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SCHUR CONVEX FUNCTIONS AND THE BONNESEN STYLE ISOPERIMETRIC INEQUALITIES FOR PLANAR CONVEX POLYGONS

JIBING QI AND WEN WANG

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Abstract. In this note, we continue to investigate Bonnesen-type isoperimetric inequalities for planar convex polygons. We shall first establish some analytic isoperimetric inequalities for a special class of Schur convex functions. Subsequently, by using these analytic isoperimetric inequalities, Bonnesen-type isoperimetric inequalities and related inverse inequalities for the planar convex polygons are obtained.

1. Introduction

Schur convex functions [4] play an important role in the study of analytic inequalities and geometric inequalities. Let us recall some notions and lemmas.

Let $I \subset \boldsymbol{R}$ and $I^n = I \times I \times \cdots \times I$ (n copies).

LEMMA 1.1. ([13]) *An $n \times n$ matrix $S = [s_{ij}]$ is said to be a doubly stochastic matrix if $s_{ij} \geqslant 0$ for $1 \leqslant i < j \leqslant n$, and*

$$\sum_{j=1}^{n} s_{ij} = 1, \quad i = 1, 2\cdots, n; \quad \sum_{i=1}^{n} s_{ij} = 1, \quad j = 1, 2, \cdots, n.$$

LEMMA 1.2. ([13])

(1). A permutation matrix is a doubly stochastic matrix.

(2). $S = [s_{ij}]$ with $s_{ij} = \frac{1}{n}$, $1 \leqslant i, j \leqslant n$, is a doubly stochastic matrix.

LEMMA 1.3. ([13]) *A real function $f : I^n \to \boldsymbol{R}$ $(n > 1)$ is called to be Schur convex function if for any doubly stochastic matrix S and all $\boldsymbol{x} \in I^n$, $f(S\boldsymbol{x}) \leqslant f(\boldsymbol{x})$. It is called to be strictly Schur convex if inequality is strict. f is said to be Schur concave (resp. strictly Schur concave) if $-f$ is Schur convex.*

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Corresponding authors: J. Qi (qijibing@ahu.edu.cn), W. Wang (wwen2014@mail.ustc.edu.cn; wenwang1985@163.com).

LEMMA 1.4. ([4]) *Let $\Omega \in \boldsymbol{R}^n$ be symmetric and convex set with nonempty interior, and let $f : \Omega \to \boldsymbol{R}$ be differentiable in the interior of Ω. Then f is Schur convex (Schur concave) on Ω if and only if f is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \geqslant 0 (\leqslant 0) \qquad \textit{for all} \quad x \in \Omega^0,$$

where Ω^0 is the interior of Ω.

The above definitions and example can be found in many references such as [4] and [14].

The classical isoperimetric inequality states that for a domain K with the boundary composing of the simple curve $\mathscr{C}$ of length L and area A

$$L^2 - 4\pi A \geqslant 0, \tag{1.1}$$

where equality holds if K is a circle. The isoperimetric deficit of K is defined as $\Delta(K) = L^2 - 4\pi A$. Bonnesen in [8] gave a low bound for the isoperimetric deficit $\Delta(K)$, as follows

$$\Delta(K) = L^2 - 4\pi A \geqslant \pi^2 (R - r)^2,$$

where R is the circumradius and r is the inradius of the curve $\mathscr{C}$.

Later Bonnesen proved a series of inequalities of the form

$$\Delta(K) = L^2 - 4\pi A \geqslant B,$$

where the equality B is an invariant of geometric significance having the following basic properties:

1. B is non-negative;
2. B is vanish only when K is a disc.

Many Bs are discovered in the last century and mathematicians are still working on those unknown invariants of geometric significance. See references [1, 2, 3, 6, 7, 8, 9, 10] for more details.

Here are some of the different forms of Bonnesen-style isoperimetric inequality.

$$L^2 - 4\pi A \geqslant 4\pi d^2; \qquad L^2 - 4\pi A \geqslant \pi^2 (r_e - r_i)^2;$$

$$L^2 - 4\pi A \geqslant (L - 2\pi r_i)^2; \qquad L^2 - 4\pi A \geqslant (L - 2\pi r_e)^2;$$

$$L^2 - 4\pi A \geqslant (\frac{A}{r} - \pi r)^2; \qquad L^2 - 4\pi A \geqslant L^2 \left(\frac{r_e - r_i}{r_e + r_i}\right)^2;$$

$$L^2 - 4\pi A \geqslant A^2 \left(\frac{1}{r_i} - \frac{1}{r_e}\right)^2; \qquad L^2 - 4\pi A \geqslant A^2 \left(\frac{1}{r} - \frac{1}{r_e}\right)^2.$$

It is difficult to compare those isoperimetric deficit lower bounds and to determine which lower bound is the best.

However, the literature on the study of Bonnesen-type isoperimetric inequalities for planar convex polygon is relatively less (see [5, 11, 12, 13]).

In 1998, Zhang [13] proved a form of Bonnesen-style isoperimetric inequality for planar convex polygon, as follows.

Let $\mathscr{C}_n$ be an n-sided plane convex polygon *inscribed in a circle* of radius R with side-length a_i $(i=1,2,\cdots,n)$ and perimeter L_n, enclosing a domain of area A_n.

$$(L_n)^2 - 4n\tan\frac{\pi}{n}A_n \geqslant \left[L_n - L_n^*\right]^2. \tag{1.2}$$

where L_n^* is the perimeter of the regular convex n-sides polygon *inscribed in the same circle* with $\mathscr{C}_n$.

In 2015, L. Ma [5] obtained a new Bonnesen-style inequality for planar convex polygon

$$(L_n)^2 - 4n\tan\frac{\pi}{n}A_n \geqslant \frac{1}{R^2}\left[A_n - A_n^*\right]^2, \tag{1.3}$$

where A_n^* is the area of the regular convex n-sides polygon *inscribed in the same circle* with $\mathscr{C}_n$.

But Zhang's result and Ma's result are for the planar convex polygon *inscribed in a circle* of radius R.

In the note, we continue to investigate the Bonnesen-type isoperimetric inequalities for the planar convex polygon, but our results are for the planar convex polygon *circumscribed in a circle* of radius r.

2. Some analytic inequalities

In order to simplify the statements. We set

$$I=(0,l); \quad H_n=\{\Theta=(\theta_1,\cdots,\theta_n)\in\mathbb{R}^n | \sum_{i=1}^n \theta_i = ml\} \quad (0<m<n);$$

$$D_n = I^n \cap H_n; \quad \Omega=(\sigma,\sigma,\cdots,\sigma) \quad \text{where} \quad \sigma = \frac{1}{n}\sum_{i=1}^n \theta_i = \frac{ml}{n}.$$

THEOREM 2.1. *Suppose that a real function $f(\theta)$ is positive and strictly convex. Then we have for $\alpha>0$*

$$\left(\sum_{i=1}^n f(\theta_i)\right)^{2\alpha} - (nf(\sigma))^\alpha \left(\sum_{i=1}^n f(\theta_i)\right)^\alpha \geqslant \left[(nf(\sigma))^\alpha - \left(\sum_{i=1}^n f(\theta_i)\right)^\alpha\right]^2. \tag{2.1}$$

In order to prove above result, we need a lemma below.

LEMMA 2.1. ([13]) *If real function $f: I^n \to \mathbb{R}$ is Schur convex, then $f(\Omega)$ is a global minimum in D_n. If f is a strictly Schur convex function, then $f(\Omega)$ is the unique global minimum in D_n.*

Proof of Theorem 2.1. Consider the function

$$F(\Theta)=\left(\sum_{i=1}^{n} f(\theta_i)\right)^{2\alpha}-(nf(\sigma))^{\alpha}\left(\sum_{i=1}^{n} f(\theta_i)\right)^{\alpha}-\left[(nf(\sigma))^{\alpha}-\left(\sum_{i=1}^{n} f(\theta_i)\right)^{\alpha}\right]^2,$$

we observe that $F(\Omega)=0$. We shall prove that $F(\Theta)$ is strictly Schur convex function on I^n where $I=(0,l)$. Obviously, $F(\Theta)$ is a symmetric function on I^n. Hence, by Lemma 1.4, to guarantee $F(\Theta)$ is strictly Schur convex, it suffices to verify that

$$\Delta=(\theta_1-\theta_2)\left(\frac{\partial F}{\partial\theta_1}-\frac{\partial F}{\partial\theta_2}\right),\ if\ \theta_1\neq\theta_2.$$

Furthermore, we set $T_n=\sum_{i=1}^{n} f(\theta_i)$. Then

$$\begin{aligned}\frac{\partial F}{\partial\theta_i}&=2\alpha(T_n)^{2\alpha-1}f'(\theta_i)-(nf(\sigma))^{\alpha}\alpha(T_n)^{\alpha-1}f'(\theta_i)+2\left[nf(\sigma))^{\alpha}-(T_n)^{\alpha}\right]\alpha(T_n)^{\alpha-1}f'(\theta_i)\\&=\alpha(nf(\sigma))^{\alpha}(T_n)^{\alpha-1}f'(\theta_i),\quad i=1,2.\end{aligned}\tag{2.2}$$

$$\Delta=(\theta_1-\theta_2)\left(\frac{\partial F}{\partial\theta_1}-\frac{\partial F}{\partial\theta_2}\right)=(\theta_1-\theta_2)\alpha(nf(\sigma))^{\alpha}(T_n)^{\alpha-1}\left[f'(\theta_1)-f'(\theta_2)\right].\tag{2.3}$$

Since f is strictly convex, then $f''>0$ and

$$(\theta_1-\theta_2)\left[f'(\theta_1)-f'(\theta_2)\right]>0.\tag{2.4}$$

Combine (2.4) and (2.3), inequality (2.1) can be derived. □

By using the strictly convex properties of $f(\theta)=\tan\theta$ and $f(\theta)=\frac{1}{\sin\theta}$ for $\theta\in(0,\pi/2)$ and Theorem 2.1, we get the following results.

COROLLARY 2.1. *Let* $\theta_i\in(0,\pi/2)$, $i=1,2,\cdots,n$; *and* $\sum_{i=1}^{n}\theta_i=\pi$. *Then for* $\alpha>0$

$$\left(\sum_{i=1}^{n}\tan\theta_i\right)^{2\alpha}-(n\tan\frac{\pi}{n})^{\alpha}\left(\sum_{i=1}^{n}\tan\theta_i\right)^{\alpha}\geqslant\left[(n\tan\frac{\pi}{n})^{\alpha}-\left(\sum_{i=1}^{n}\tan\theta_i\right)^{\alpha}\right]^2.\tag{2.5}$$

In particular, take $\alpha=1$, *we have*

$$\left(\sum_{i=1}^{n}\tan\theta_i\right)^{2}-n\tan\frac{\pi}{n}\left(\sum_{i=1}^{n}\tan\theta_i\right)\geqslant\left[n\tan\frac{\pi}{n}-\sum_{i=1}^{n}\tan\theta_i\right]^2.\tag{2.6}$$

COROLLARY 2.2. *Let* $\theta_i\in(0,\pi/2)$, $i=1,2,\cdots,n$; *and* $\sum_{i=1}^{n}\theta_i=\pi$. *Then for* $\alpha>0$

$$\left(\sum_{i=1}^{n}\frac{1}{\sin\theta_i}\right)^{2\alpha}-\left(\frac{n}{\sin\frac{\pi}{n}}\right)^{\alpha}\left(\sum_{i=1}^{n}\frac{1}{\sin\theta_i}\right)^{\alpha}\geqslant\left[\left(\frac{n}{\sin\frac{\pi}{n}}\right)^{\alpha}-\left(\sum_{i=1}^{n}\frac{1}{\sin\theta_i}\right)^{\alpha}\right]^2.\tag{2.7}$$

COROLLARY 2.3. *Let* $x_i \in (0,1)$, $i = 1,2,\cdots,n$*; and* $\sum_{i=1}^{n} x_i = m$*. Then for* $\alpha > 0$

$$\left(\sum_{i=1}^{n} x_i^2\right)^{2\alpha} - \left(\frac{m^2}{n}\right)^{\alpha}\left(\sum_{i=1}^{n} x_i^2\right)^{\alpha} \geqslant \left[\left(\frac{m^2}{n}\right)^{\alpha} - \left(\sum_{i=1}^{n} x_i^2\right)^{\alpha}\right]^2. \tag{2.8}$$

Where we use the fact that $f(x) = x^2$ in $(0,1)$ is strictly convex function.

3. Bonnensen style isoperimetric inequalities of plane convex polygon

In this section, by using above analytic isoperimetric inequalities, we establish some Bonnesen-type isoperimetric inequalities and related inverse inequalities for the planar convex polygon. Our first main result is stated as follows.

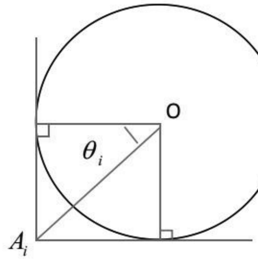
THEOREM 3.1. *Let* $\mathscr{C}_n$ *be an* n*-sided plane convex polygon circumscribed in a circle of radius* r *with perimeter* L_n*, enclosing a domain of area* A_n*. If* $\alpha > 0$*, then*

$$(L_n)^{2\alpha} - 4^{\alpha}\left(n\tan\frac{\pi}{n}\right)^{\alpha}(A_n)^{\alpha} \geqslant \frac{4^{\alpha}}{r^{2\alpha}}\left[(A_n^*)^{\alpha} - (A_n)^{\alpha}\right]^2, \tag{3.1}$$

$$\left(\frac{A_n}{r^2}\right)^{2\alpha} - \left(n\tan\frac{\pi}{n}\right)^{\alpha}\left(\frac{L_n}{2r}\right)^{\alpha} \geqslant \left[\left(\frac{A_n^*}{r^2}\right)^{\alpha} - \left(\frac{A_n}{r^2}\right)^{\alpha}\right]^2, \tag{3.2}$$

where A_n^* *is the area of the regular convex* n*-sides polygon circumscribed in the same circle with* $\mathscr{C}_n$*.*

Proof. We denote a_i the length of the ith side of $\mathscr{C}_n$, and θ_i the half of the central angle subtended by the ith vertex A_i of $\mathscr{C}_n$, $i = 1,2,\cdots,n$, then



$$L_n = \sum_{i=1}^{n} a_i = 2r\sum_{i=1}^{n}\tan\theta_i; \qquad A_n = \frac{1}{2}\sum_{i=1}^{n} a_i \cdot r = r^2\sum_{i=1}^{n}\tan\theta_i; \tag{3.3}$$

$$\sum_{i=1}^{n}\theta_i = \pi; \qquad A_n^* = nr^2\tan\frac{\pi}{n}. \tag{3.4}$$

Substituting (3.3) and (3.4) into (2.5), thus (3.1) and (3.2) are valid. □

REMARK 1. Inequality (3.2) can be regarded as inverse inequality of (3.1).

THEOREM 3.2. *Let $\mathscr{C}_n$ be an n-sided plane convex polygon circumscribed in a circle of radius r with perimeter L_n, enclosing a domain of area A_n. If $\alpha > 0$, Then*

$$(L_n)^{2\alpha} - 4^{\alpha}\left(n\tan\frac{\pi}{n}\right)^{\alpha}(A_n)^{\alpha} \geqslant \left[(l_n^*)^{\alpha} - (L_n)^{\alpha}\right]^2, \tag{3.5}$$

$$\left(\frac{A_n}{r^2}\right)^{2\alpha} - \left(n\tan\frac{\pi}{n}\right)^{\alpha}\left(\frac{L_n}{2r}\right)^{\alpha} \geqslant \left[\left(\frac{l_n^*}{r^2}\right)^{\alpha} - \left(\frac{L_n}{r^2}\right)^{\alpha}\right]^2, \tag{3.6}$$

where l_n^* is the perimeter of the regular convex n-sides polygon circumscribed in the same circle with $\mathscr{C}_n$.

Proof. Similar to the proof of theorem 3.1 and pay attention to the equation $l_n^* = 2nr\tan\frac{\pi}{n}$. □

REMARK 2. Inequality (3.6) can be considered as inverse inequality of (3.5).
Taking $\alpha = 1$, we can derive the following inequalities.

COROLLARY 3.1. *Let $\mathscr{C}_n$ be an n-sided plane convex polygon circumscribed in a circle of radius r with perimeter L_n, enclosing a domain of area A_n. Then*

$$L_n^2 - 4\left(n\tan\frac{\pi}{n}\right)A_n \geqslant \frac{4}{r^2}\left[(A_n^*) - (A_n)\right]^2, \tag{3.7}$$

$$\left(\frac{A_n}{r^2}\right)^2 - \left(n\tan\frac{\pi}{n}\right)\frac{L_n}{2r} \geqslant \left[\left(\frac{A_n^*}{r^2}\right) - \left(\frac{A_n}{r^2}\right)\right]^2, \tag{3.8}$$

$$L_n^2 - 4\left(n\tan\frac{\pi}{n}\right)A_n \geqslant \left[(l_n^*) - (L_n)\right]^2, \tag{3.9}$$

$$\left(\frac{A_n}{r^2}\right)^2 - \left(n\tan\frac{\pi}{n}\right)\frac{L_n}{2r} \geqslant \left[\left(\frac{l_n^*}{r^2}\right) - \left(\frac{L_n}{r^2}\right)\right]^2. \tag{3.10}$$

REMARK 3. Our results (3.7) and (3.9) are different from (1.2) (Zhang's result) and (1.3) (Ma's result). Their results are mainly about an n-sided plane convex polygon *inscribed in a circle* of radius R, while our results in Theorem 3.1 and 3.2 are mainly about an n-sided plane convex polygon *circumscribed in a circle of radius* r.

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Jibing Qi
Department of Mathematics
Shanghai University
Shanghai, 200444, P. R. China
and
School of Mathematics and Statistics
Hefei Normal University
Hefei 230601, P. R. China
e-mail: qijibing@ahu.edu.cn

Wen Wang
School of Mathematics and Statistics
Hefei Normal University
Hefei 230601, P. R. China
and
School of mathematical Science
and
University of Science and Technology of China
Hefei 230026, P. R. China
e-mail: wwen2014@mail.ustc.edu.cn; wenwang1985@163.com