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A GENERALIZED NONLINEAR SUMS–DIFFERENCE INEQUALITY AND ITS APPLICATIONS

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Abstract. In this paper, we established a generalized sums difference inequality with two variables, which included five sums. By using a lemma, we turned the inequality into a common form. We applied our result to boundary value problem of a partial difference equation for boundedness, uniqueness.

1. Introduction

Gronwall-Bellman inequality is an important tool in the study of existence, uniqueness, boundedness of solutions of differential equations and integral equation. Various generalizations of Gronwall-Bellman type inequality [4, 14] and their applications have attracted great attention of many mathematicians (e.g., [5, 6, 9, 15, 20]). Some recent works can be found, e.g., in [2, 3, 7, 8, 11, 12, 16, 17, 27] and some references therein. In 2005, Agarwal et al. [2] investigated the inequality

$$u(t) \leqslant a(t) + \sum_{i=1}^{n} \int_{b_i(t_0)}^{b_i(t)} g_i(t,s) w_i(u(s)) ds, \quad t_0 \leqslant t < t_1.$$

In 2008, Agarwal et al. [3] discussed the retarded integral inequality

$$\varphi(u(t)) \leqslant c + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} u^q(s)[f_i(s)\varphi(u(s)) + g_i(s)] ds,$$

where c is a constant. In 2009, Chen et al. [7] studied the following retarded integral inequality

$$\psi(u(x,y)) \leqslant c + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} g(s,t) u(s,t)\, dt ds + \int_{\gamma(x_0)}^{\gamma(x)} \int_{\delta(y_0)}^{\delta(y)} f(s,t) u(s,t) \varphi(u(s,t))\, dt ds,$$

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where c is a constant. In 2016, Qin et al. [22] studied the following retarded integral inequality

$$u(t) \leqslant c_1 + c_2 \int_{\alpha(t_0)}^{\alpha(t)} f(s)w(u(s))\,ds + c_3 \int_{\beta(t_0)}^{\beta(t)} g(s)w(u(s))ds + c_4 \int_{t_0}^{T} u(s)^{\lambda}\,ds.$$

With the development of the theory of difference equations, more attentions are paid on some discrete versions of Gronwall type inequalities (e.g., [1, 19, 25, 26, 27, 28] for some early works). Some recent works can be found, e.g., in [10, 18, 21, 23] and some references therein. In 2006, Chueng et al. [10] discussed the inequality

$$u^p(m,n) \leqslant c + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)u(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)u(s,t)\varphi(u(s,t)),$$

where $c \geqslant 0$, and a,b are nonnegative real-valued functions in $\mathbb{Z}_+^2$, and φ is a continuous nondecreasing function with $\varphi(r) > 0$, for $r > 0$. In 2007, Ma and Cheung [18] studied the inequality

$$\psi(u(m,n)) \leqslant a(m,n) + c(m,n)\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} \psi'(u(s,t))[d(s,t)w(u(s,t)) + e(s,t)].$$

In 2009, Wang et al. [24] investigated the inequality

$$\psi(u(m,n)) \leqslant c(m,n) + \sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_i(m,n,s,t)\varphi_i(u(s,t)).$$

In 2013, Feng et al. [13] discussed the inequalities including four sums

$$u^p(m,n) \leqslant c(m,n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\Big[b(s,t,m,n)u^q(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c(\xi,\eta,m,n)u^r(\xi,\eta)\Big] + \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\Big[d(s,t,m,n)u^h(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e(\xi,\eta,m,n)u^j(\xi,\eta)\Big].$$

Motivated by the ideas in [2, 10, 13, 18, 24], in this paper, we establish a more general form of sum-difference inequality

$$\psi(u(m,n)) \leqslant c(m,n) + \sum_{i=1}^{k}\Big(\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1} f_i(s,t,j,l)u^p(s,t) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1} h_i(s,t,j,l)u^q(s,t)\varphi_i(u(j,l))\Big). \tag{1.1}$$

2. Lemma

LEMMA 1. *Suppose w is a continuous and positive functions on $\mathbb{R}_+$, f is a nonnegative function on $\Lambda \times \Lambda$, u is a nonnegative function on Λ, then we can obtain the following formula:*

$$\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1} f(s,t,j,l)w(u(j,l)) = \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} w(u(s,t)) \sum_{j=s+1}^{m-1}\sum_{l=t+1}^{n-1} f(j,l,s,t).$$

where Λ is defined in section 3. Main result.

Proof. We use mathematical induction with respect to m and n. If $m = n = 2$, we obtain

$$\sum_{s=0}^{1}\sum_{t=0}^{1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1} f(s,t,j,l)w(u(j,l)) = f(1,1,0,0)w(u(0,0)),$$

$$\sum_{s=0}^{1}\sum_{t=0}^{1} w(u(s,t)) \sum_{j=s+1}^{1}\sum_{l=t+1}^{1} f(j,l,s,t) = w(u(0,0))f(1,1,0,0).$$

Thus

$$\sum_{s=0}^{1}\sum_{t=0}^{1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1} f(s,t,j,l)w(u(j,l)) = \sum_{s=0}^{1}\sum_{t=0}^{1} w(u(s,t)) \sum_{j=s+1}^{1}\sum_{l=t+1}^{1} f(j,l,s,t).$$

It means that the lemma is true for $m = n = 2$. Suppose that the lemma is true for $m = m_1$, $n = n_1$, that is

$$\sum_{s=0}^{m_1-1}\sum_{t=0}^{n_1-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1} f(s,t,j,l)w(u(j,l)) = \sum_{s=0}^{m_1-1}\sum_{t=0}^{n_1-1} w(u(s,t)) \sum_{j=s+1}^{m_1-1}\sum_{l=t+1}^{n_1-1} f(j,l,s,t).$$

Consider $m = m_1 + 1$, $n = n_1 + 1$, then we have

$$\begin{aligned}
\sum_{s=0}^{m_1}\sum_{t=0}^{n_1} w(u(s,t)) \sum_{j=s+1}^{m_1}\sum_{l=t+1}^{n_1} f(j,l,s,t) &= \sum_{s=0}^{m_1-1}\sum_{t=0}^{n_1-1} w(u(s,t)) \sum_{j=s+1}^{m_1}\sum_{l=t+1}^{n_1} f(j,l,s,t) \\
&= \sum_{s=0}^{m_1-1}\sum_{t=0}^{n_1-1} w(u(s,t)) \sum_{j=s+1}^{m_1-1}\sum_{l=t+1}^{n_1-1} f(j,l,s,t) \\
&\quad + \sum_{s=0}^{m_1-1}\sum_{t=0}^{n_1-1} w(u(s,t))f(m_1,n_1,s,t) \\
&= \sum_{s=0}^{m_1-1}\sum_{t=0}^{n_1-1} w(u(s,t)) \sum_{j=s+1}^{m_1-1}\sum_{l=t+1}^{n_1-1} f(j,l,s,t) \\
&\quad + \sum_{j=0}^{m_1-1}\sum_{l=0}^{n_1-1} f(m_1,n_1,j,l)w(u(j,l)) \\
&= \sum_{s=0}^{m_1}\sum_{t=0}^{n_1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1} f(s,t,j,l)w(u(j,l))
\end{aligned}$$

Using the inductive assumption, thus

$$\sum_{s=0}^{m_1}\sum_{t=0}^{n_1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1} f(s,t,j,l)w(u(j,l)) = \sum_{s=0}^{m_1}\sum_{t=0}^{n_1} w(u(s,t)) \sum_{j=s+1}^{m_1}\sum_{l=t+1}^{n_1} f(j,l,s,t).$$

It implies that it is true for $m = m_1+1, n = n_1+1$. Therefore, it is true for any natural number $m \geqslant 2, n \geqslant 2$. $\square$

3. Main result

Throughout this paper, $\mathbb{R}$ denote the set of all real numbers, let $\mathbb{R}_+ := [0,\infty)$ and $\mathbb{N}_0 := \{0,1,\cdots\}$. $m_1, n_1 \in \mathbb{N}_0$ are given numbers, $I := [0,m_1)\cap\mathbb{N}_0$ and $J := [0,n_1)\cap\mathbb{N}_0$ are two fixed lattices of integer points in $\mathbb{R}$, $\Lambda := I\times J \subset \mathbb{N}_0^2$. For any $(s,t)\in\Lambda$, let $\Lambda_{(s,t)}$ denote the sublattice $[0,s)\times[0,t)\cap\Lambda$ of Λ. For functions $w(m), z(m,n), m,n \in \mathbb{N}_0$, let $\Delta w(m) := w(m+1)-w(m)$ and $\Delta_1 z(m,n) := z(m+1,n)-z(m,n)$. Obviously, the linear difference equation $\Delta x(m) = b(m)$ with the initial condition $x(0) = 0$ has the solution $\sum_{s=0}^{m-1} b(s)$. For convenience, in the sequel we define that $\sum_{s=0}^{0-1} b(s) = 0$.

Consider (1.1)and suppose that

(H_1) ψ is a strictly increasing continuous function on $\mathbb{R}_+$, $\psi(u) > 0$ for all $u > 0$;

(H_2) all $\varphi_i, (i = 1,2,\cdots,k)$ are continuous functions on $\mathbb{R}_+$ and positive on $(0,\infty)$;

(H_3) $c(m,n) > 0$ on $I\times J$, and $c(m,n)$ is nondecreasing in each variable;

(H_4) $p > 0, q > 0$ are constants and $p > q$;

(H_5) all $f_i, h_i (i = 1,2,\cdots,k)$ are nonnegative functions on $\Lambda\times\Lambda$.

We technically consider a sequence of functions $w_i(s)$, which can be calculated recursively by

$$\begin{cases} w_1(s) := \max_{\tau\in[0,s]} \varphi_1(\tau), \\ w_{i+1}(s) := \max_{\tau\in[0,s]}\{\frac{\varphi_{i+1}(\tau)}{w_i(\tau)}\} w_i(s), \quad i = 1,2,\cdots,k-1. \end{cases} \tag{3.1}$$

we define the functions:

$$\Psi_p(u) := \int_0^u \frac{ds}{(\psi^{-1}(s))^p}, \qquad u > 0, \tag{3.2}$$

$$\Psi_q(u) := \int_0^u \frac{ds}{(\psi^{-1}(s))^q}, \qquad u > 0, \tag{3.3}$$

$$W_i(u) := \int_1^u \frac{ds}{w_i(\psi^{-1}(\Psi_p^{-1}(s)))}, \quad i = 1,2,\cdots,k, \ u > 0, \tag{3.4}$$

$$\tilde{W}_i(u) := \int_1^u \frac{ds}{w_i(\psi^{-1}(\Psi_p^{-1}(s)))}, \quad i = 1,2,\cdots,k, \ u > 0.$$

Obviously both Ψ_p and W_i are strictly increasing and continuous functions, let Ψ_p^{-1}, W_i^{-1} denote Ψ, W_i inverse function, respectively, then both Ψ_p^{-1} and W_i^{-1} are also continuous and increasing functions. Furthermore, let

$$\tilde{f}_i(m,n,s,t) := \max_{(\tau,\xi)\in[0,m]\times[0,n]} f_i(\tau,\xi,s,t), \tag{3.5}$$

$$\tilde{h}_i(m,n,s,t) := \max_{(\tau,\xi)\in[0,m]\times[0,n]} h_i(\tau,\xi,s,t), \tag{3.6}$$

which are nondecreasing in m and n for each fixed s and t and satisfies $\tilde{f}_i(m,n,s,t) \geqslant f_i(m,n,s,t) \geqslant 0$, $\tilde{h}_i(m,n,s,t) \geqslant h_i(m,n,s,t) \geqslant 0$, for all $i = 1,2,\cdots,k$.

THEOREM 3.1. *Suppose that $(H_1$–$H_5)$ hold and $u(m,n)$ is a nonnegative function on Λ satisfying (1.1). Then, case one: if $\psi^{-1}(z(m,t)) > 1$,*

$$u(m,n) \leqslant \psi^{-1}\Big\{\Psi_p^{-1}\big[W_k^{-1}\big(W_k(E_k(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_k(m,n,s,t)\big)\big]\Big\}, \tag{3.7}$$

for $(m,n) \in \Lambda_{(M_1,N_1)}$, where

$$E_1(m,n) := \Psi_p(c(m,n)) + \sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{f}_i(s,t,j,l),$$

$$E_i(m,n) := W_{i-1}^{-1}\Big(W_{i-1}(E_{i-1}(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_{i-1}(m,n,s,t)\Big), \quad i = 2,3,\ldots,k,$$

and $(M_1,N_1) \in \Lambda$ is arbitrarily given on the boundary of the lattice

$$\begin{aligned}\mathscr{R} := \Big\{&(m,n)\in\Lambda : W_i(E_i(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{h}_i(m,n,s,t) \leqslant \int_1^{\infty}\frac{ds}{w_i(\psi^{-1}(\Psi_p^{-1}(s)))},\\ &W_i^{-1}\Big(W_i(E_i(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{h}_i(m,n,s,t)\Big) \leqslant \int_1^{\infty}\frac{ds}{\psi^{-1}(s)}, i = 1,2,\cdots,k\Big\}.\end{aligned}$$

Case two: if $\psi^{-1}(z(m,t)) < 1$,

$$u(m,n) \leqslant \psi^{-1}\Big\{\Psi_q^{-1}\big[\tilde{W}_k^{-1}\big(\tilde{W}_k(E_k(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_k(m,n,s,t)\big)\big]\Big\}, \tag{3.8}$$

for $(m,n) \in \Lambda_{(M_1,N_1)}$, where

$$E_1(m,n) := \Psi_q(c(m,n)) + \sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{f}_i(s,t,j,l),$$

$$E_i(m,n) := \tilde{W}_{i-1}^{-1}\Big(\tilde{W}_{i-1}(E_{i-1}(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_{i-1}(m,n,s,t)\Big), \quad i = 2,3,\ldots,k,$$

and $(M_1,N_1)\in\Lambda$ *is arbitrarily given on the boundary of the lattice*

$$\mathscr{R}{:=}\Big\{(m,n)\in\Lambda:\tilde{W}_i(E_i(m,n))+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{h}_i(m,n,s,t)\leqslant\int_1^{\infty}\frac{ds}{\tilde{w}_i(\psi^{-1}(\Psi_q^{-1}(s)))},$$
$$W_i^{-1}\Big(\tilde{W}_i(E_i(m,n))+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{h}_i(m,n,s,t)\Big)\leqslant\int_1^{\infty}\frac{ds}{\psi^{-1}(s)},i=1,2,\cdots,k\Big\}.$$

Proof of Theorem 3.1. First of all, we monotonize some given functions φ_i in the sums. Obviously the sequence $w_i(s)$ defined by $\varphi_i(s)$ in (3.1) are nondecreasing and nonnegative functions and satisfy $w_i(s)\geqslant\varphi_i(s)$, $i=1,2,\cdots,k$. Moreover, the ratio $w_{i+1}(s)/w_i(s)$ are also nondecreasing, $i=1,2,\cdots,k$. By (1.1), (3.5) and (3.6), from (3.1), we have

$$\psi(u(m,n))\leqslant c(m,n)+\sum_{i=1}^{k}\Big(\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{f}_i(s,t,j,l)u^p(s,t)$$
$$+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{h}_i(s,t,j,l)u^q(s,t)w_i(u(j,l))\Big).\tag{3.9}$$

We consider the case that $c(m,n)>0$, for all $(m,n)\in\Lambda$. By H_3, from(3.9), we have

$$\psi(u(m,n))\leqslant c(M,N)+\sum_{i=1}^{k}\Big(\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{f}_i(s,t,j,l)u^p(s,t)$$
$$+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{h}_i(s,t,j,l)u^q(s,t)w_i(u(j,l))\Big).\tag{3.10}$$

for all $(m,n)\in\Lambda_{(M,N)}$, where $0\leqslant M\leqslant M_1$ and $0\leqslant N\leqslant N_1$ are chosen arbitrarily. Let $z(m,n)$ denote the function on the right-hand side of (3.10), which is a nonnegative and nondecreasing function on $\Lambda_{(M,N)}$ and $z(0,n)=c(M,N)$. Then we obtain the equivalent form of (3.10)

$$u(m,n)\leqslant\psi^{-1}(z(m,n)),\quad\forall(m,n)\in\Lambda_{(M,N)}.\tag{3.11}$$

Since w_i is nondecreasing and satisfy $w_i(u)>0$, for $u>0$. By the definition of z and (3.11), from (3.10), we have

$$\begin{aligned}\Delta_1 z(m,n)&=\sum_{i=1}^{k}\sum_{t=0}^{n-1}\sum_{j=0}^{m-1}\sum_{l=0}^{t-1}\tilde{f}_i(m,t,j,l)u^p(m,t)\\&\quad+\sum_{i=1}^{k}\sum_{t=0}^{n-1}\sum_{j=0}^{m-1}\sum_{l=0}^{t-1}\tilde{h}_i(m,t,j,l)u^q(m,t)w_i(u(j,l))\\&\leqslant\sum_{i=1}^{k}\sum_{t=0}^{n-1}\sum_{j=0}^{m-1}\sum_{l=0}^{t-1}\tilde{f}_i(m,t,j,l)(\psi^{-1}(z(m,t)))^p\\&\quad+\sum_{i=1}^{k}\sum_{t=0}^{n-1}\sum_{j=0}^{m-1}\sum_{l=0}^{t-1}\tilde{h}_i(m,t,j,l)(\psi^{-1}(z(m,t)))^q w_i(\psi^{-1}(z(j,l))).\end{aligned}\tag{3.12}$$

Case one: if $\psi^{-1}(z(m,t)) > 1$. Using the monotonicity of ψ^{-1} and z, from (3.12), we have

$$\Delta_1 z(m,n) \leqslant (\psi^{-1}(z(m,n)))^p \Big(\sum_{i=1}^{k} \sum_{t=0}^{n-1} \sum_{j=0}^{m-1} \sum_{l=0}^{t-1} \tilde{f}_i(m,t,j,l) + \sum_{i=1}^{k} \sum_{t=0}^{n-1} \sum_{j=0}^{m-1} \sum_{l=0}^{t-1} \tilde{h}_i(m,t,j,l) w_i(\psi^{-1}(z(j,l))) \Big). \tag{3.13}$$

that is

$$\frac{\Delta_1 z(m,n)}{(\psi^{-1}(z(m,n)))^p} \leqslant \Big(\sum_{i=1}^{k} \sum_{t=0}^{n-1} \sum_{j=0}^{m-1} \sum_{l=0}^{t-1} \tilde{f}_i(m,t,j,l) + \sum_{i=1}^{k} \sum_{t=0}^{n-1} \sum_{j=0}^{m-1} \sum_{l=0}^{t-1} \tilde{h}_i(m,t,j,l) w_i(\psi^{-1}(z(j,l))) \Big). \tag{3.14}$$

On the other hand, by the mean-value theorem for integrals, for arbitrarily given (m,n), $(m+1,n) \in \Lambda_{(M,N)}$, in the open interval $(z(m,n), z(m+1,n))$, there exists ξ, which satisfies

$$\Psi_p(z(m+1,n)) - \Psi_p(z(m,n)) = \int_{z(m,n)}^{z(m+1,n)} \frac{ds}{(\psi^{-1}(s))^p} = \frac{\Delta_1 z(m,n)}{(\psi^{-1}(\xi))^p} \leqslant \frac{\Delta_1 z(m,n)}{(\psi^{-1}(z(m,n)))^p} \tag{3.15}$$

where we use the definition of Ψ_p in (3.2). From (3.14) and (3.15), we obtain

$$\Psi_p(z(m+1,n)) \leqslant \Psi_p(z(m,n)) + \Big(\sum_{i=1}^{k} \sum_{t=0}^{n-1} \sum_{j=0}^{m-1} \sum_{l=0}^{t-1} \tilde{f}_i(m,t,j,l) + \sum_{i=1}^{k} \sum_{t=0}^{n-1} \sum_{j=0}^{m-1} \sum_{l=0}^{t-1} \tilde{h}_i(m,t,j,l) w_i(\psi^{-1}(z(j,l))) \Big). \tag{3.16}$$

Keep n fixed and substitute m with s in (3.16). Then, taking the sums on both sides of (3.16) over $s = 0, 1, \cdots, m-1$, we have

$$\Psi_p(z(m,n)) \leqslant \Psi_p(z(0,n)) + \sum_{i=1}^{k} \Big(\sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{j=0}^{s-1} \sum_{l=0}^{t-1} \tilde{f}_i(s,t,j,l) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{j=0}^{s-1} \sum_{l=0}^{t-1} \tilde{h}_i(s,t,j,l) w_i(\psi^{-1}(z(j,l))) \Big)$$

$$\leqslant \Psi_p(c(M,N)) + \sum_{i=1}^{k}\Big(\sum_{s=0}^{M-1}\sum_{t=0}^{N-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{f}_i(s,t,j,l)$$
$$+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{h}_i(s,t,j,l)w_i(\psi^{-1}(z(j,l)))\Big)$$
$$= C_k(M,N) + \sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{h}_i(s,t,j,l)w_i(\psi^{-1}(z(j,l))), \quad (3.17)$$

where

$$C_k(M,N) = \Psi_p(c(M,N)) + \sum_{i=1}^{k}\sum_{s=0}^{M-1}\sum_{t=0}^{N-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{f}_i(s,t,j,l). \quad (3.18)$$

Let

$$v(m,n) = \Psi_p(z(m,n)).$$

From (3.17), we have

$$v(m,n) \leqslant C_k(M,N) + \sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{h}_i(s,t,j,l)w_i(\psi^{-1}(\Psi_p^{-1}(v(j,l)))), \quad (3.19)$$

for all $(m,n) \in \Lambda_{(M,N)}$. Using the lemma 1, (3.19) can be written as

$$v(m,n) \leqslant C_k(M,N) + \sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_i(m,n,s,t)w_i(\psi^{-1}(\Psi_p^{-1}(v(s,t)))), \quad (3.20)$$

where $\tilde{g}_i(m,n,s,t) = \sum_{j=s+1}^{m-1}\sum_{l=t+1}^{n-1}\tilde{h}_i(j,l,s,t)$. Obviously, $\tilde{g}_i(m,n,s,t)$, $i = 1,2,\cdots k$ are nondecreasing in m and n for each fixed s and t and $\tilde{g}_i(m,n,s,t) \geqslant 0$. Then from (3.20), we have

$$v(m,n) \leqslant C_k(M,N) + \sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_i(M,N,s,t)w_i(\psi^{-1}(\Psi_p^{-1}(v(s,t)))), \quad (3.21)$$

for all $(m,n) \in \Lambda_{(M,N)}$.

From (3.21), we can conclude that

$$v(m,n) \leqslant W_k^{-1}\Big(W_k(E_k(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_k(M,N,s,t)\Big), \quad (3.22)$$

for all $(m,n) \in \Lambda_{(M,N)}$, where

$$E_i(M,N) := W_{i-1}^{-1}\Big(W_{i-1}(E_{i-1}(M,N)) + \sum_{s=0}^{M-1}\sum_{t=0}^{N-1}\tilde{g}_{i-1}(M,N,s,t)\Big), \quad i = 2,\cdots,k, \quad (3.23)$$

$$E_1(M,N) := C_1(M,N).$$

For $k=1$, let $z_1(m,n)$ denote the function on the right-hand side of (3.21), which is a nonnegative and nondecreasing function on $\Lambda_{(M,N)}$, $z_1(0,n)=C_1(M,N)$ and $v(m,n)\leqslant z_1(m,n)$. Then we get

$$\begin{aligned}\Delta_1 z_1(m,n) &= \sum_{t=0}^{n-1}\tilde{g}_1(M,N,m,t)w_1(\psi^{-1}(\Psi_p^{-1}(v(m,t))))\\ &\leqslant \sum_{t=0}^{n-1}\tilde{g}_1(M,N,m,t)w_1(\psi^{-1}(\Psi_p^{-1}(z_1(m,t)))),\end{aligned} \tag{3.24}$$

for all $(m,n)\in\Lambda_{(M,N)}$. From (3.24), we have

$$\frac{\Delta_1 z_1(m,n)}{w_1(\psi^{-1}(\Psi_p^{-1}(z_1(m,n))))}\leqslant\sum_{t=n_0}^{n-1}\tilde{g}_1(M,N,m,t). \tag{3.25}$$

By the mean-value theorem for integrals, there exists ξ in the open interval $(z_1(m,n),z_1(m+1,n))$, for arbitrarily given (m,n), $(m+1,n)\in\Lambda_{(M,N)}$ such that

$$\begin{aligned}W_1(z_1(m+1,n))-W_1(z_1(m,n)) &= \int_{z_1(m,n)}^{z_1(m+1,n)}\frac{ds}{w_1(\psi^{-1}(\Psi_p^{-1}(s)))}\\ &= \frac{\Delta_1 z_1(m,n)}{w_1(\psi^{-1}(\Psi_p^{-1}(\xi)))}\\ &\leqslant \frac{\Delta_1 z_1(m,n)}{w_1(\psi^{-1}(\Psi_p^{-1}(z_1(m,n))))}.\end{aligned} \tag{3.26}$$

From (3.25) and (3.26), we have

$$W_1(z_1(m+1,n))\leqslant W_1(z_1(m,n))+\sum_{t=0}^{n-1}\tilde{g}_1(M,N,m,t). \tag{3.27}$$

Keep n fixed and substitute m with s in (3.27). Then, taking the sums on both sides of (3.27) over $s=0,1,\cdots,m-1$, we have

$$\begin{aligned}W_1(z_1(m,n)) &\leqslant W_1(z_1(0,n))+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_1(M,N,s,t)\\ &= W_1(C_1(M,N))+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_1(M,N,s,t),\end{aligned} \tag{3.28}$$

for all $(m,n)\in\Lambda_{(M,N)}$. Using $v(m,n)\leqslant z_1(m,n)$, from (3.28), we get

$$v(m,n)\leqslant z_1(m,n)\leqslant W_1^{-1}\Big(W_1(C_1(M,N))+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_1(M,N,s,t)\Big), \tag{3.29}$$

for all $(m,n)\in\Lambda_{(M,N)}$. This proves that (3.22) is true for $k=1$.

Next, we make the inductive assumption that (3.22) is true for $k=l$, then

$$v(m,n)\leqslant W_l^{-1}\Big(W_l(E_l(M,N))+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_l(M,N,s,t)\Big), \tag{3.30}$$

for all $(m,n)\in\Lambda_{(M,N)}$, where

$$E_1(M,N):=C_1(M,N),$$
$$E_i(M,N):=W_{i-1}^{-1}\Big(W_{i-1}(E_{i-1}(M,N))+\sum_{s=0}^{M-1}\sum_{t=0}^{N-1}\tilde{g}_{i-1}(M,N,s,t)\Big),$$
$$i=2,3,\cdots,l.$$

Now we consider

$$v(m,n)\leqslant C_{l+1}(M,N)+\sum_{i=1}^{l+1}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_i(M,N,s,t)w_i(\psi^{-1}(\Psi_p^{-1}(v(s,t)))), \tag{3.31}$$

for all $(m,n)\in\Lambda_{(M,N)}$. Let $z_2(m,n)$ denote the nonnegative and nondecreasing function of the right-hand of (3.31), then $z_2(0,n)=C_{l+1}(M,N)$ and $v(m,n)\leqslant z_2(m,n)$. Let

$$\phi_i(u):=w_i(u)/w_1(u),\quad i=1,2,\cdots,l+1. \tag{3.32}$$

By (3.1), we conclude that ϕ_i $i=1,2,\cdots,l$ are nondecreasing functions. From (3.31), we have

$$\begin{aligned}
&\frac{\Delta_1 z_2(m,n)}{w_1(\psi^{-1}(\Psi^{-1}(z_2(m,n))))}\\
&=\frac{\sum_{i=1}^{l+1}\sum_{t=0}^{n-1}\tilde{g}_i(M,N,m,t)w_i(\psi^{-1}(\Psi_p^{-1}(v(m,t))))}{w_1(\psi^{-1}(\Psi^{-1}(z_2(m,n))))}\\
&\leqslant\frac{\sum_{i=1}^{l+1}\sum_{t=0}^{n-1}\tilde{g}_i(M,N,m,t)w_i(\psi^{-1}(\Psi_p^{-1}(z_2(m,t))))}{w_1(\psi^{-1}(\Psi^{-1}(z_2(m,n))))}\\
&\leqslant\sum_{t=0}^{n-1}\tilde{g}_1(M,N,m,t)+\sum_{i=2}^{l+1}\sum_{t=0}^{n-1}\tilde{g}_i(M,N,m,t)\phi_i(\psi^{-1}(\Psi_p^{-1}(z_2(m,t))))\\
&=\sum_{t=0}^{n-1}\tilde{g}_1(M,N,m,t)+\sum_{i=1}^{l}\sum_{t=0}^{n-1}\tilde{g}_{i+1}(M,N,m,t)\phi_{i+1}(\psi^{-1}(\Psi_p^{-1}(z_2(m,t)))).
\end{aligned} \tag{3.33}$$

By the mean-value theorem for integrals, there exists ξ in the open interval $(z_2(m,n),z_2(m+1,n))$, for arbitrarily given (m,n), $(m+1,n)\in\Lambda_{(M,N)}$, then we can obtain the following formula:

$$\begin{aligned}
W_1(z_2(m+1,n))-W_1(z_2(m,n))&=\int_{z_2(m,n)}^{z_2(m+1,n)}\frac{ds}{w_1(\psi^{-1}(\Psi_p^{-1}(s)))}\\
&=\frac{\Delta_1 z_2(m,n)}{w_1(\psi^{-1}(\Psi_p^{-1}(\xi)))}\\
&\leqslant\frac{\Delta_1 z_2(m,n)}{w_1(\psi^{-1}(\Psi_p^{-1}(z_2(m,n))))}.
\end{aligned} \tag{3.34}$$

From (3.33) and (3.34), we get

$$\begin{aligned} & W_1(z_2(m+1,n)) - W_1(z_2(m,n)) \\ \leqslant & \sum_{t=0}^{n-1} \tilde{g}_1(M,N,m,t) + \sum_{i=1}^{l}\sum_{t=0}^{n-1} \tilde{g}_{i+1}(M,N,m,t)\phi_{i+1}(\psi^{-1}(\Psi_p^{-1}(z_2(m,t)))) \end{aligned} \quad (3.35)$$

Substitute m with s in (3.35) and keep n fixed, then taking the sum on both sides of (3.35) over $s = 0,1,\cdots,m-1$, we have

$$\begin{aligned} W_1(z_2(m,n)) \leqslant & W_1(C_{l+1}(M,N)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} \tilde{g}_1(M,N,s,t) \\ & + \sum_{i=1}^{l}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1} \tilde{g}_{i+1}(M,N,s,t)\phi_{i+1}(\psi^{-1}(\Psi_p^{-1}(z_2(s,t)))), \end{aligned} \quad (3.36)$$

for all $(m,n) \in \Lambda_{(M,N)}$.

Let

$$\theta(m,n) := W_1(z_2(m,n)), \quad (3.37)$$

$$\rho_1(M,N) := W_1(C_{l+1}(M,N)) + \sum_{s=0}^{M-1}\sum_{t=0}^{N-1} \tilde{g}_1(M,N,s,t). \quad (3.38)$$

Using (3.37) and (3.38), from (3.36) we have

$$\theta(m,n)) \leqslant \rho_1(M,N) + \sum_{i=1}^{l}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1} \tilde{g}_{i+1}(M,N,s,t)\phi_{i+1}[\psi^{-1}(\Psi_p^{-1}(W_1^{-1}(\theta(s,t))))]. \quad (3.39)$$

It has the same form as (3.21). We are ready to use the inductive assumption for (3.39). Let $\delta(s) := \psi^{-1}(\Psi_p^{-1}(W_1^{-1}(s)))$. Since $\psi^{-1}, \Psi^{-1}, W_1^{-1}, \phi_i$ are continuous, nondecreasing and positive on $(0,\infty)$, each $\phi_i(\delta(s))$ is continuous and nondecreasing on $(0,\infty)$. Moreover

$$\frac{\phi_{i+1}(\delta(s))}{\phi_i(\delta(s))} = \frac{w_{i+1}(\delta(s))}{w_i(\delta(s))} = \max_{\tau\in[0,\delta(s)]}\left\{\frac{\varphi_{i+1}(\tau)}{w_i(\tau)}\right\}, \quad i = 1,2,\cdots,l, \quad (3.40)$$

which is also continuous and nondecreasing on $[0,\infty)$ and positive on $(0,\infty)$. Therefore, by the inductive assumption in (3.30), from (3.39), we have

$$\theta(m,n) \leqslant \Phi_l^{-1}\Big(\Phi_l(\rho_l(M,N)) + \sum_{s=0}^{m-1}\sum_{n=0}^{n-1} \tilde{g}_{l+1}(M,N,s,t)\Big), \quad (3.41)$$

for all $(m,n) \in \Lambda_{(M,N)}$, where

$$\Phi_i(u) := \int_0^u \frac{ds}{\phi_{i+1}(\psi^{-1}(\Psi_p^{-1}(W_1^{-1}(s))))}, \quad u > 0, \quad i = 1,2,\cdots,l \quad (3.42)$$

$$\rho_i(M,N) := \Phi_{i-1}^{-1}\Big(\Phi_{i-1}(\rho_{i-1}(M,N))+\sum_{s=0}^{M-1}\sum_{n=0}^{N-1}\tilde{g}_i(M,N,s,t)\Big),\ \ i=2,3,\cdots,l. \tag{3.43}$$

Note that

$$\begin{aligned}\Phi_i(u)=&\int_0^u \frac{w_1(\psi^{-1}(\Psi_p^{-1}(W_1^{-1}(s))))ds}{w_{i+1}(\psi^{-1}(\Psi_p^{-1}(W_1^{-1}(s))))},\\ =&\int_1^{W_1^{-1}(u)}\frac{ds}{w_{i+1}(\psi^{-1}(\Psi_p^{-1}(s)))}\\ =&\ W_{i+1}(W_1^{-1}(u)),\quad i=1,2,\cdots,l.\end{aligned} \tag{3.44}$$

Thus, from (3.37), (3.41) and (3.44), we have

$$\begin{aligned}v(m,n)\leqslant z_2(m,n)&=W_1^{-1}(\theta(m,n))\\ &\leqslant W_1^{-1}\Big(\Phi_l^{-1}\Big(\Phi_l(\rho_l(M,N))+\sum_{s=0}^{m-1}\sum_{n=0}^{n-1}\tilde{g}_{l+1}(M,N,s,t)\Big)\Big)\\ &=W_{l+1}^{-1}\Big(W_{l+1}\Big(W_1^{-1}(\rho_l(M,N))\Big)+\sum_{s=0}^{m-1}\sum_{n=0}^{n-1}\tilde{g}_{l+1}(M,N,s,t)\Big),\end{aligned} \tag{3.45}$$

for all $(m,n)\in\Lambda_{(M,N)}$. We can prove that the term of $W_1^{-1}(\rho_l(M,N))$ in (3.45) is just the same as $E_{l+1}(M,N)$ defined in (3.23). Let $\tilde{\rho}_i(M,N) := W_1^{-1}(\rho_i(M,N))$. By (3.38), we have

$$\begin{aligned}\tilde{\rho}_1(M,N)&=W_1^{-1}(\rho_1(M,N))\\ &=W_1^{-1}\Big(W_1(C_{l+1}(M,N))+\sum_{s=0}^{M-1}\sum_{t=0}^{N-1}\tilde{g}_1(M,N,s,t)\Big)\\ &=E_2(M,N).\end{aligned}$$

Then by the mathematical induction for i, using (3.43) and (3.44), we get

$$\begin{aligned}\tilde{\rho}_i(M,N)&=W_1^{-1}\Big(\Phi_{i-1}^{-1}\Big(\Phi_{i-1}(\rho_{i-1}(M,N))+\sum_{s=0}^{M-1}\sum_{t=0}^{N-1}\tilde{g}_i(M,N,s,t)\Big)\Big)\\ &=W_i^{-1}\big[W_i(W_1^{-1}(\rho_{i-1}(M,N)))+\sum_{s=0}^{M-1}\sum_{t=0}^{N-1}\tilde{g}_i(M,N,s,t)\big]\\ &=W_i^{-1}\big[W_i(\tilde{\rho}_{i-1}(M,N))+\sum_{s=0}^{M-1}\sum_{t=0}^{N-1}\tilde{g}_i(M,N,s,t)\big]\\ &=E_{i+1}(M,N),\quad i=2,3\cdots,l.\end{aligned}$$

This prove that $W_1^{-1}(\rho_l(M,N))$ in (3.45) is just the same as $E_{l+1}(M,N)$ defined in (3.23). Hence (3.45) can be equivalently written as

$$v(m,n)\leqslant W_{l+1}^{-1}\Big(W_{l+1}(E_{l+1}(M,N))+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_{l+1}(M,N,s,t)\Big),\forall(m,n)\in\Lambda_{(M,N)}. \tag{3.46}$$

The estimation (3.22) of unknown function v in the inequality (3.19) is proved by induction. By (3.11), (3.22) and (3.46), we have

$$u(m,n) \leqslant \psi^{-1}(z(m,n)) \leqslant \psi^{-1}\Big(\Psi_p^{-1}\Big(v(m,n)\Big)\Big)$$
$$\leqslant \psi^{-1}\Big(\Psi_p^{-1}\Big(W_k^{-1}\Big(W_k(E_k(M,N)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_k(M,N,s,t)\Big)\Big)\Big), \quad (3.47)$$

for all $(m,n) \in \Lambda_{(M,N)}$. Let $m = M, n = N$, from (3.47), we have

$$u(M,N) \leqslant \psi^{-1}\Big(\Psi_p^{-1}\Big(W_k^{-1}\Big(W_k(E_k(M,N)) + \sum_{s=0}^{M-1}\sum_{t=0}^{N-1}\tilde{g}_k(M,N,s,t)\Big)\Big)\Big).$$

This proves (3.7), since M and N are chosen arbitrarily.

Case two: if $\psi^{-1}(z(m,t)) < 1$. Using the monotonicity of ψ^{-1} and z, we can deduce $(\psi^{-1}(z(m,n)))^p < (\psi^{-1}(z(m,n)))^q$, from (3.12), we have

$$\Delta_1 z(m,n) \leqslant (\psi^{-1}(z(m,n)))^q \Big(\sum_{i=1}^{k}\sum_{t=0}^{n-1}\sum_{j=0}^{m-1}\sum_{l=0}^{t-1}\tilde{f}_i(m,t,j,l)$$
$$+\sum_{i=1}^{k}\sum_{t=0}^{n-1}\sum_{j=0}^{m-1}\sum_{l=0}^{t-1}\tilde{h}_i(m,t,j,l)w_i(\psi^{-1}(z(j,l)))\Big). \quad (3.48)$$

Using the similar proof process, we get

$$u(m,n) \leqslant \psi^{-1}\Big(\Psi_q^{-1}\Big(\tilde{W}_k^{-1}\Big(\tilde{W}_k(E_k(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_k(m,n,s,t)\Big)\Big)\Big). \quad \square$$

REMARK 1. When $f_i = 0$, $i = 1,2,\cdots,k$, $q = 0$, Theorem 2.1 reduces to Theorem 2.1 in [24].

REMARK 2. When $f_i = 0$, $i = 1,2,\cdots,k$, $q = 0$, $\varphi_1(u) = u^q$, $\varphi_2(u) = u^r$, $\varphi_3(u) = u^h$, $\varphi_4(u) = u^j$, Theorem 2.1 reduces to Theorem 5 in [13].

4. Corollary

COROLLARY 1. Suppose that (H_2, H_3, H_5) hold and $p = q = 1$, $\psi(u(m,n)) = u(m,n)$ is a nonnegative function on Λ satisfying

$$u(m,n) \leqslant c(m,n) + \sum_{i=1}^{k}\Big(\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1} f_i(s,t,j,l)u(s,t)$$
$$+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1} h_i(s,t,j,l)u(s,t)\varphi_i(u(j,l))\Big).$$

Then

$$u(m,n) \leqslant \exp\left[\hat{W}_k^{-1}\left(\hat{W}_k(\hat{E}_k(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_k(m,n,s,t)\right)\right],$$

for $(m,n) \in \Lambda_{(M_1,N_1)}$, where

$$\hat{W}_i(u) := \int_1^u \frac{ds}{w_i(\exp(s))}, \quad i = 1,2,\cdots,k, \ \ u > 0.$$

$$\hat{E}_1(m,n) := \ln(c(m,n)) + \sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{f}_i(s,t,j,l),$$

$$\hat{E}_i(m,n) := \hat{W}_{i-1}^{-1}\Big(\hat{W}_{i-1}(\hat{E}_{i-1}(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_{i-1}(m,n,s,t)\Big), \quad i = 2,3,\ldots,k,$$

and $(M_1,N_1) \in \Lambda$ is arbitrarily given on the boundary of the lattice.

COROLLARY 2. Suppose that $(H_2–H_5)$ hold and $f_i = 0$, $0 < p < 1$, $\psi(u(m,n)) = u(m,n)$ is a nonnegative function on Λ satisfying

$$u(m,n) \leqslant c(m,n) + \sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1} h_i(s,t,j,l)u^p(s,t)\varphi_i(u(j,l)).$$

Then

$$u(m,n) \leqslant \left[\bar{W}_k^{-1}\left(\bar{W}_k(\bar{E}_k(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_k(m,n,s,t)\right)\right]^{\frac{1}{1-p}},$$

for $(m,n) \in \Lambda_{(M_1,N_1)}$, where

$$\bar{W}_i(u) := \int_1^u \frac{ds}{w_i(s^{\frac{1}{1-p}})}, \quad i = 1,2,\cdots,k, \ \ u > 0.$$

$$\bar{E}_1(m,n) := c(m,n)^{\frac{1}{1-p}}$$

$$\bar{E}_i(m,n) := \bar{W}_{i-1}^{-1}\Big(\bar{W}_{i-1}(\bar{E}_{i-1}(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_{i-1}(m,n,s,t)\Big), \quad i = 2,3,\ldots,k,$$

and $(M_1,N_1) \in \Lambda$ is arbitrarily given on the boundary of the lattice.

5. Applications

In this section, we apply our result to study the boundedness, uniqueness of the solutions of boundary value problems to certain difference equations. We consider the partial difference equation with the initial boundary value conditions.

$$\Delta_2\Delta_1\psi(z(m,n)) = F(m,n,\varphi_1(z(m,n)),\cdots,\varphi_k(z(m,n))), \tag{5.1}$$

$$\psi(z(m,0)) = a_1(m), \psi(z(0,n)) = a_2(n), a_1(0) = a_2(0) = 0, \tag{5.2}$$

for all $(m,n) \in \Lambda$, where $\Lambda = I \times J$ is defined as in the section 2, ψ is a continuous and strictly increasing odd function on $\mathbb{R}$, satisfying $\psi(0) = 0$ and $\psi(u) > 0$ for $u > 0$, $F : \Lambda \times \mathbb{R}^k \to \mathbb{R}$, $a_1 : I \to \mathbb{R}$ and $a_2 : J \to \mathbb{R}$, $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing continuous functions and the ratio φ_{i+1}/φ_i are also nondecreasing, $\varphi_i(u) > 0$ for $u > 0$ $i = 1, 2, \cdots, k$.

In the following corollary, we firstly apply our result to discuss boundedness on the solution of problem (5.1).

COROLLARY 3. Assume that $F : \Lambda \times \mathbb{R}^k \to \mathbb{R}$ is a continuous function satisfying

$$|F(m,n,\varphi_1(u),\cdots,\varphi_k(u))| \leqslant \sum_{i=1}^{k} \left[f_i(M,N,m,n)|u|^p + g_i(M,N,m,n)|u|^q \varphi_i(|u|) \right], \tag{5.3}$$

$$|a_1(m) + a_2(n)| \leqslant a(m,n), \tag{5.4}$$

for all $(m,n) \in \Lambda$, where $p > q > 0$ is a constant, $f_i(M,N,m,n), g_i(M,N,m,n)$, $i = 1, 2, \cdots, k$, are continuous nonnegative functions and nondecreasing in M and N for each fixed m and n, $a(m,n) : \Lambda \to \mathbb{R}_+$ is nondecreasing in each variable. If $z(m,n)$ is any solution of (5.1) with the condition (5.2), then, case one: if $\psi^{-1}(\tilde{z}(m,t)) > 1$,

$$|z(m,n)| \leqslant \psi^{-1}\left\{ \Psi_p^{-1}\left[G_k^{-1}\left(G_k(H_k(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} g_k(M,N,s,t) \right) \right] \right\}, \tag{5.5}$$

for all $(m,n) \in \Lambda_{(M,N)}$, where $\Psi_p(u)$ is defined by (3.2), and

$$G_i(u) := \int_1^u \frac{ds}{\varphi_i(\psi^{-1}(\Psi_p^{-1}(s)))}, \; u > 0,$$

$$H_1(m,n) := \Psi_p(a(m,n)),$$

$$H_i(m,n) := G_{i-1}^{-1}[G_{i-1}(H_{i-1}(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} g_{i-1}(M,N,s,t)],$$

Ψ_p^{-1} and G_k^{-1} denote the inverse function of Ψ_p and G.

Case two: if $\psi^{-1}(\tilde{z}(m,t)) < 1$,

$$|z(m,n)| \leqslant \psi^{-1}\left\{ \Psi_q^{-1}\left[\tilde{G}_k^{-1}\left(\tilde{G}_k(\tilde{H}_k(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} g_k(M,N,s,t) \right) \right] \right\}, \tag{5.6}$$

for all $(m,n) \in \Lambda_{(M,N)}$, where $\Psi_q(u)$ is defined by (3.3), and

$$\tilde{G}_i(u) := \int_1^u \frac{ds}{\varphi_i(\psi^{-1}(\Psi_q^{-1}(s)))}, \; u > 0,$$

$$\tilde{H}_1(m,n) := \Psi_q(a(m,n)),$$

$$\tilde{H}_i(m,n) := \tilde{G}_{i-1}^{-1}[\tilde{G}_{i-1}(\tilde{H}_{i-1}(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} g_{i-1}(M,N,s,t)],$$

Ψ_q^{-1} and $\tilde{G}_k^{-1}$ denote the inverse function of Ψ_q and $\tilde{G}$.

Proof. The solution $z(m,n)$ of (5.1) satisfies the following equivalent difference equation:

$$\psi(z(m,n)) = a_1(m) + a_2(n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} F(s,t,\varphi_1(z(s,t)),\cdots,\varphi_k(z(s,t))). \tag{5.7}$$

By (5.3), (5.4) and (5.7), we obtain

$$\begin{aligned}|\psi(z(m,n))| &= |a_1(m)+a_2(n)| + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} |F(s,t,\varphi_1(z(s,t)),\cdots,\varphi_k(z(s,t)))| \\ &\leqslant a(m,n) + \sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1} f_i(M,N,s,t)|z(s,t)^p| \\ &\quad + \sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1} |z(s,t)|^q g_i(M,N,s,t)\big]\,\varphi_i(|z(s,t)|).\end{aligned} \tag{5.8}$$

Since $|\psi(z(m,n))| = \psi(|z(m,n)|)$, (5.8) has the same form of (1.1). Let $\tilde{z}(m,n)$ denote the function on the right-hand side of (5.8), then $|z(m,n)| \leqslant \psi^{-1}(\tilde{z}(m,n))$. Applying Theorem 3.1 to inequality (5.8), we obtain the estimation of $z(m,n)$ as given in (5.5) and (5.6).

If there exists a constant $M > 0$,

$$H_i(m,n) < M, \quad \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} g_i(M,N,s,t) < M, \quad i = 1,2,\cdots,k, \tag{5.9}$$

for all $(m,n) \in \Lambda_{(M,N)}$, then every solution $z(m,n)$ of (5.1) is bounded on $\Lambda_{(M,N)}$. □

Next, we discuss the uniqueness of the solutions of (5.1).

COROLLARY 4. Assume additionally that

$$\begin{aligned}&|F(m,n,\varphi_1(u_1),\cdots,\varphi_k(u_1)) - F(m,n,\varphi_1(u_2),\cdots,\varphi_k(u_2))| \\ &\quad\leqslant \sum_{i=1}^{k} h_i(M,N,m,n)|\psi(u_1)-\psi(u_2)|^q \varphi_i(|\psi(u_1)-\psi(u_2)|)\end{aligned} \tag{5.10}$$

for $u_1, u_2 \in \mathbb{R}$ and $(m,n) \in \Lambda$, where Λ is defined in the section 2, $h_i : \Lambda \to \mathbb{R}_+$ are nonnegative functions, $i = 1,2,\cdots,k$, $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous nondecreasing with the nondecreasing ratio φ_{i+1}/φ_i such that $\varphi_i(u) > 0$ for all $u > 0$, and $\int_0^1 \frac{ds}{\varphi_i(s)} = \infty$, for $i = 1,2,\cdots,k$, and $\psi : \mathbb{R} \to \mathbb{R}$ is a strictly increasing odd function satisfying $\psi(u) > 0$, for all $u > 0$. Then, (5.1) has at most one solution on Λ.

Proof. Let $z(m,n)$ and $\tilde{z}(m,n)$ are two solutions of (5.1). By (5.7) and (5.10), we have

$$\begin{aligned}&|\psi(z(m,n)) - \psi(\tilde{z}(m,n))| \\ &\leqslant \sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1} h_i(M,N,s,t)|\psi(z(s,t))-\psi(\tilde{z}(s,t))|^q \varphi_i(|\psi(z(s,t))-\psi(\tilde{z}(s,t))|)\end{aligned} \tag{5.11}$$

for all $(m,n) \in \Lambda$, (5.11) is the special form of (1.1), where $f_i = 0$, $i = 1,2,\cdots,k$, $a(m,n) = 0$, $h_i(M,N,s,t)$, $i = 1,2,\cdots,k$, are continuous nonnegative functions and nondecreasing in M and N for each fixed s and t. Applying Theorem 3.1, we obtain an estimation of the difference $|\psi(z(m,n)) - \psi(\tilde{z}(m,n))|$ in the form (5.5), where $H_1(m,n) = 0$, because $\Psi_p(0) = 0$. Furthermore, by the definition of G_i, we conclude that

$$\lim_{u\to 0} G_i(u) = -\infty, \quad \lim_{u\to\infty} G_i^{-1}(u) = 0, \; i = 1,2,\cdots,k.$$

It follows that

$$G_i(H_i(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} h_i(M,N,s,t) = -\infty,$$

$$G_i^{-1}[G_i(\tilde{H}_i(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} h_i(M,N,s,t)] = 0, \; i = 1,2,\cdots,k.$$

From (5.5), we deduce that $|\psi(z(m,n)) - \psi(\tilde{z}(m,n))| \leqslant 0$, implying that $z(m,n) = \tilde{z}(m,n)$, for all $(m,n) \in \Lambda$. □

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