

UNIQUENESS THEOREMS ON FUNCTIONAL INEQUALITIES CONCERNING CUBIC–QUADRATIC–ADDITIVE EQUATION

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Abstract. We prove uniqueness theorems concerning the functional inequalities in connection with an n -dimensional cubic-quadratic-additive equation $\sum_{i=1}^m c_i f(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n) = 0$ by applying the direct method.

1. Introduction

Let V and W be real vector spaces. For a given mapping $f : V \rightarrow W$, we define

$$\begin{aligned} Af(x, y) &:= f(x + y) - f(x) - f(y), \\ Qf(x, y) &:= f(x + y) + f(x - y) - 2f(x) - 2f(y), \\ Cf(x, y) &:= f(x + 2y) - 3f(x + y) - f(x - y) + 3f(x) - 6f(y) \end{aligned}$$

for all $x, y \in V$. A mapping $f : V \rightarrow W$ is called an additive mapping, a quadratic mapping, or a cubic mapping provided f satisfies the functional equation $Af(x, y) = 0$ for all $x, y \in V$, $Qf(x, y) = 0$ for all $x, y \in V$, or $Cf(x, y) = 0$ for all $x, y \in V$, respectively. We note that the mappings $g, h, k : \mathbb{R} \rightarrow \mathbb{R}$ given by $g(x) = ax$, $h(x) = ax^2$, and $k(x) = ax^3$ are solutions of $Ag(x, y) = 0$, $Qh(x, y) = 0$, and $Ck(x, y) = 0$, respectively.

A mapping $f : V \rightarrow W$ is called a cubic-quadratic-additive mapping if and only if f is represented by the sum of an additive mapping, a quadratic mapping, and a cubic mapping. A functional equation is called a cubic-quadratic-additive functional equation provided that each of its solutions is a cubic-quadratic-additive mapping and every cubic-quadratic-additive mapping is also a solution of that equation. The mapping $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = ax^3 + bx^2 + cx$ is a solution of the cubic-quadratic-additive functional equation.

For the study of functional inequalities concerning the cubic-quadratic-additive equations and a broad variety of other types of functional inequalities, the reader is referred to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18].

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Throughout this paper, let V and W be real vector spaces, X a real normed space, Y a real Banach space, and let \mathbb{N}_0 denote the set of all nonnegative integers. For a given mapping $f : V \rightarrow W$, we define $Df : V^n \rightarrow W$ by

$$Df(x_1, x_2, \dots, x_n) := \sum_{i=1}^m c_i f(a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n) \quad (1.1)$$

for all $x_1, x_2, \dots, x_n \in V$, where m is a positive integer and c_i, a_{ij} are real constants.

In this paper, we prove uniqueness theorems that can be easily applied to the investigation of functional inequalities concerning a large class of functional equations of the form $Df(x_1, x_2, \dots, x_n) = 0$, which includes the cubic-quadratic-additive functional equation as a special case. This theorem is particularly useful for proving the Hyers-Ulam stability of a variety of functional equations.

2. Preliminaries

For a given mapping $f : V \rightarrow W$, we use the following abbreviations:

$$\begin{aligned} f_o(x) &:= \frac{f(x) - f(-x)}{2}, & f_e(x) &:= \frac{f(x) + f(-x)}{2}, \\ f_o^{(1)}(x) &:= \frac{a^3 f_o(x) - f_o(ax)}{a^3 - a}, & f_o^{(2)}(x) &:= -\frac{a f_o(x) - f_o(ax)}{a^3 - a} \end{aligned}$$

for all $x \in V$. We will now introduce a lemma that was proved in [14, Corollary 2].

LEMMA 2.1. *Let $k > 1$ be a real constant, let $\phi : V \setminus \{0\} \rightarrow [0, \infty)$ be a function satisfying either*

$$\Phi(x) := \sum_{i=0}^{\infty} \frac{1}{k^i} \phi(k^i x) < \infty \quad (2.1)$$

for all $x \in V \setminus \{0\}$ or

$$\Phi(x) := \sum_{i=0}^{\infty} k^{3i} \phi\left(\frac{x}{k^i}\right) < \infty \quad (2.2)$$

for all $x \in V \setminus \{0\}$, and let $f : V \rightarrow Y$ be an arbitrarily given mapping. If there exists a mapping $F : V \rightarrow Y$ satisfying

$$\|f(x) - F(x)\| \leq \Phi(x) \quad (2.3)$$

for all $x \in V \setminus \{0\}$ and

$$F_o^{(1)}(kx) = kF_o^{(1)}(x), \quad F_e(kx) = k^2F_e(x), \quad F_o^{(2)}(kx) = k^3F_o^{(2)}(x) \quad (2.4)$$

for all $x \in V$, then F is a unique mapping satisfying (2.3) and (2.4).

We introduce lemmas that were proved in [14, Corollary 3].

LEMMA 2.2. *Let $k > 1$ be a real number, let $\phi, \psi : V \setminus \{0\} \rightarrow [0, \infty)$ be functions satisfying each of the following conditions*

$$\begin{aligned} \sum_{i=0}^{\infty} k^i \psi\left(\frac{x}{k^i}\right) < \infty, & \quad \sum_{i=0}^{\infty} \frac{1}{k^{2i}} \phi(k^i x) < \infty, \\ \tilde{\Phi}(x) := \sum_{i=0}^{\infty} k^i \phi\left(\frac{x}{k^i}\right) < \infty, & \quad \tilde{\Psi}(x) := \sum_{i=0}^{\infty} \frac{1}{k^{2i}} \psi(k^i x) < \infty \end{aligned}$$

for all $x \in V \setminus \{0\}$, and let $f : V \rightarrow Y$ be an arbitrarily given mapping. If there exists a mapping $F : V \rightarrow Y$ satisfying the inequality

$$\|f(x) - F(x)\| \leq \tilde{\Phi}(x) + \tilde{\Psi}(x) \quad (2.5)$$

for all $x \in V \setminus \{0\}$ and the conditions in (2.4) for all $x \in V$, then F is a unique mapping satisfying the conditions (2.4) for all $x \in V$ and the inequality (2.5) for all $x \in V \setminus \{0\}$.

LEMMA 2.3. *Let $k > 1$ be a real number, let $\phi, \psi : V \setminus \{0\} \rightarrow [0, \infty)$ be functions satisfying each of the following conditions*

$$\begin{aligned} \sum_{i=0}^{\infty} k^{2i} \psi\left(\frac{x}{k^i}\right) < \infty, & \quad \sum_{i=0}^{\infty} \frac{1}{k^{3i}} \phi(k^i x) < \infty, \\ \tilde{\Phi}(x) := \sum_{i=0}^{\infty} k^{2i} \phi\left(\frac{x}{k^i}\right) < \infty, & \quad \tilde{\Psi}(x) := \sum_{i=0}^{\infty} \frac{1}{k^{3i}} \psi(k^i x) < \infty \end{aligned}$$

for all $x \in V \setminus \{0\}$, and let $f : V \rightarrow Y$ be an arbitrarily given mapping. If there exists a mapping $F : V \rightarrow Y$ satisfying the inequality

$$\|f(x) - F(x)\| \leq \tilde{\Phi}(x) + \tilde{\Psi}(x) \quad (2.6)$$

for all $x \in V \setminus \{0\}$ and the conditions in (2.4) for all $x \in V$, then F is a unique mapping satisfying the conditions (2.4) for all $x \in V$ and the inequality (2.6) for all $x \in V \setminus \{0\}$.

3. Main results

In the following four theorems, we prove that there exists only one exact solution near every approximate solution to $Df(x_1, x_2, \dots, x_n) = 0$.

THEOREM 3.1. *Let a be a real constant with $a \notin \{-1, 0, 1\}$, let n be a fixed integer greater than 1, let $\mu, \nu : V \setminus \{0\} \rightarrow [0, \infty)$ be functions satisfying the conditions*

$$\begin{cases} \sum_{i=0}^{\infty} \frac{\mu(a^i x)}{|a|^{3i}} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} \frac{\nu(a^i x)}{|a|^{3i}} < \infty \quad \text{when } |a| < 1, \\ \sum_{i=0}^{\infty} \frac{\mu(a^i x)}{|a|^i} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} \frac{\nu(a^i x)}{|a|^i} < \infty \quad \text{when } |a| > 1 \end{cases} \quad (3.1)$$

for all $x \in V \setminus \{0\}$, and let $\varphi : (V \setminus \{0\})^n \rightarrow [0, \infty)$ be a function satisfying the condition

$$\begin{cases} \sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \dots, a^i x_n)}{|a|^{3i}} < \infty \text{ when } |a| < 1, \\ \sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \dots, a^i x_n)}{|a|^i} < \infty \text{ when } |a| > 1 \end{cases} \quad (3.2)$$

for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$. If a mapping $f : V \rightarrow Y$ satisfies $f(0) = 0$,

$$\|f_e(ax) - a^2 f_e(x)\| \leq \mu(x) \quad \text{and} \quad \|f_o(a^2 x) - (a + a^3)f_o(ax) + a^4 f_o(x)\| \leq \nu(x) \quad (3.3)$$

for all $x \in V \setminus \{0\}$, and if f moreover satisfies

$$\|Df(x_1, x_2, \dots, x_n)\| \leq \varphi(x_1, x_2, \dots, x_n) \quad (3.4)$$

for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$, then there exists a unique mapping $F : V \rightarrow Y$ such that

$$DF(x_1, x_2, \dots, x_n) = 0 \quad (3.5)$$

for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$, and

$$F_e(ax) = a^2 F_e(x), \quad F_o^{(1)}(ax) = a F_o^{(1)}(x), \quad F_o^{(2)}(ax) = a^3 F_o^{(2)}(x) \quad (3.6)$$

for all $x \in V$, and such that

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \left(\frac{\mu(a^i x)}{a^{2i+2}} + \frac{|a^{2i+2} - 1| \nu(a^i x)}{|a^3 - a| |a|^{3i+3}} \right) \quad (3.7)$$

for all $x \in V \setminus \{0\}$.

Proof. First, we define $A := \{f : V \rightarrow Y \mid f(0) = 0\}$ and a mapping $J_m : A \rightarrow A$ by

$$J_m f(x) := \frac{f_o^{(2)}(a^m x)}{a^{3m}} + \frac{f_e(a^m x)}{a^{2m}} + \frac{f_o^{(1)}(a^m x)}{a^m}$$

for $x \in V$ and $m \in \mathbb{N}_0$. It follows from (3.3) that

$$\begin{aligned} & \|J_m f(x) - J_{m+l} f(x)\| \\ & \leq \sum_{i=m}^{m+l-1} \|J_i f(x) - J_{i+1} f(x)\| \\ & = \sum_{i=m}^{m+l-1} \left\| \frac{f_o(a^{i+1}x) - a f_o(a^i x)}{(a^3 - a)a^{3i}} - \frac{f_o(a^{i+2}x) - a f_o(a^{i+1}x)}{(a^3 - a)a^{3i+3}} \right. \\ & \quad \left. + \frac{f_e(a^i x)}{a^{2i}} - \frac{f_e(a^{i+1}x)}{a^{2i+2}} - \frac{f_o(a^{i+1}x) - a^3 f_o(a^i x)}{(a^3 - a)a^i} \right. \\ & \quad \left. + \frac{f_o(a^{i+2}x) - a^3 f_o(a^{i+1}x)}{(a^3 - a)a^{i+1}} \right\| \end{aligned} \quad (3.8)$$

$$\begin{aligned}
&\leq \sum_{i=m}^{m+l-1} \left\| \frac{f_e(a^{i+1}x) - a^2 f_e(a^i x)}{a^{2i+2}} \right\| \\
&\quad + \sum_{i=m}^{m+l-1} \left\| -\frac{f_o(a^2 a^i x) - (a+a^3)f_o(a^{i+1}x) + a^4 f_o(a^i x)}{(a^3-a)a^{3i+3}} \right. \\
&\quad \quad \left. + \frac{f_o(a^2 a^i x) - (a+a^3)f_o(a^{i+1}x) + a^4 f_o(a^i x)}{(a^3-a)a^{i+1}} \right\| \\
&\leq \sum_{i=m}^{m+l-1} \left(\frac{\mu(a^i x)}{a^{2i+2}} + \frac{|a^{2i+2} - 1| \nu(a^i x)}{|a^3 - a| |a|^{3i+3}} \right)
\end{aligned}$$

for all $x \in V \setminus \{0\}$.

In view of (3.1) and (3.8), the sequence $\{J_m f(x)\}$ is a Cauchy sequence for all $x \in V \setminus \{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_m f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \rightarrow Y$ by

$$F(x) := \lim_{m \rightarrow \infty} J_m f(x) = \lim_{m \rightarrow \infty} \left(\frac{f_o^{(2)}(a^m x)}{a^{3m}} + \frac{f_e(a^m x)}{a^{2m}} + \frac{f_o^{(1)}(a^m x)}{a^m} \right)$$

for all $x \in V$.

We easily obtain from the definition of F that

$$\begin{aligned}
F_o^{(1)}(ax) &= -\frac{F_o(a^2 x) - a^3 F_o(ax)}{a^3 - a} \\
&= -\lim_{m \rightarrow \infty} \left(\frac{f_o(a^{m+3}x) - a f_o(a^{m+2}x)}{a^{3m}(a^3 - a)^2} - \frac{f_o(a^{m+3}x) - a^3 f_o(a^{m+2}x)}{a^m(a^3 - a)^2} \right) \\
&\quad + \lim_{m+1 \rightarrow \infty} a^3 \left(\frac{f_o(a^{(m+1)+2}x) - a f_o(a^{(m+1)+1}x)}{a^{3(m+1)}(a^3 - a)^2} \right. \\
&\quad \quad \left. - \frac{f_o(a^{(m+1)+2}x) - a^3 f_o(a^{(m+1)+1}x)}{a^{(m+1)}(a^3 - a)^2} \right) \\
&= -\lim_{m \rightarrow \infty} \left(\frac{f_o(a^{m+3}x) - a^3 f_o(a^{m+2}x)}{a^{m+1}(a^3 - a)} \right) \\
&= -\lim_{m \rightarrow \infty} \left(\frac{f_o(a^{m+2}ax) - a^3 f_o(a^{m+1}ax)}{a^{m+1}(a^3 - a)} \right) \\
&= -\lim_{m+1 \rightarrow \infty} a \left(\frac{f_o(a^{(m+1)+2}x) - a^3 f_o(a^{(m+1)+1}x)}{a^{(m+1)+1}(a^3 - a)} \right) \\
&= a F_o^{(1)}(x),
\end{aligned}$$

$$\begin{aligned}
F_e(ax) &= \frac{F(ax) + F(-ax)}{2} \\
&= \lim_{m \rightarrow \infty} \frac{f(a^{m+1}x) + f(-a^{m+1}x)}{2a^{2m}}
\end{aligned}$$

$$\begin{aligned}
&= a^2 \lim_{m+1 \rightarrow \infty} \frac{f(a^{m+1}x) + f(-a^{m+1}x)}{2a^{2(m+1)}} \\
&= a^2 F_e(x),
\end{aligned}$$

$$\begin{aligned}
F_o^{(2)}(ax) &= \frac{F(a^2x) - aF(ax)}{a^3 - a} \\
&= \lim_{m \rightarrow \infty} \left(\frac{f_o(a^{m+3}x) - af_o(a^{m+2}x)}{a^{3m}(a^3 - a)^2} - \frac{f_o(a^{m+3}x) - a^3 f_o(a^{m+2}x)}{a^m(a^3 - a)^2} \right) \\
&\quad - \lim_{m+1 \rightarrow \infty} a \left(\frac{f_o(a^{(m+1)+2}x) - af_o(a^{(m+1)+1}x)}{a^{3(m+1)}(a^3 - a)^2} \right. \\
&\quad \quad \left. - \frac{f_o(a^{(m+1)+2}x) - a^3 f_o(a^{(m+1)+1}x)}{a^{(m+1)}(a^3 - a)^2} \right) \\
&= \lim_{m \rightarrow \infty} \left(\frac{f_o(a^{m+2}ax) - af_o(a^{m+1}ax)}{a^{3m+3}(a^3 - a)} \right) \\
&= a^3 \lim_{m+1 \rightarrow \infty} \left(\frac{f_o(a^{(m+1)+2}x) - af_o(a^{(m+1)+1}x)}{a^{3(m+1)+3}(a^3 - a)} \right) \\
&= a^3 F_o^{(2)}(x)
\end{aligned}$$

for all $x \in V$, and by (1.1) and (3.2), we get

$$\begin{aligned}
&\|DF(x_1, x_2, \dots, x_n)\| \\
&= \lim_{m \rightarrow \infty} \left\| \frac{Df_o(a^{m+1}x_1, \dots, a^{m+1}x_n) - aDf_o(a^m x_1, \dots, a^m x_n)}{a^{3m}(a^3 - a)} \right. \\
&\quad \left. + \frac{Df_e(a^m x_1, \dots, a^m x_n)}{a^{2m}} \right. \\
&\quad \left. - \frac{Df_o(a^{m+1}x_1, \dots, a^{m+1}x_n) - a^3 Df_o(a^m x_1, \dots, a^m x_n)}{a^m(a^3 - a)} \right\| \\
&\leq \lim_{m \rightarrow \infty} \left(\left(\frac{1}{|a|^{2m}} + \frac{|a|^{2m+3} + a}{|a^3 - a||a|^{3m}} \right) \varphi_e(a^m x_1, \dots, a^m x_n) \right. \\
&\quad \left. + \frac{|a|^{2m} + 1}{|a^3 - a||a|^{3m}} \varphi_e(a^{m+1} x_1, \dots, a^{m+1} x_n) \right) \\
&= 0
\end{aligned}$$

for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$, where $\varphi_e(x_1, \dots, x_n) := \frac{\varphi(x_1, \dots, x_n) + \varphi(-x_1, \dots, -x_n)}{2}$; *i.e.*,

$$DF(x_1, x_2, \dots, x_n) = 0$$

for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$. Moreover, if we put $m = 0$ and let $l \rightarrow \infty$ in (3.8), then we obtain the inequality (3.7).

Notice that the equalities

$$\begin{aligned} F_o^{(1)}(|a|x) &= |a|F_o^{(1)}(x), & F_o^{(1)}\left(\frac{x}{|a|}\right) &= \frac{F_o^{(1)}(x)}{|a|}, \\ F_e(|a|x) &= |a|^2F_e(x), & F_e\left(\frac{x}{|a|}\right) &= \frac{F_e(x)}{|a|^2}, \\ F_o^{(2)}(|a|x) &= |a|^3F_o^{(2)}(x), & F_o^{(2)}\left(\frac{x}{|a|}\right) &= \frac{F_o^{(2)}(x)}{|a|^3} \end{aligned} \quad (3.9)$$

are true in view of (3.6). Therefore, the equalities in (2.4) hold, for all $x \in V$, with $k = |a|$ if $|a| > 1$ or $k = \frac{1}{|a|}$ if $|a| < 1$.

When $|a| > 1$, in view of Lemma 2.1, there exists a unique mapping $F : V \rightarrow Y$ satisfying the equalities in (3.6) and the inequality (3.7), since the inequality

$$\begin{aligned} \|f(x) - F(x)\| &\leq \sum_{i=0}^{\infty} \left(\frac{\mu(a^i x)}{a^{2i+2}} + \frac{|a^{2i+2} - 1|v(a^i x)}{|a^3 - a||a|^{3i+3}} \right) \\ &\leq \sum_{i=0}^{\infty} \left(\frac{\mu(a^i x)}{|a|^i} + \frac{v(a^i x)}{|a^3 - a||a|^i} \right) \\ &\leq \sum_{i=0}^{\infty} \frac{\phi(k^i x)}{k^i} \end{aligned}$$

holds for all $x \in V \setminus \{0\}$, where we set $k := |a|$ and $\phi(x) := \mu(x) + \mu(-x) + \frac{v(x) + v(-x)}{|a^3 - a|}$.

When $|a| < 1$, in view of Lemma 2.1, there exists a unique mapping $F : V \rightarrow Y$ satisfying the equalities in (3.6) and the inequality (3.7), since the inequality

$$\begin{aligned} \|f(x) - F(x)\| &\leq \sum_{i=0}^{\infty} \left(\frac{\mu(a^i x)}{a^{2i+2}} + \frac{|a^{2i+2} - 1|v(a^i x)}{|a^3 - a||a|^{3i+3}} \right) \\ &\leq \sum_{i=0}^{\infty} \left(\frac{\mu(a^i x)}{|a|^{3i+3}} + \frac{v(a^i x)}{|a^3 - a||a|^{3i+3}} \right) \\ &\leq \sum_{i=0}^{\infty} k^{3i} \phi\left(\frac{x}{k^i}\right) \end{aligned}$$

holds for all $x \in V \setminus \{0\}$, where $k := \frac{1}{|a|}$ and $\phi(x) := \frac{\mu(x) + \mu(-x)}{|a|^3} + \frac{v(x) + v(-x)}{|a^3 - a||a|^3}$. \square

In the following theorem, we assume that μ , ν and φ satisfy other conditions than those of Theorem 3.1 and we prove that there exists a unique exact solution near every approximate solution to $Df(x_1, x_2, \dots, x_n) = 0$.

THEOREM 3.2. *Assume that a is a real constant with $a \notin \{-1, 0, 1\}$. Let n be a fixed integer greater than 1, let $\mu, \nu : V \setminus \{0\} \rightarrow [0, \infty)$ be functions satisfying the*

conditions

$$\begin{cases} \sum_{i=0}^{\infty} |a|^i \mu\left(\frac{x}{a^i}\right) < \infty \text{ and } \sum_{i=0}^{\infty} |a|^i \nu\left(\frac{x}{a^i}\right) < \infty \text{ when } |a| < 1, \\ \sum_{i=0}^{\infty} |a|^{3i} \mu\left(\frac{x}{a^i}\right) < \infty \text{ and } \sum_{i=0}^{\infty} |a|^{3i} \nu\left(\frac{x}{a^i}\right) < \infty \text{ when } |a| > 1 \end{cases} \quad (3.10)$$

for all $x \in V \setminus \{0\}$, and let $\varphi : (V \setminus \{0\})^n \rightarrow [0, \infty)$ be a function satisfying the condition

$$\begin{cases} \sum_{i=0}^{\infty} |a|^i \varphi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \dots, \frac{x_n}{a^i}\right) < \infty \text{ when } |a| < 1, \\ \sum_{i=0}^{\infty} |a|^{3i} \varphi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \dots, \frac{x_n}{a^i}\right) < \infty \text{ when } |a| > 1 \end{cases} \quad (3.11)$$

for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$. If a mapping $f : V \rightarrow Y$ satisfies $f(0) = 0$, the inequalities in (3.3) for all $x \in V \setminus \{0\}$, and (3.4) for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$, then there exists a unique mapping $F : V \rightarrow Y$ satisfying (3.5) for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ and (3.6) for all $x \in V$, and such that

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \left(a^{2i} \mu\left(\frac{x}{a^{i+1}}\right) + \frac{|a^{3i+3} - a^{i+1}|}{|a^3 - a|} \nu\left(\frac{x}{a^{i+2}}\right) \right) \quad (3.12)$$

for all $x \in V \setminus \{0\}$.

Proof. First, we define the mappings $J_m f : V \rightarrow Y$ by

$$J_m f(x) := a^{3m+3} f_o^{(2)}\left(\frac{x}{a^{m+1}}\right) + a^{2m} f_e\left(\frac{x}{a^m}\right) + a^{m+1} f_o^{(1)}\left(\frac{x}{a^{m+1}}\right)$$

for all $x \in V$ and $m \in \mathbb{N}_0$. It follows from (3.3) that

$$\begin{aligned} & \|J_m f(x) - J_{m+1} f(x)\| \\ & \leq \sum_{i=m}^{m+1-1} a^{2i} \left\| f_e\left(\frac{x}{a^{i+1}}\right) - a^2 f_e\left(\frac{x}{a^{i+1}}\right) \right\| \\ & \quad + \frac{1}{|a^3 - a|} \sum_{i=m}^{m+1-1} \left\| a^{3i+3} \left(f_o\left(\frac{a^2 x}{a^{i+2}}\right) - (a + a^3) f_o\left(\frac{ax}{a^{i+2}}\right) + a^4 f\left(\frac{x}{a^{i+2}}\right) \right) \right. \\ & \quad \left. - a^{i+1} \left(f_o\left(\frac{a^2 x}{a^{i+2}}\right) - (a + a^3) f\left(\frac{ax}{a^{i+2}}\right) + a^4 f_o\left(\frac{x}{a^{i+2}}\right) \right) \right\| \\ & \leq \sum_{i=m}^{m+1-1} \left(a^{2i} \mu\left(\frac{x}{a^{i+1}}\right) + \frac{|a^{3i+3} - a^{i+1}|}{|a^3 - a|} \nu\left(\frac{x}{a^{i+2}}\right) \right) \end{aligned} \quad (3.13)$$

for all $x \in V \setminus \{0\}$.

On account of (3.10) and (3.13), the sequence $\{J_m f(x)\}$ is a Cauchy sequence for all $x \in V \setminus \{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_m f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \rightarrow Y$ by

$$F(x) := \lim_{m \rightarrow \infty} \left(a^{3m+3} f_o^{(2)} \left(\frac{x}{a^{m+1}} \right) + a^{2m} f_e \left(\frac{x}{a^m} \right) + a^{m+1} f_o^{(1)} \left(\frac{x}{a^{m+1}} \right) \right)$$

for all $x \in V$. Moreover, if we put $m = 0$ and let $l \rightarrow \infty$ in (3.13), we obtain the inequality (3.12).

In view of the definition of F , (3.4), (3.11), and

$$\begin{aligned} DF(x_1, x_2, \dots, x_n) &= \lim_{m \rightarrow \infty} \left(a^{2m} Df_e \left(\frac{x_1}{a^m}, \dots, \frac{x_n}{a^m} \right) + \frac{a^{3m+3} - a^{m+1}}{a^3 - a} Df_o \left(\frac{x_1}{a^m}, \dots, \frac{x_n}{a^m} \right) \right. \\ &\quad \left. + \frac{a^{m+4} - a^{3m+4}}{a^3 - a} Df_o \left(\frac{x_1}{a^{m+1}}, \dots, \frac{x_n}{a^{m+1}} \right) \right) \end{aligned}$$

for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$, we get the equalities in (3.6) for all $x \in V$ and we further obtain $DF(x_1, x_2, \dots, x_n) = 0$ for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$. We notice that the equalities in (3.9) hold in view of (3.6). Therefore, the equalities in (2.4) hold, for all $x \in V$, with $k = |a|$ if $|a| > 1$ or $k = \frac{1}{|a|}$ if $|a| < 1$.

When $|a| > 1$, according to Lemma 2.1, there exists a unique mapping $F : V \rightarrow Y$ satisfying the equalities in (3.6) and the inequality (3.12), since the inequality

$$\begin{aligned} &\|f(x) - F(x)\| \\ &\leq \sum_{i=0}^{\infty} \left(a^{2i} \mu \left(\frac{x}{a^{i+1}} \right) + \frac{|a^{3i+3} - a^{i+1}|}{|a^3 - a|} \nu \left(\frac{x}{a^{i+2}} \right) \right) \\ &\leq \sum_{i=0}^{\infty} a^{2i} \left(\mu \left(\frac{x}{a^{i+1}} \right) + \mu \left(\frac{-x}{a^{i+1}} \right) \right) + \sum_{i=0}^{\infty} \frac{|a^{3i+3}|}{|a^3 - a|} \left(\nu \left(\frac{x}{a^{i+2}} \right) + \nu \left(\frac{-x}{a^{i+2}} \right) \right) \\ &= \sum_{i=0}^{\infty} k^{3i} \phi \left(\frac{x}{k^i} \right) \end{aligned}$$

holds for all $x \in V \setminus \{0\}$, where $k := |a|$ and $\phi(x) := \mu \left(\frac{x}{a} \right) + \mu \left(\frac{-x}{a} \right) + \frac{|a^3|}{|a^3 - a|} \nu \left(\frac{x}{a^2} \right) + \frac{|a^3|}{|a^3 - a|} \nu \left(\frac{-x}{a^2} \right)$.

When $|a| < 1$, according to Lemma 2.1, there exists a unique mapping $F : V \rightarrow Y$ satisfying the equalities in (3.6) and the inequality (3.12), since the inequality

$$\begin{aligned} &\|f(x) - F(x)\| \\ &\leq \sum_{i=0}^{\infty} \left(a^{2i} \mu \left(\frac{x}{a^{i+1}} \right) + \frac{|a^{3i+3} - a^{i+1}|}{|a^3 - a|} \nu \left(\frac{x}{a^{i+2}} \right) \right) \end{aligned}$$

$$\begin{aligned} &\leq a^2 \sum_{i=0}^{\infty} \left(\mu \left(\frac{x}{a^{i+1}} \right) + \mu \left(\frac{-x}{a^{i+1}} \right) \right) + \frac{|a^i|}{|a^3 - a|} \sum_{i=0}^{\infty} \left(\nu \left(\frac{x}{a^{i+2}} \right) + \nu \left(\frac{-x}{a^{i+2}} \right) \right) \\ &\leq \sum_{i=0}^{\infty} \frac{\phi(k^i x)}{k^i} \end{aligned}$$

holds for all $x \in V \setminus \{0\}$, where $k := \frac{1}{|a|}$ and $\phi(x) := \mu\left(\frac{x}{a}\right) + \mu\left(\frac{-x}{a}\right) + \frac{1}{|a^3 - a|} \nu\left(\frac{x}{a^2}\right) + \frac{1}{|a^3 - a|} \nu\left(\frac{-x}{a^2}\right)$. \square

We assume that μ , ν and ϕ satisfy different conditions from those of Theorems 3.1 and 3.2 and prove that there exists a unique exact solution near every approximate solution to $Df(x_1, x_2, \dots, x_n) = 0$.

THEOREM 3.3. *Let a be a real constant with $a \notin \{-1, 0, 1\}$, let n be a fixed integer greater than 1, let $\mu : V \setminus \{0\} \rightarrow [0, \infty)$ be a function satisfying the conditions*

$$\begin{cases} \sum_{i=0}^{\infty} \frac{\mu(a^i x)}{|a|^{2i}} < \infty & \text{when } |a| > 1, \\ \sum_{i=0}^{\infty} |a|^{2i} \mu\left(\frac{x}{a^i}\right) < \infty & \text{when } |a| < 1 \end{cases} \quad (3.14)$$

for all $x \in V \setminus \{0\}$, let $\nu : V \setminus \{0\} \rightarrow [0, \infty)$ be a function satisfying the conditions

$$\begin{cases} \sum_{i=0}^{\infty} \frac{\nu(a^i x)}{|a|^{2i}} < \infty \text{ and } \sum_{i=0}^{\infty} |a|^i \nu\left(\frac{x}{a^i}\right) < \infty & \text{when } |a| > 1, \\ \sum_{i=0}^{\infty} \frac{\nu(a^i x)}{|a|^i} < \infty \text{ and } \sum_{i=0}^{\infty} |a|^{2i} \nu\left(\frac{x}{a^i}\right) < \infty & \text{when } |a| < 1 \end{cases} \quad (3.15)$$

for all $x \in V \setminus \{0\}$, and let $\phi : (V \setminus \{0\})^n \rightarrow [0, \infty)$ be a function satisfying the conditions

$$\begin{cases} \sum_{i=0}^{\infty} \frac{\phi(a^i x_1, a^i x_2, \dots, a^i x_n)}{|a|^{2i}} < \infty \text{ and } \sum_{i=0}^{\infty} |a|^i \phi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \dots, \frac{x_n}{a^i}\right) < \infty \\ \text{when } |a| > 1, \\ \sum_{i=0}^{\infty} \frac{\phi(a^i x_1, a^i x_2, \dots, a^i x_n)}{|a|^i} < \infty \text{ and } \sum_{i=0}^{\infty} |a|^{2i} \phi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \dots, \frac{x_n}{a^i}\right) < \infty \\ \text{when } |a| < 1 \end{cases} \quad (3.16)$$

for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$. If a mapping $f : V \rightarrow Y$ satisfies $f(0) = 0$ and the inequalities in (3.3) for all $x \in V \setminus \{0\}$ and (3.4) for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$, then there exists a unique mapping $F : V \rightarrow Y$ satisfying the equality (3.5) for all

$x_1, x_2, \dots, x_n \in V \setminus \{0\}$, the equalities in (3.6) for all $x \in V$, and

$$\|f(x) - F(x)\| \leq \begin{cases} \sum_{i=0}^{\infty} \frac{\mu(a^i x)}{a^{2i+2}} + \frac{1}{|a^3 - a|} \sum_{i=0}^{\infty} \left(\frac{\nu(a^i x)}{|a|^{3i+3}} + |a|^i \nu\left(\frac{x}{a^{i+1}}\right) \right) \\ \text{when } |a| > 1, \\ \sum_{i=0}^{\infty} |a|^{2i} \mu\left(\frac{x}{a^{i+1}}\right) + \frac{1}{|a^3 - a|} \sum_{i=0}^{\infty} \left(\frac{\nu(a^i x)}{|a|^{i+1}} + |a|^{3i} \nu\left(\frac{x}{a^{i+1}}\right) \right) \\ \text{when } |a| < 1 \end{cases} \quad (3.17)$$

for all $x \in V \setminus \{0\}$.

Proof. We will divide the proof of this theorem into two cases, the case for $|a| > 1$ and the other case for $|a| < 1$.

Case 1. Assume that $|a| > 1$. We define a set $A := \{f : V \rightarrow Y \mid f(0) = 0\}$ and a mapping $J_m : A \rightarrow A$ by

$$J_m f(x) := \frac{f_o^{(2)}(a^m x)}{a^{3m}} + \frac{f_e(a^m x)}{a^{2m}} + a^m f_o^{(1)}\left(\frac{x}{a^m}\right)$$

for all $x \in V$ and $m \in \mathbb{N}_0$. It follows from (3.3) that

$$\begin{aligned} & \|J_m f(x) - J_{m+l} f(x)\| \\ & \leq \sum_{i=m}^{m+l-1} \left\| \frac{f_e(a^{i+1} x) - a^2 f_e(a^i x)}{a^{2i+2}} \right\| \\ & \quad + \frac{1}{|a^3 - a|} \sum_{i=m}^{m+l-1} \left\| -\frac{f_o(a^2 \cdot a^i x) - (a + a^3) f_o(a^{i+1} x) + a^4 f(a^i x)}{a^{3i+3}} \right. \\ & \quad \quad \left. - a^i \left(f_o\left(\frac{a^2 x}{a^{i+1}}\right) - (a + a^3) f_o\left(\frac{ax}{a^{i+1}}\right) + a^4 f_o\left(\frac{x}{a^{i+1}}\right) \right) \right\| \\ & \leq \sum_{i=m}^{m+l-1} \frac{\mu(a^i x)}{a^{2i+2}} + \frac{1}{|a^3 - a|} \sum_{i=m}^{m+l-1} \left(\frac{\nu(a^i x)}{|a|^{3i+3}} + |a|^i \nu\left(\frac{x}{a^{i+1}}\right) \right) \end{aligned} \quad (3.18)$$

for all $x \in V \setminus \{0\}$.

In view of (3.14), (3.15), and (3.18), the sequence $\{J_m f(x)\}$ is a Cauchy sequence for all $x \in V \setminus \{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_m f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \rightarrow Y$ by

$$F(x) := \lim_{m \rightarrow \infty} \frac{f_o^{(2)}(a^m x)}{a^{3m}} + \frac{f_e(a^m x)}{a^{2m}} + a^m f_o^{(1)}\left(\frac{x}{a^m}\right)$$

for all $x \in V$. Moreover, if we put $m = 0$ and let $l \rightarrow \infty$ in (3.18), we obtain the first inequality of (3.17).

Using the definition of F , (3.4), (3.16), and

$$\begin{aligned} & DF(x_1, x_2, \dots, x_n) \\ &= \lim_{m \rightarrow \infty} \left(\frac{Df_o(a^{m+1}x_1, \dots, a^{m+1}x_n) - aDf_o(a^m x_1, \dots, a^m x_n)}{a^{3m}(a^3 - a)} \right. \\ &\quad \left. + \frac{Df_e(a^m x_1, \dots, a^m x_n)}{a^{2m}} \right. \\ &\quad \left. - \frac{a^m}{(a^3 - a)} \left(Df_o\left(\frac{ax_1}{a^m}, \dots, \frac{ax_n}{a^m}\right) - a^3 Df_o\left(\frac{x_1}{a^m}, \dots, \frac{x_n}{a^m}\right) \right) \right) \end{aligned}$$

for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$, we get the equalities in (3.6) for all $x \in V$ and we further get $DF(x_1, x_2, \dots, x_n) = 0$ for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$. We notice that the equalities in (2.4) are true in view of (3.6), where $k = |a|$.

Using Lemma 2.2, we conclude that there exists a unique mapping $F : V \rightarrow Y$ satisfying the equalities in (3.6) and the first inequality in (3.17), since the inequality

$$\begin{aligned} \|f(x) - F(x)\| &\leq \sum_{i=0}^{\infty} \frac{\mu(a^i x)}{a^{2i+2}} + \frac{1}{|a^3 - a|} \sum_{i=0}^{\infty} \left(\frac{\nu(a^i x)}{|a|^{3i+3}} + |a|^i \nu\left(\frac{x}{a^{i+1}}\right) \right) \\ &\leq \sum_{i=0}^{\infty} \frac{\mu(a^i x)}{a^{2i}} + \frac{1}{|a^3 - a|} \sum_{i=0}^{\infty} \left(\frac{\nu(a^i x)}{|a|^{2i}} + |a|^i \nu\left(\frac{x}{a^{i+1}}\right) \right) \\ &\leq \sum_{i=0}^{\infty} \left(\frac{\psi(k^i x)}{k^{2i}} + k^i \phi\left(\frac{x}{k^i}\right) \right) \end{aligned}$$

holds for all $x \in V \setminus \{0\}$, where $k := |a|$, $\phi(x) := \frac{\nu(\frac{x}{a}) + \nu(\frac{-x}{a})}{|a^3 - a|}$, and $\psi(x) := \mu(x) + \mu(-x) + \frac{\nu(x) + \nu(-x)}{|a^3 - a|a^3}$.

Case 2. We now consider the case of $|a| < 1$ and define a mapping $J_m : A \rightarrow A$ by

$$J_m f(x) := a^{3m} f_o^{(2)}\left(\frac{x}{a^m}\right) + a^{2m} f_e\left(\frac{x}{a^m}\right) + \frac{f_o^{(1)}(a^m x)}{a^m}$$

for all $x \in V$ and $n \in \mathbb{N}_0$. It follows from (3.3) that

$$\begin{aligned} & \|J_m f(x) - J_{m+l} f(x)\| \\ & \leq \sum_{i=m}^{m+l-1} a^{2i} \left\| f_e\left(a \frac{x}{a^{i+1}}\right) - a^2 f_e\left(\frac{x}{a^{i+1}}\right) \right\| \\ & \quad + \frac{1}{|a^3 - a|} \sum_{i=m}^{m+l-1} \left\| a^{3i} \left(f_o\left(\frac{a^2 x}{a^{i+1}}\right) - (a + a^3) f_o\left(\frac{ax}{a^{i+1}}\right) + a^4 f_o\left(\frac{x}{a^{i+1}}\right) \right) \right. \\ & \quad \left. + \frac{f_o(a^2 a^i x) - (a + a^3) f_o(a^{i+1} x) + a^4 f_o(a^i x)}{a^{i+1}} \right\| \quad (3.19) \\ & \leq \sum_{i=m}^{m+l-1} |a|^{2i} \mu\left(\frac{x}{a^{i+1}}\right) + \frac{1}{|a^3 - a|} \sum_{i=m}^{m+l-1} \left(\frac{\nu(a^i x)}{|a|^{i+1}} + |a|^{3i} \nu\left(\frac{x}{a^{i+1}}\right) \right) \end{aligned}$$

for all $x \in V \setminus \{0\}$.

On account of (3.14), (3.15), and (3.19), the sequence $\{J_m f(x)\}$ is a Cauchy sequence for all $x \in V \setminus \{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_m f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \rightarrow Y$ by

$$F(x) := \lim_{m \rightarrow \infty} \left(a^{3m} f_o^{(2)} \left(\frac{x}{a^m} \right) + a^{2m} f_e \left(\frac{x}{a^m} \right) + \frac{f_o^{(1)}(a^m x)}{a^m} \right)$$

for all $x \in V$. Moreover, if we put $m = 0$ and let $l \rightarrow \infty$ in (3.19), we obtain the second inequality in (3.17).

By the definition of F , (3.4), (3.16), and

$$\begin{aligned} & DF(x_1, x_2, \dots, x_n) \\ &= \lim_{m \rightarrow \infty} \left(a^{2m} Df_e \left(\frac{x_1}{a^m}, \dots, \frac{x_n}{a^m} \right) \right. \\ &\quad + \frac{a^{3m}}{a^3 - a} Df_o \left(\frac{x_1}{a^{m-1}}, \dots, \frac{x_n}{a^{m-1}} \right) - \frac{a^{3m+1}}{a^3 - a} Df_o \left(\frac{x_1}{a^m}, \dots, \frac{x_n}{a^m} \right) \\ &\quad \left. - \frac{Df_o(a^{m+1}x_1, \dots, a^{m+1}x_n) - a^3 Df_o(a^m x_1, \dots, a^m x_n)}{a^m(a^3 - a)} \right) \end{aligned}$$

for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$, we get the equalities in (3.6) for all $x \in V$ and we moreover obtain $DF(x_1, x_2, \dots, x_n) = 0$ for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$. We remark that the equalities in (2.4) hold by considering (3.6) with $k = \frac{1}{|a|}$.

Using Lemma 2.2, we conclude that there exists a unique mapping $F : V \rightarrow Y$ satisfying the equalities in (3.6) and the second inequality in (3.17), since the inequality

$$\begin{aligned} \|f(x) - F(x)\| &\leq \sum_{i=0}^{\infty} |a|^{2i} \mu \left(\frac{x}{a^{i+1}} \right) + \frac{1}{|a^3 - a|} \sum_{i=0}^{\infty} \left(\frac{\nu(a^i x)}{|a|^{i+1}} + |a|^{3i} \nu \left(\frac{x}{a^{i+1}} \right) \right) \\ &= \sum_{i=0}^{\infty} \left(\frac{\nu(a^i x) + \nu(-a^i x)}{|a^3 - a| |a|^{i+1}} + |a|^{2i} \left(\mu \left(\frac{x}{a^{i+1}} \right) + \mu \left(\frac{-x}{a^{i+1}} \right) \right) \right. \\ &\quad \left. + \frac{|a|^{2i}}{|a^3 - a|} \left(\nu \left(\frac{x}{a^{i+1}} \right) + \nu \left(\frac{-x}{a^{i+1}} \right) \right) \right) \\ &\leq \sum_{i=0}^{\infty} \left(\frac{\psi(k^i x)}{k^{2i}} + k^i \phi \left(\frac{x}{k^i} \right) \right) \end{aligned}$$

holds for all $x \in V \setminus \{0\}$, where $k := \frac{1}{|a|}$, $\phi(x) := \frac{1}{|a^3 - a| |a|} (\nu(x) + \nu(-x))$, and $\psi(x) := \frac{1}{|a^3 - a|} (\nu(\frac{x}{a}) + \nu(\frac{-x}{a})) + \mu(\frac{x}{a}) + \mu(\frac{-x}{a})$. \square

Suppose μ , ν and ϕ satisfy other conditions from those of preceding three theorems. In the following theorem, we prove that there exists a unique exact solution near every approximate solution to $Df(x_1, x_2, \dots, x_n) = 0$.

THEOREM 3.4. *Suppose a is a real constant with $a \notin \{-1, 0, 1\}$. Let n be a fixed integer greater than 1, let $\mu : V \setminus \{0\} \rightarrow [0, \infty)$ be a function satisfying the conditions*

$$\begin{cases} \sum_{i=0}^{\infty} |a|^{2i} \mu\left(\frac{x}{a^i}\right) < \infty \text{ when } |a| > 1, \\ \sum_{i=0}^{\infty} \frac{\mu(a^i x)}{|a|^{2i}} < \infty \text{ when } |a| < 1 \end{cases} \quad (3.20)$$

for all $x \in V \setminus \{0\}$, let $\nu : V \setminus \{0\} \rightarrow [0, \infty)$ be a function satisfying the conditions

$$\begin{cases} \sum_{i=0}^{\infty} \frac{\nu(a^i x)}{|a|^{3i}} < \infty \text{ and } \sum_{i=0}^{\infty} |a|^{2i} \nu\left(\frac{x}{a^i}\right) < \infty \text{ when } |a| > 1, \\ \sum_{i=0}^{\infty} \frac{\nu(a^i x)}{|a|^{2i}} < \infty \text{ and } \sum_{i=0}^{\infty} |a|^{3i} \nu\left(\frac{x}{a^i}\right) < \infty \text{ when } |a| < 1 \end{cases} \quad (3.21)$$

for all $x \in V \setminus \{0\}$, and let $\varphi : (V \setminus \{0\})^n \rightarrow [0, \infty)$ be a function satisfying the conditions

$$\begin{cases} \sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \dots, a^i x_n)}{|a|^{3i}} < \infty \text{ and } \sum_{i=0}^{\infty} |a|^{2i} \varphi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \dots, \frac{x_n}{a^i}\right) < \infty \\ \text{when } |a| > 1, \\ \sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \dots, a^i x_n)}{|a|^{2i}} < \infty \text{ and } \sum_{i=0}^{\infty} |a|^{3i} \varphi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \dots, \frac{x_n}{a^i}\right) < \infty \\ \text{when } |a| < 1 \end{cases} \quad (3.22)$$

for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$. If a mapping $f : V \rightarrow Y$ satisfies $f(0) = 0$ and the inequalities in (3.3) for all $x \in V \setminus \{0\}$ and (3.4) for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$, then there exists a unique mapping $F : V \rightarrow Y$ satisfying the equality (3.5) for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$, the equalities in (3.6) for all $x \in V$, and

$$\begin{aligned} & \|f(x) - F(x)\| \\ & \leq \begin{cases} |a|^{2i} \mu\left(\frac{x}{a^{i+1}}\right) + \frac{1}{|a^3 - a|} \left(\frac{\nu(a^i x)}{|a|^{3i+3}} + |a|^i \nu\left(\frac{x}{a^{i+1}}\right) \right) \text{ when } |a| > 1, \\ \frac{\mu(a^i x)}{|a|^{2i+2}} + \frac{1}{|a^3 - a|} \left(\frac{\nu(a^i x)}{|a|^{i+1}} + |a|^{3i} \nu\left(\frac{x}{a^{i+1}}\right) \right) \text{ when } |a| < 1 \end{cases} \end{aligned} \quad (3.23)$$

for all $x \in V \setminus \{0\}$.

Proof. We will divide the proof of this theorem into two cases; namely, the case for $|a| > 1$ and the other case for $|a| < 1$.

Case 1. Assume that $|a| > 1$. We define a set $A := \{f : V \rightarrow Y \mid f(0) = 0\}$ and a mapping $J_m : A \rightarrow A$ by

$$J_m f(x) := \frac{f_o^{(2)}(a^m x)}{a^{3m}} + a^{2m} f_e\left(\frac{x}{a^m}\right) + a^m f_o^{(1)}\left(\frac{x}{a^m}\right)$$

for all $x \in V$ and $m \in \mathbb{N}_0$. It follows from (3.3) that

$$\begin{aligned}
& \|J_m f(x) - J_{m+l} f(x)\| \\
& \leq \sum_{i=m}^{m+l-1} \left\| a^{2i} \left(f_e \left(\frac{ax}{a^{i+1}} \right) - a^2 f_e \left(\frac{x}{a^{i+1}} \right) \right) \right\| \\
& \quad + \frac{1}{|a^3 - a|} \sum_{i=m}^{m+l-1} \left\| - \frac{f_o(a^2 a^i x) - (a + a^3) f_o(a^{i+1} x) + a^4 f(a^i x)}{a^{3i+3}} \right. \\
& \quad \quad \left. - a^i \left(f_o \left(\frac{a^2 x}{a^{i+1}} \right) - (a + a^3) f_o \left(\frac{ax}{a^{i+1}} \right) + a^4 f \left(\frac{x}{a^{i+1}} \right) \right) \right\| \\
& \leq \sum_{i=m}^{m+l-1} \left(|a|^{2i} \mu \left(\frac{x}{a^{i+1}} \right) + \frac{1}{|a^3 - a|} \left(\frac{\nu(a^i x)}{|a|^{3i+3}} + |a|^i \nu \left(\frac{x}{a^{i+1}} \right) \right) \right)
\end{aligned} \tag{3.24}$$

for all $x \in V \setminus \{0\}$.

In view of (3.20), (3.21), and (3.24), the sequence $\{J_m f(x)\}$ is a Cauchy sequence for all $x \in V \setminus \{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_m f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \rightarrow Y$ by

$$F(x) := \lim_{m \rightarrow \infty} \left(\frac{f_o^{(2)}(a^m x)}{a^{3m}} + a^{2m} f_e \left(\frac{x}{a^m} \right) + a^m f_o^{(1)} \left(\frac{x}{a^m} \right) \right)$$

for all $x \in V$. Moreover, if we put $m = 0$ and let $l \rightarrow \infty$ in (3.24), we obtain the first inequality of (3.23).

Using the definition of F , (3.4), (3.22), and

$$\begin{aligned}
& DF(x_1, x_2, \dots, x_n) \\
& = \lim_{m \rightarrow \infty} \left(\frac{Df_o(a^{m+1} x_1, \dots, a^{m+1} x_n) - a Df_o(a^m x_1, \dots, a^m x_n)}{a^{3m}(a^3 - a)} + a^{2m} Df_e \left(\frac{x_1}{a^m}, \dots, \frac{x_n}{a^m} \right) \right. \\
& \quad \left. + \frac{a^m}{(a^3 - a)} \left(Df_o \left(\frac{x_1}{a^{m-1}}, \dots, \frac{x_n}{a^{m-1}} \right) - a^3 Df_o \left(\frac{x_1}{a^m}, \dots, \frac{x_n}{a^m} \right) \right) \right)
\end{aligned}$$

for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$, we obtain the equalities in (3.6) for all $x \in V$ and we further get $DF(x_1, x_2, \dots, x_n) = 0$ for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$. We notice that the equalities in (2.4) are true in view of (3.6), where $k = |a|$.

Using Lemma 2.3, we conclude that there exists a unique mapping $F : V \rightarrow Y$ satisfying the equalities in (3.6) and the first inequality in (3.23), since the inequality

$$\begin{aligned}
& \|f(x) - F(x)\| \\
& \leq \sum_{i=0}^{\infty} \left(|a|^{2i} \mu \left(\frac{x}{a^{i+1}} \right) + \frac{1}{|a^3 - a|} \left(\frac{\nu(a^i x)}{|a|^{3i+3}} + |a|^i \nu \left(\frac{x}{a^{i+1}} \right) \right) \right)
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=0}^{\infty} \left(|a|^{2i} \mu \left(\frac{x}{a^{i+1}} \right) + |a|^{2i} \mu \left(\frac{-x}{a^{i+1}} \right) + \frac{|a|^{2i}}{|a^3 - a|} \left(\nu \left(\frac{x}{a^{i+1}} \right) + \nu \left(\frac{-x}{a^{i+1}} \right) \right) \right. \\
&\quad \left. + \frac{1}{|a^3 - a|} \left(\frac{\nu(a^i x)}{|a|^{3i+3}} + \frac{\nu(-a^i x)}{|a|^{3i+3}} \right) \right) \\
&\leq \sum_{i=0}^{\infty} \left(\frac{\Psi(k^i x)}{k^{3i}} + k^{2i} \phi \left(\frac{x}{k^i} \right) \right)
\end{aligned}$$

holds for all $x \in V \setminus \{0\}$, where $k := |a|$, $\phi(x) := \mu\left(\frac{x}{a}\right) + \mu\left(\frac{-x}{a}\right) + \frac{\nu\left(\frac{x}{a}\right) + \nu\left(\frac{-x}{a}\right)}{|a^3 - a|}$, and $\Psi(x) := \frac{\nu(x) + \nu(-x)}{|a^3 - a||a|^3}$.

Case 2. We now consider the case of $|a| < 1$ and define a mapping $J_m : A \rightarrow A$ by

$$J_m f(x) := a^{3m} f_o^{(2)} \left(\frac{x}{a^m} \right) + \frac{f_e(a^m x)}{a^{2m}} + \frac{f_o^{(1)}(a^m x)}{a^m}$$

for all $x \in V$ and $n \in \mathbb{N}_0$. It follows from (3.3) that

$$\begin{aligned}
&\|J_m f(x) - J_{m+l} f(x)\| \\
&\leq \sum_{i=m}^{m+l-1} \left\| \frac{a^2 f_e(a^i x) - f_e(a^i a x)}{a^{2i+2}} \right\| \\
&\quad + \frac{1}{|a^3 - a|} \sum_{i=m}^{m+l-1} \left\| a^{3i} \left(f_o \left(\frac{a^2 x}{a^{i+1}} \right) - (a + a^3) f_o \left(\frac{a x}{a^{i+1}} \right) + a^4 f_o \left(\frac{x}{a^{i+1}} \right) \right) \right. \\
&\quad \quad \left. + \frac{f_o(a^2 a^i x) - (a + a^3) f_o(a^{i+1} x) + a^4 f_o(a^i x)}{a^{i+1}} \right\| \quad (3.25) \\
&\leq \sum_{i=m}^{m+l-1} \left(\frac{\mu(a^i x)}{|a|^{2i+2}} + \frac{1}{|a^3 - a|} \left(\frac{\nu(a^i x)}{|a|^{i+1}} + |a|^{3i} \nu \left(\frac{x}{a^{i+1}} \right) \right) \right)
\end{aligned}$$

for all $x \in V \setminus \{0\}$.

On account of (3.20), (3.21) and (3.25), the sequence $\{J_m f(x)\}$ is a Cauchy sequence for all $x \in V \setminus \{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_m f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \rightarrow Y$ by

$$F(x) := \lim_{m \rightarrow \infty} a^{3m} f_o^{(2)} \left(\frac{x}{a^m} \right) + \frac{f_e(a^m x)}{a^{2m}} + \frac{f_o^{(1)}(a^m x)}{a^m}$$

for all $x \in V$. Moreover, if we put $m = 0$ and let $l \rightarrow \infty$ in (3.25), we obtain the second inequality in (3.23).

By the definition of F , (3.4), (3.22), and

$$\begin{aligned} & DF(x_1, x_2, \dots, x_n) \\ &= \lim_{m \rightarrow \infty} \left(\frac{a^{3m}}{a^3 - a} Df_o \left(\frac{x_1}{a^{m-1}}, \dots, \frac{x_n}{a^{m-1}} \right) - \frac{a^{3m+1}}{a^3 - a} Df_o \left(\frac{x_1}{a^m}, \dots, \frac{x_n}{a^m} \right) \right. \\ &\quad \left. + \frac{Df_e(a^m x_1, \dots, a^m x_n)}{a^{2m}} \right. \\ &\quad \left. - \frac{Df_o(a^{m+1} x_1, \dots, a^{m+1} x_n) - a^3 Df_o(a^m x_1, \dots, a^m x_n)}{a^m (a^3 - a)} \right) \end{aligned}$$

for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$, we get the equalities in (3.6) for all $x \in V$ and we moreover have $DF(x_1, x_2, \dots, x_n) = 0$ for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$. We remark that the equalities in (2.4) hold by considering (3.6) with $k = \frac{1}{|a|}$.

Using Lemma 2.3, we conclude that there exists a unique mapping $F : V \rightarrow Y$ satisfying the equalities in (3.6) and the second inequality in (3.23), since the inequality

$$\begin{aligned} & \|f(x) - F(x)\| \\ &\leq \sum_{i=0}^{\infty} \left(\frac{\mu(a^i x)}{|a|^{2i+2}} + \frac{1}{|a^3 - a|} \left(\frac{\nu(a^i x)}{|a|^{i+1}} + |a|^{3i} \nu \left(\frac{x}{a^{i+1}} \right) \right) \right) \\ &= \sum_{i=0}^{\infty} \left(\frac{\mu(a^i x) + \mu(-a^i x)}{|a|^{2i+2}} + \frac{\nu(a^i x) + \nu(-a^i x)}{|a^3 - a| |a|^{2i+1}} + \frac{|a|^{3i}}{|a^3 - a|} \left(\nu \left(\frac{x}{a^{i+1}} \right) + \nu \left(-\frac{x}{a^{i+1}} \right) \right) \right) \\ &\leq \sum_{i=0}^{\infty} \left(\frac{\psi(k^i x)}{k^{3i}} + k^{2i} \phi \left(\frac{x}{k^i} \right) \right) \end{aligned}$$

holds for all $x \in V \setminus \{0\}$, where $k := \frac{1}{|a|}$, $\phi(x) := \frac{1}{|a|^2} (\mu(x) + \mu(-x)) + \frac{1}{|a^3 - a| |a|} (\nu(x) + \nu(-x))$, and $\psi(x) := \frac{1}{|a^3 - a|} (\mu(\frac{x}{a}) + \mu(-\frac{x}{a}))$. \square

By using Theorems 3.1, 3.2, 3.3, and 3.4, we can prove the following corollary.

COROLLARY 3.5. *Let X be a normed space and let $p, \varepsilon, \theta, \xi$ be real constants such that $p \notin \{1, 2, 3\}$, $a \notin \{-1, 0, 1\}$, $\xi > 0$, and $\theta > 0$. If a mapping $f : X \rightarrow Y$ satisfies $f(0) = 0$,*

$$\|f_e(ax) - a^2 f_e(x)\| \leq \varepsilon \|x\|^p, \quad (3.26)$$

and

$$\|f_o(a^2 x) - (a + a^3) f_o(ax) + a^4 f_o(x)\| \leq \theta \|x\|^p \quad (3.27)$$

for all $x \in X \setminus \{0\}$, as well as if f satisfies the inequality

$$\|Df(x_1, x_2, \dots, x_n)\| \leq \xi (\|x_1\|^p + \dots + \|x_n\|^p) \quad (3.28)$$

for all $x_1, x_2, \dots, x_n \in X \setminus \{0\}$, then there exists a unique mapping $F : X \rightarrow Y$ satisfying (3.5) for all $x_1, x_2, \dots, x_n \in X \setminus \{0\}$, (3.6) for all $x \in X$, as well as

$$\|f(x) - F(x)\| \leq \begin{cases} \left(\frac{|a|}{||a| - |a|^p|} + \frac{|a|^3}{||a|^3 - |a|^p|} \right) \frac{\theta \|x\|^p}{|a|^p |a^3 - a|} + \frac{\varepsilon \|x\|^p}{|a^2 - |a|^p|} \\ \text{when either } |a| \text{ and } p < 1 \text{ or } |a| > 1 \text{ and } p > 3, \\ \left(\frac{1}{||a| - |a|^p|} + \frac{1}{||a|^3 - |a|^p|} \right) \frac{\theta \|x\|^p}{|a^3 - a|} + \frac{\varepsilon \|x\|^p}{|a^2 - |a|^p|} \\ \text{for the other cases} \end{cases} \quad (3.29)$$

for all $x \in X \setminus \{0\}$.

Proof. Let us put $\mu(x) := \varepsilon \|x\|^p$, $\nu(x) := \theta \|x\|^p$, and $\varphi(x_1, x_2, \dots, x_n) := \theta (\|x_1\|^p + \dots + \|x_n\|^p)$ for all $x_1, x_2, \dots, x_n \in X \setminus \{0\}$. Then φ , μ , ν satisfy (3.1) and (3.2) when either $|a| > 1$ and $p < 1$ or when $|a| < 1$ and $p > 3$. If either $|a| > 1$ and $p > 3$ or if $|a| < 1$ and $p < 1$, then φ , μ , ν satisfy (3.10) and (3.11). Moreover, φ , μ , ν satisfy (3.14), (3.15), and (3.16) when $1 < p < 2$ and φ , μ , ν satisfy (3.20), (3.21), and (3.22) when $2 < p < 3$. Therefore, by Theorems 3.1, 3.2, 3.3, and 3.4, there exists a unique mapping $F : X \rightarrow Y$ such that (3.5) holds for all $x_1, x_2, \dots, x_n \in X \setminus \{0\}$, and (3.6) holds for all $x \in X$, and such that (3.29) holds for all $x \in X \setminus \{0\}$. \square

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REFERENCES

- [1] S. CZERWIK, *Functional Equations and Inequalities in Several Variables*, World Sci. Publ., Singapore, 2002.
- [2] P. GÄVRUȚA, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
- [3] L. GÄVRUȚA AND P. GÄVRUȚA, *On a problem of John M. Rassias concerning the stability in Ulam sense of Euler-Lagrange equation*, in: *Functional Equations, Difference Inequalities and Ulam Stability Notions* (J. M. Rassias ed.), Nova Science Publishers, 2010 (pp. 47–53).
- [4] A. GILANYI, *Hyers-Ulam stability of monomial functional equations on a general domain*, Proc. Natl. Sci. USA **96** (1999), 10588–10590.
- [5] M. E. GORDJI, S. K. GHARETAPEH, J. M. RASSIAS AND S. ZOLFAGHARI, *Solution and stability of a mixed type additive, quadratic, and cubic functional equation*, Adv. Difference Equ. **2009** (2009), Article ID 826130, 17 pages.

- [6] M. E. GORDJI AND H. KHODAEI, *Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces*, *Nonlinear Anal.* **71** (2009), 5629–5643.
- [7] D. H. HYERS, *On the stability of the linear functional equation*, *Proc. Natl. Acad. Sci. USA* **27** (1941), 222–224.
- [8] S.-M. JUNG, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer Optimization and Its Applications, vol. 48, Springer, New York, 2011.
- [9] S.-M. JUNG, D. POPA AND M. TH. RASSIAS, *On the stability of the linear functional equation in a single variable on complete metric groups*, *J. Global Optim.* **59** (2014), no. 1, 165–171.
- [10] S.-M. JUNG AND M. TH. RASSIAS, *A linear functional equation of third order associated with the Fibonacci numbers*, *Abstr. Appl. Anal.* **2014** (2014), Article ID 137468, 7 pages.
- [11] S.-M. JUNG, M. TH. RASSIAS AND C. MORTICI, *On a functional equation of trigonometric type*, *Appl. Math. Comput.* **252** (2015), 294–303.
- [12] P. L. KANNAPPAN, *Functional Equations and Inequalities with Applications*, Springer, New York, 2009.
- [13] Y.-H. LEE, *On the generalized Hyers-Ulam stability of the generalized polynomial function of degree 3*, *Tamsui Oxf. J. Math. Sci.* **24** (4) (2008), 429–444.
- [14] Y.-H. LEE AND S.-M. JUNG, *General uniqueness theorem concerning the stability of additive, quadratic, and cubic functional equations*, *Adv. Difference Equ.* 2016 (2016), Article no. 75, 12 pages.
- [15] C. MORTICI, M. TH. RASSIAS AND S.-M. JUNG, *On the stability of a functional equation associated with the Fibonacci numbers*, *Abstr. Appl. Anal.* **2014** (2014), Article ID 546046, 6 pages.
- [16] C. PARK, *A fixed point approach to the Fuzzy stability of an additive-quadratic-cubic functional equation*, *Fixed Point Theory Appl.* **2009** (2009), Article ID 918785, 24 pages.
- [17] TH. M. RASSIAS, *On the stability of the linear mapping in Banach spaces*, *Proc. Amer. Math. Soc.* **72** (1978), 297–300.
- [18] S. M. ULAM, *A Collection of Mathematical Problems*, Interscience Publ., New York, 1960.

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