mean of positive a and b is defined by

$$M_t(a,b) = M^{1/t}(a^t, b^t) \quad \text{if} \quad t \neq 0 \quad \text{and} \quad M_0(a,b) = \sqrt{ab}. \tag{2}$$

The t-order logarithmic mean of $a,b > 0$ is defined by

$$L_t(a,b) = \begin{cases} \left(\frac{b^t - a^t}{t(\ln b - \ln a)}\right)^{\frac{1}{t}}, & t \neq 0 \\ \sqrt{ab}, & t = 0, \end{cases} \tag{3}$$

where $L_1(a,b) = L(a,b) = \frac{b-a}{\ln b - \ln a}$ is the logarithmic mean.

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We note, that AGM is connected with elliptic integrals by Gauss's fantastic result:

$$AGM(1,r') = \frac{\pi}{2k(r)}$$

where $k(r) = \int_0^{\pi/2} (1-r^2\sin^2 x)^{-1/2}dx,\ r' = \sqrt{1-r^2},\ 0 \leqslant r \leqslant 1$ (see [2]).

In the paper [8], the author presented new sophisticated proofs for famous inequalities

$$L(a,b) < AGM(a,b) < L^{2/3}(a^{3/2}, b^{3/2})$$

where the first inequality was proved by B. C. Carlson and M. Vuorinen [4], while the second one is due to J. and P. Borwein [2].

Denote by

$$D_n(t) = b_{n+1}^{-t}\left(\frac{b_n^t - a_n^t}{\ln b_n - \ln a_n} - \frac{b_{n+1}^t - a_{n+1}^t}{\ln b_{n+1} - \ln a_{n+1}}\right), \tag{4}$$

where $a_n, b_n$ is defined by (1). It was showed in [8] that $t \mapsto D_n(t)$ is continuous on $\mathbb{R}$ and increasing on $(1,\infty)$ with $D_n(1) < 0$, $D_n(3/2) > 0$. So $D_n(t)$ has an unique zero $t_0^{(n)}(a,b)$ on $(1, 3/2)$.

Denote by $t_{0i}(a,b) = \inf_{n\in N}\{t_0^{(n)}(a,b)\}$ and $t_{0s}(a,b) = \sup_{n\in N}\{t_0^{(n)}(a,b)\}$. It is evident that $1 \leqslant t_{0i}(a,b) < 3/2,\ 1 < t_{0s}(a,b) \leqslant 3/2$.

At the end in the same paper [8], the author posed two open problems and remarks as follows:

PROBLEM 1. Is $t_{0i}(a,b)$ equal to 1?

REMARK 1. If the answer is "no", then there must exist $t_1 \in (1, t_{0i}(a,b))$ such that $D_n(t_1) < 0$ for all $n$, whence

$$L(a_n^{t_1}, b_n^{t_1}) < L(a_{n+1}^{t_1}, b_{n+1}^{t_1}) \text{ for all } n,$$

which implies

$$L_{t_1}(a,b) < AGM(a,b), \tag{5}$$

where $L_t(a,b)$ is defined by (3).

Furthermore, are there some concrete constants $t_1 \in (1, t_{0i}(a,b))$ such that (5) holds?

PROBLEM 2. Is $t_{0s}(a,b)$ equal to $3/2$?

REMARK 2. If the answer is "no", then there must exist $t_2 \in (t_{0s}(a,b), 3/2)$ such that $D_n(t_2) > 0$ for all $n$, whence

$$L(a_n^{t_2}, b_n^{t_2}) > L(a_{n+1}^{t_2}, b_{n+1}^{t_2}) \text{ for all } n,$$

which implies

$$AGM(a,b) < L_{t_2}(a,b), \tag{6}$$

where $L_t(a,b)$ is defined by (3).

Furthermore, are there some concrete constants $t_2 \in (t_{0s}(a,b), 3/2)$ such that (6) holds?

REMARK 3. If replacing $a^t$, $b^t$ by $a,b$ in $L_t(a,b)$ and $a$, $b$ by $a^{1/t}$, $b^{1/t}$ in $AGM(a,b)$, then the inequality (5) is equivalent to

$$L(a,b) < AGM_{1/t_1}(a,b) \text{ for } 1/t_1 \in (1/t_{0i}(a,b), 1). \tag{7}$$

This shows that the open problem presented by Vamanamurthy is equivalent to the problem 1.

In this note, we will prove that

- for $\forall a,b > 0$ with $a \neq b$ we have $t_{0i}(a,b) > 1$, and $\inf_{a,b>0,\, a\neq b}\{t_{0i}(a,b)\} = 1$;
- for $\forall a,b > 0$ with $a \neq b$ we have $t_{0s}(a,b) = 3/2$.

These solve two open problems as above.

## 2. Two lemmas

In order to solve two open problems posed in [8], we need two important lemmas. Without loss of generality, we assume that $b < a$. Denote by $s_n = s_n(a,b) = b_n/a_n$, then $0 < s_n = s_n(a,b) < 1$. We have

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{1 + b_n/a_n}{2} = \frac{1+s_n}{2}, \\
\frac{b_{n+1}}{a_{n+1}} &= \frac{\sqrt{a_n b_n}}{(a_n+b_n)/2} = 2\frac{\sqrt{b_n/a_n}}{1+b_n/a_n} = 2\frac{\sqrt{s_n}}{1+s_n}, \\
\frac{a_n}{b_{n+1}} &= \frac{a_n}{\sqrt{a_n b_n}} = \frac{1}{\sqrt{b_n/a_n}} = \frac{1}{\sqrt{s_n}}.
\end{aligned}$$

Then $D_n(t)$ and $D_n'(t)$ can be expressed as

$$\begin{aligned}
D_n(t) &= b_{n+1}^{-t}\left(\frac{b_n^t - a_n^t}{\ln b_n - \ln a_n} - \frac{b_{n+1}^t - a_{n+1}^t}{\ln b_{n+1} - \ln a_{n+1}}\right) \\
&= \left(\frac{a_n}{b_{n+1}}\right)^t \left(\frac{(b_n/a_n)^t - 1}{\ln(b_n/a_n)}\right) - \frac{1 - (a_{n+1}/b_{n+1})^t}{\ln(b_{n+1}/a_{n+1})} \\
&= \frac{s_n^{t/2} - s_n^{-t/2}}{\ln s_n} - \frac{1 - \left(\frac{1+s_n}{2\sqrt{s_n}}\right)^t}{\ln \frac{2\sqrt{s_n}}{1+s_n}},
\end{aligned} \tag{8}$$

$$D_n'(t) = \frac{s_n^{-t/2} + s_n^{t/2}}{2} - \left(\frac{s_n^{-1/2} + s_n^{1/2}}{2}\right)^t.$$

We now show that $t \mapsto D_n(t),\ n \in \mathbb{N}$ is a convex function on $[1,\infty)$.

LEMMA 1. *Let $D_n(t)\ \ n \in N$ be defined in (2). Then $D_n''(t) > 0$ for $n \in \mathbb{N},\ t \in [1,\infty)$.*

*Proof.* Differentiation gives

$$s_n^{t/2} D_n''(t) = \frac{s_n^t - 1}{4} \ln s_n - \left(\frac{1+s_n}{2}\right)^t \ln\left(\frac{1+s_n}{2\sqrt{s_n}}\right) := g(t),$$
$$g'(t) = \left(\frac{1+s_n}{2}\right)^t \ln\left(\frac{2\sqrt{s_n}}{1+s_n}\right) \ln\left(\frac{1+s_n}{2}\right) + \frac{s_n^t \ln^2 s_n}{4} \geqslant 0,$$

where the inequality holds due to $0 < s_n < 1$.

Clearly, the proof will be done if we prove that

$$g(1) = h(s_n) = \left(\frac{1+s_n}{2}\right) \ln\left(\frac{2\sqrt{s_n}}{1+s_n}\right) + \frac{s_n - 1}{4} \ln s_n \geqslant 0.$$

Differentiation again yields that for $0 < s_n < 1$,

$$h'(s_n) = \frac{1}{2} \ln\left(\frac{2s_n}{1+s_n}\right) \leqslant 0,$$

which indicates that $h(s_n) > h(1) = 0$ and the proof of our lemma is complete. □

LEMMA 2. *For $n \in \mathbb{N}$ and $a, b > 0$ with $a \neq b$, let*

$$r_n = 1 - \frac{1}{2} \frac{D_n(1)}{D_n(3/2) - D_n(1)},$$

*where $D_n(t)$ is given by (2). Then we have*

$$\lim_{n\to\infty} r_n = \frac{3}{2}.$$

*Proof.* To complete the proof, we write $r_n$ as

$$r_n = 1 + \frac{1}{2} \frac{1}{D_n(3/2)/(-D_n(1)) + 1}.$$

By (8) we have

$$\begin{aligned}
\frac{D_n(3/2)}{-D_n(1)} &= \left[\frac{s_n^{-3/4}-s_n^{3/4}}{\ln s_n} - \frac{\left(\frac{1+s_n}{2\sqrt{s_n}}\right)^{3/2}-1}{\ln\left(\frac{2\sqrt{s_n}}{1+s_n}\right)}\right] \Bigg/ \left[\frac{\frac{1+s_n}{2\sqrt{s_n}}-1}{\ln\left(\frac{2\sqrt{s_n}}{1+s_n}\right)} - \frac{\frac{1}{\sqrt{s_n}}-\sqrt{s_n}}{\ln s_n}\right] \\
&= \frac{1}{s_n^{1/4}} \frac{\left(1-s_n^{3/2}\right)\frac{\ln\left(\frac{2\sqrt{s_n}}{1+s_n}\right)}{\ln s_n} - \left(\left(\frac{1+s_n}{2}\right)^{3/2} - s_n^{3/4}\right)}{\frac{1+s_n}{2} - \sqrt{s_n} - \frac{\ln\left(\frac{2\sqrt{s_n}}{1+s_n}\right)}{\ln s_n}(1-s_n)} \\
&= \frac{1}{s_n^{1/4}} \frac{\left(1-s_n^{3/2}\right)\ln\left(\frac{2\sqrt{s_n}}{1+s_n}\right) - \left(\left(\frac{1+s_n}{2}\right)^{3/2} - s_n^{3/4}\right)\ln s_n}{\left(\frac{1+s_n}{2} - \sqrt{s_n}\right)\ln s_n - \ln\left(\frac{2\sqrt{s_n}}{1+s_n}\right)(1-s_n)} \\
&:= \frac{1}{s_n^{1/4}} \frac{f_1(s_n)}{f_2(s_n)}.
\end{aligned}$$

From

$$s_{n+1} = \frac{b_{n+1}}{a_{n+1}} = \frac{2\sqrt{s_n}}{1+s_n},$$

we see that $1 > s_{n+1}(a,b) > s_n(a,b) > 0$, which is due to $\sqrt{x}(1+x) \leqslant 2$ for $0 < x < 1$. This implies that sequence $\{s_n(a,b)\}_{n=1}^{\infty}$ is convergent, and it is easily deduced that $\lim_{n\to\infty} s_n(a,b) = 1$.

To prove $\lim_{n\to\infty} r_n = \lim_{s_n\to 1} r_n = \frac{3}{2}$, it suffices to show that $\lim_{s_n\to 1}[D_n(3/2)/(-D_n(1))] = 0$. To this end, we use power series expansion to get

$$\begin{aligned}
f_1(s_n) &= \left(1-s_n^{3/2}\right)\ln\left(\frac{2\sqrt{s_n}}{1+s_n}\right) - \left(\left(\frac{1+s_n}{2}\right)^{3/2} - s_n^{3/4}\right)\ln s_n \\
&= \frac{21}{5\times 2^{15}}(s_n-1)^7 + O\left((s_n-1)^8\right), \\
f_2(s_n) &= \left(\frac{1+s_n}{2} - \sqrt{s_n}\right)\ln s_n - \ln\left(\frac{2\sqrt{s_n}}{1+s_n}\right)(1-s_n) \\
&= \frac{1}{384}(s_n-1)^5 + O\left((s_n-1)^6\right).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\lim_{s_n\to 1}\frac{D_n(3/2)}{-D_n(1)} &= \lim_{s_n\to 1}\frac{1}{s_n^{1/4}}\frac{f_1(s_n)}{f_2(s_n)} \\
&= \lim_{s_n\to 1}\frac{\frac{21}{5\times 2^{15}}(s_n-1)^7 + O\left((s_n-1)^8\right)}{\frac{1}{384}(s_n-1)^5 + O\left((s_n-1)^6\right)} = 0.
\end{aligned}$$

This completes the proof. □

## 3. Main result

Now we are in a position to state and prove our main result.

THEOREM 1. *For* $n \in \mathbb{N}$, *let* $t_0^{(n)} = t_0^{(n)}(a,b)$ *denote the unique zero of* $D_n(t)$ *defined by (2) on* $(1, 3/2)$. *Then we have*

$$t_{0s} = \sup_{n\in\mathbb{N}}\{t_0^{(n)}(a,b)\} = \frac{3}{2}, \quad t_{0i} = \inf_{n\in\mathbb{N}}\{t_0^{(n)}(a,b)\} > 1$$
$$\text{and} \qquad \inf_{a>0,\,b>0,\,a\neq b}\{t_{0i}(a,b)\} = 1.$$

*Proof.* Lemma 1 tells us that $t \mapsto D_n(t)$ is convex on $[1,\infty)$, which implies that for $t_0^{(n)} \in [1, 3/2]$ we have

$$\frac{D_n(3/2) - D_n(1)}{3/2 - 1} > \frac{D_n\left(t_0^{(n)}\right) - D_n(1)}{t_0^{(n)} - 1} = \frac{-D_n(1)}{t_0^{(n)} - 1}.$$

This gives

$$r_n = 1 - \frac{1}{2}\frac{D_n(1)}{D_n(3/2) - D_n(1)} < t_0^{(n)} < \frac{3}{2}.$$

It follows from Lemma 2 that $\lim_{n\to\infty} t_0^{(n)}(a,b) = 3/2$, and therefore, $t_{0s} = \sup_{n\in\mathbb{N}}\{t_0^{(n)}(a,b)\} = \frac{3}{2}$.

Denote by

$$F(s_n,t) = \left(\frac{2\sqrt{s_n}}{1+s_n}\right)^t \ln\left(\frac{2\sqrt{s_n}}{1+s_n}\right) \times D_n(t)$$
$$= \left(\ln\left(\frac{2\sqrt{s_n}}{1+s_n}\right)\right)\left(\frac{2}{1+s_n}\right)^t\left(\frac{s_n^t - 1}{\ln s_n}\right) - \left(\frac{2\sqrt{s_n}}{1+s_n}\right)^t + 1.$$

Put $t_0 = \inf_{a>0,\,b>0,\,a\neq b.} t_{0i}(a,b)$. Let $\{s_k\}_{k=1}^{\infty}$ be a sequence such that $0 < s_{k+1} < s_k < 1$, $\lim_{k\to\infty} s_k = 0$. Let $\{t(s_k)\}_{k=1}^{\infty}$ be a sequence of solutions $F(s_k, t(s_k)) = 0$ in $(1, 3/2)$. Then $\{t(s_k)\}_{k=1}^{\infty}$ is a bounded sequence. It implies that the sequence has a convergent subsequence $\{t(s_{k_l})\}_{l=1}^{\infty}$. Denote by $t_0^* = \lim_{l\to\infty} t(s_{k_l})$. It is clear that

$$D_n(t(a,b)) = 0 \quad \Longleftrightarrow \quad F(s_n(a,b), t(a,b)) = 0.$$

So from $\lim_{l\to\infty} F(s_{k_l}, t(s_{k_l})) = 0$ we obtain $2^{t_0^*-1} = 1$. It implies $t_0 = 1$. □

CONCLUSSION 1. For all $a, b > 0$ with $a \neq b$ there is $t_1 \in (1, t_{0i}(a,b))$ such that

$$L_{t_1}(a,b) < AGM(a,b),$$

but $\sup_{a>0,\, b>0,\, a\neq b.} \{t_1(a,b)\} = 1.$

For all $a,b>0$ with $a\neq b$ there is no $t_2 \in (1,3/2)$ such that

$$L_{t_2}(a,b) > AGM(a,b).$$

So $L_1(a,b)$ and $L_{3/2}(a,b)$ are optimal lower and upper bounds for $AGM(a,b)$, respectively.

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