ANOTHER LOOK AT VORONOVSKAJA TYPE FORMULAS

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Abstract. Voronovskaja type formulas are usually established for sequences of positive linear operators, i.e., operators $P_n$ with $P_n f \geq 0$ whenever the function $f$ is positive. The aim of this paper is twofold. First, we obtain some general formulas concerning compositions of operators on Banach spaces, without any assumption of positivity. Next, we establish Voronovskaja type formulas for operators which are manifestly nonpositive. Combining these two approaches we recover some known results and obtain new applications. The final section is devoted to inequalities accompanying our formulas.

1. Introduction

Voronovskaja type formulas are important tools in Approximation Theory. Rates of convergence and saturation properties for sequences of positive linear operators can be established by using such formulas. The prototypical Voronovskaja formula is related with the classical Bernstein operators $B_n : C[0, 1] \to C[0, 1]$,

$$B_n f (t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} f \left( \frac{k}{n} \right), \quad f \in C[0, 1], \quad t \in [0, 1],$$

and reads as follows:

$$\lim_{n \to \infty} n \left( B_n f(t) - f(t) \right) = \frac{t(1-t)}{2} f^{(2)}(t), \quad f \in C^2[0, 1], \quad (1.1)$$

uniformly on $[0, 1]$. Moreover (see, e.g. [1]),

$$\lim_{n \to \infty} n \left[ n \left( B_n f(t) - f(t) \right) - \frac{t(1-t)}{2} f^{(2)}(t) \right] = \frac{t(1-t)}{24} \left( 3t (1-t) f^{(4)}(t) + 4 (1-2t) f^{(3)}(t) \right), \quad f \in C^4[0, 1], \quad (1.2)$$

uniformly on $[0, 1]$.


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Consider also the Beta operators of Lupas and Mühlbach, \( \overline{B}_n : C[0, 1] \to C[0, 1] \), defined by (see, e.g. [12], [13], [14], [4])

\[
\overline{B}_n f(x) = \begin{cases} 
    f(0), & x = 0, \\
    \frac{1}{\overline{B}(nx, n-nx)} \int_0^1 t^{nx-1}(1-t)^{n-1-nx} f(t) dt, & 0 < x < 1, \\
    f(1), & x = 1,
\end{cases}
\]

where \( B \) is Euler’s Beta function.

In this case, see [9],

\[
\lim_{n \to \infty} n \left( \overline{B}_n f(t) - f(t) \right) = \frac{t(1-t)}{2} f^{(2)}(t), \quad f \in C^2[0, 1], \quad (1.3)
\]

\[
\lim_{n \to \infty} n \left[ n \left( \overline{B}_n f(t) - f(t) \right) - \frac{t(1-t)}{2} f^{(2)}(t) \right] = \frac{t(1-t)}{24} \left( 3t(1-t)f^{(4)}(t) + 8(1-2t)f^{(3)}(t) - 12f^{(2)}(t) \right), \quad f \in C^4[0, 1], \quad (1.4)
\]

uniformly on \([0, 1]\).

It is easy to compute the images of monomials under \( \overline{B}_n \); see (3.1) below. Convenient expressions for the images of monomials under \( \overline{B}_n^{-1} \) are also known, see [3]. Due to these circumstances, the operators \( \overline{B}_n \) are used in order to represent other classical operators by means of composition and decomposition; see [3], [10], [11] and the references given there.

In Section 2 we determine the Voronovskaja formula for the sequence \((P_n Q_n)_{n \geq 1}\) in terms of the corresponding formulas for the (quite general) sequences \((P_n)_{n \geq 1}\) and \((Q_n)_{n \geq 1}\). Some examples illustrate this general result.

At the beginning of Section 3 we present a new formula describing the action of \( \overline{B}_n^{-1} \) on polynomials.

Combining this formula with the results of Section 2, we give new proofs and extend some Voronovskaja type results for \( \overline{B}_n^{-1} \) obtained in [3] and [10].

Section 4 is devoted to the operator \( F_n := \overline{B}_n^{-1} \circ B_n \). It was investigated in [3], [8], [10], [11] in relation with the problem of decomposing \( B_n \) into simpler blocks. With the methods already used in Section 3, we extend and give new proofs to the Voronovskaja type formulas for \( F_n \) and \( F_n^{-1} \) obtained in [3].

Voronovskaja type formulas are usually accompanied by inequalities. For example, (1.1) is accompanied by the inequality \( B_n f \geq f \), \( n \geq 1 \), \( f \in C[0, 1] \) convex.

In Section 5 we present a similar inequality for the operator \( \overline{B}_n^{-1} \).

Throughout the paper we consider the spaces \( C^k[0, 1] \), \( k \geq 0 \), endowed with the topology of uniform convergence.
2. A general result

Let $X$ be a Banach space and $W \subset Z \subset Y$ linear subspaces of $X$. Let $A, B : Y \to X$; $U, V : Z \to X$; $S, T : W \to X$ be linear operators.

Consider also two sequences of linear operators $P_n : X \to X$, $Q_n : Y \to X$, $n \geq 1$, and suppose that each $P_n$ is bounded.

**THEOREM 2.1.** (i) Suppose that

$$\lim_{n \to \infty} P_n x = x, \quad x \in X,$$

$$\lim_{n \to \infty} n(P_n y - y) = Ay; \quad \lim_{n \to \infty} n(Q_n y - y) = By, \quad y \in Y. \quad (2.1)$$

Then

$$\lim_{n \to \infty} n(P_n Q_n y - y) = Ay + By, \quad y \in Y. \quad (2.2)$$

(ii) In addition to (2.1) and (2.2), suppose that

$$Bz \in Y, \quad z \in Z, \quad (2.4)$$

Then

$$\lim_{n \to \infty} n(P_n z - z) - Az = Uz; \quad \lim_{n \to \infty} n(Q_n z - z) - Bz = Vz, \quad z \in Z. \quad (2.5)$$

Then

$$\lim_{n \to \infty} n(P_n Q_n z - z) - Az - Bz = Uz + Vz + ABz, \quad z \in Z. \quad (2.6)$$

(iii) Let (2.1), (2.2), (2.4), (2.5) be satisfied. Moreover, suppose that for each $w \in W$ we have $Vw \in Y$, $Bw \in Z$, and

$$\lim_{n \to \infty} n(P_n w - w) - Aw = Uw \quad Sw,$$

$$\lim_{n \to \infty} n(Q_n w - w) - Bw = Vw \quad Tw.$$ Then, for all $w \in W$,

$$\lim_{n \to \infty} n(P_n Q_n w - w) - Aw - Bw = Uw - Vw - ABw \quad Sw + Tw + AVw + UBw.$$

**Proof.** (i) By using (2.1) and the Banach-Steinhaus Theorem we infer that the sequence $(\|P_n\|)_{n \geq 1}$ is bounded, i.e., there exists $M > 0$ such that $\|P_n\| \leq M$, $n \geq 1$. Let $y \in Y$. Then

$$n(P_n Q_n y - y) = n(P_n y - y) + P_n By + P_n [n(Q_n y - y) - By].$$

According to (2.2) and (2.1), the first two summands tend to $Ay$, respectively $By$, while

$$\|P_n [n(Q_n y - y) - By]\| \leq M \|n(Q_n y - y) - By\| \to 0.$$
This proves (2.3).

(ii) Let \( z \in Z \). Then

\[
n\left[ n(P_n Q_n z - z) - Az - Bz \right]
\]

\[
= n\left[ n(P_n z - z) - Az \right] + P_n V z + n(P_n Bz - Bz) + P_n \left\{ n\left[ n(Q_n z - z) - Bz \right] - V z \right\}.
\]

According to (2.5), (2.1), (2.4) and (2.2), the first three summands tend, respectively, to \( U z, V z, ABz \).

Moreover,

\[
\left\| P_n \left\{ n\left[ n(Q_n z - z) - Bz \right] - V z \right\} \right\| \leq M \left\| n\left[ n(Q_n z - z) - Bz \right] - V z \right\| \to 0.
\]

This proves (2.6).

(iii) The proof of (iii) is similar and we omit the details. \( \Box \)

**Example 2.2.** With notation from Section 1, let \( P_n := B_n \) and \( Q_n := \overline{B}_n \). Then, according to (1.1)–(1.4), we have

\[
Af(t) = Bf(t) = \frac{t(1-t)}{2} f^{(2)}(t), \quad f \in C^2[0,1], \quad t \in [0,1],
\]

\[
UF(t) = \frac{t(1-t)}{2} \left( 3t(1-t)f^{(4)}(t) + 4(1-2t)f^{(3)}(t) \right),
\]

\[
VF(t) = \frac{t(1-t)}{24} \left( 3t(1-t)f^{(4)}(t) + 8(1-2t)f^{(3)}(t) - 12f^{(2)}(t) \right), \quad f \in C^4[0,1], \quad t \in [0,1].
\]

Let \( U_n := P_n Q_n = B_n \overline{B}_n \); it is called the genuine Bernstein-Durrmeyer operator, see, e.g., [5], [6] and the references therein. By using Theorem 2.1 we get

\[
\lim_{n \to \infty} \left( U_nf(t) - f(t) \right) = t(1-t)f^{(2)}(t), \quad f \in C^2[0,1], \quad t \in [0,1],
\]

\[
\lim_{n \to \infty} n\left[ U_nf(t) - f(t) \right] - t(1-t)f^{(2)}(t)
\]

\[
= \frac{t(1-t)}{2} \left( t(1-t)f^{(4)}(t) + 2(1-2t)f^{(3)}(t) - 2f^{(2)}(t) \right), \quad f \in C^4[0,1], \quad t \in [0,1].
\]

**Example 2.3.** Let now \( P_n := \overline{B}_n \) and \( Q_n := B_n \). Then \( S_n := P_n Q_n = \overline{B}_n B_n \) is a special Stancu operator; see, e.g., [7] and the references given there.

As in the previous example, we get

\[
\lim_{n \to \infty} \left( S_nf(t) - f(t) \right) = t(1-t)f^{(2)}(t), \quad f \in C^2[0,1], \quad t \in [0,1].
\]

\[
\lim_{n \to \infty} n\left[ S_nf(t) - f(t) \right] - t(1-t)f^{(2)}(t)
\]

\[
= \frac{t(1-t)}{2} \left( t(1-t)f^{(4)}(t) + 2(1-2t)f^{(3)}(t) - 2f^{(2)}(t) \right), \quad f \in C^4[0,1], \quad t \in [0,1].
\]
EXAMPLE 2.4. Let $P_n := B_n$, $Q_n := B_{n+1}$. The operator $D_n := P_nQ_n = B_nB_{n+1}$ was considered in [5] and [7]. By using Theorem 2.1 we infer

$$
\lim_{n \to \infty} n \left( D_n f(t) - f(t) \right) = t(1-t)f'(t), \quad f \in C^2[0,1], \quad t \in [0,1],
$$

$$
\lim_{n \to \infty} n \left( D_n f(t) - f(t) \right) = t(1-t)f'(t)
$$

$$
= \frac{t(1-t)}{6} \left( 3t(1-t)f''(t) + 5(1-2t)f'''(t) - 3f''(t) \right), \quad f \in C^4[0,1], \quad t \in [0,1].
$$

3. The operator $B_n^{-1}$

Let $\Pi_n$ be the space of all polynomial functions of degree at most $n$, defined on $\mathbb{R}$, and $\Pi = \bigcup_{n \geq 0} \Pi_n$. For $k \geq 0$ consider the monomial $e_k(t) = t^k$. Moreover, let $(a)_k := a(a+1)\ldots(a+k-1)$ for $a \in \mathbb{R}$, $k \geq 1$, and $(a)_0 := 1$. It is easy to check that

$$
B_n e_k(t) = \frac{(nt)_k}{(n)_k}, \quad n \geq 1, \quad k \geq 0. \tag{3.1}
$$

Let $[a_0, a_1, \ldots, a_k; f]$ be the divided difference of the function $f$ on the knots $a_0, a_1, \ldots, a_k$.

The Lagrange interpolation polynomial associated with $f$ and the knots $a_0, \ldots, a_m$ has the Newton form

$$
L_m f(t; a_0, \ldots, a_m) = \sum_{k=0}^{m} [a_0, \ldots, a_k; f] (t-a_0)\ldots(t-a_{k-1}).
$$

If $p \in \Pi_m$, then $p(t) = L_m p(t; a_0, \ldots, a_m)$. In particular,

$$
p(t) = L_m p(t; 0, \ldots, -\frac{1}{n}, \ldots, -\frac{m}{n}),
$$

which leads to

$$
p(t) = \sum_{k=0}^{m} \frac{(n)_k}{n^k} \left[ 0, -\frac{1}{n}, \ldots, -\frac{k}{n} ; p \right] \frac{(nt)_k}{(n)_k}, \quad p \in \Pi_m, \quad n \geq 1. \tag{3.2}
$$

From (3.1) and (3.2) we deduce

$$
p(t) = \sum_{k=0}^{m} \frac{(n)_k}{n^k} \left[ 0, -\frac{1}{n}, \ldots, -\frac{k}{n} ; p \right] B_n e_k(t), \quad p \in \Pi_m, \quad n \geq 1. \tag{3.3}
$$

By a slight abuse of notation, let us consider the operators $B_n^{-1} : \Pi \to \Pi$, $n \geq 1$.

Then (3.3) can be written as

$$
B_n^{-1} p(t) = \sum_{k=0}^{m} \frac{(n)_k}{n^k} \left[ 0, -\frac{1}{n}, \ldots, -\frac{k}{n} ; p \right] t^k, \quad p \in \Pi_m, \quad n \geq 1. \tag{3.4}
$$
Theorem 3.1. Let \( m \geq 0 \) and \( p_n \in \Pi_m \), \( n \geq 1 \). Suppose that the sequence \((p_n)\) is uniformly convergent on \([0,1]\) to \( p \in \Pi_m \). Then

\[
\lim_{n \to \infty} n \left( \mathbb{B}_n^{-1} p_n(t) - p_n(t) \right) = -\frac{t(1-t)}{2} p^{(2)}(t), \quad t \in [0,1]. \tag{3.5}
\]

Proof. It is easy to verify that

\[
\left[ 0, -\frac{1}{n}, \ldots, -\frac{k}{n}; e_j \right] = C_{k,j} n^{k-j}
\]

for some real numbers \( C_{k,j} \), with \( C_{j,j} = 1 \). Then, according to (3.4),

\[
\lim_{n \to \infty} n \left( \mathbb{B}_n^{-1} e_j(t) - e_j(t) \right) = \lim_{n \to \infty} \left( \sum_{k=0}^{j} \frac{(n)_k}{n^j} C_{k,j} t^k - t^j \right)
\]

\[
= \lim_{n \to \infty} \sum_{k=0}^{j-1} \frac{(n)_k}{n^j} C_{k,j} t^k + t^j \lim_{n \to \infty} \left( \frac{(n)_j}{n^j} - 1 \right)
\]

\[
= C_{j-1,j} t^{j-1} + \frac{j(j-1)}{2} t^j.
\]

Denote

\[
Be_j := C_{j-1,j} e_{j-1} + \frac{j(j-1)}{2} e_j, \quad j \geq 0. \tag{3.6}
\]

Then \( B \) can be extended as a linear operator \( B : \Pi \to \Pi \), and

\[
\lim_{n \to \infty} n \left( \mathbb{B}_n^{-1} q - q \right) = Bq, \quad q \in \Pi. \tag{3.7}
\]

In Theorem 2.1 (i) let \( P_n := \mathbb{B}_n \) and \( Q_n := \mathbb{B}_n^{-1} \). Then, according to (1.3),

\[
Aq(t) = \frac{t(1-t)}{2} q^{(2)}(t).
\]

From (2.3) we deduce that \( Aq + Bq = 0 \), \( q \in \Pi \), so that (3.7) yields

\[
\lim_{n \to \infty} n \left( \mathbb{B}_n^{-1} q(t) - q(t) \right) = -\frac{t(1-t)}{2} q^{(2)}(t), \quad q \in \Pi. \tag{3.8}
\]

Now let \( p_n = \sum_{j=0}^{m} a_{nj} e_j \in \Pi_m \) be uniformly convergent on \([0,1]\) to

\[ p = \sum_{j=0}^{m} a_j e_j \in \Pi_m. \]

Then \( a_{nj} \to a_j \) and (3.8) leads to

\[
\lim_{n \to \infty} n \left( \mathbb{B}_n^{-1} p_n(t) - p_n(t) \right) = \lim_{n \to \infty} \sum_{j=0}^{m} a_{nj} n \left( \mathbb{B}_n^{-1} e_j(t) - e_j(t) \right)
\]

\[
= -\frac{t(1-t)}{2} \left( \sum_{j=0}^{m} a_j e_j \right)^{(2)}(t) = -\frac{t(1-t)}{2} p^{(2)}(t).
\]

This proves (3.5). \( \square \)
Remark 3.2. After establishing (3.6), another continuation of the above proof can be based on the fact that $C_{j-1,j} = -\frac{j(j-1)}{2}$.

Remark 3.3. (3.8) was obtained in [10] by using the eigenstructure of $\overline{B}_n$ described in [3] and [4].

With notation from [2] and [3], let $p_k^{(n)}$ be the eigenpolynomials of $B_n : \Pi_n \to \Pi_n$, $0 \leq k \leq n$. Then $p_k^{(n)} \in \Pi_k$. According to [2, Th. 4.1], for each $k$ there exists the limit $p_k^* := \lim_{n \to \infty} p_k^{(n)}$. From Theorem 3.1 we infer

**Corollary 3.4.** For each $k \geq 0$,

$$\lim_{n \to \infty} n \left( \overline{B}_n^{-1} p_k^{(n)}(t) - p_k^{(n)}(t) \right) = -\frac{t(1-t)}{2} (p_k^*)^2(t),$$

uniformly on $[0,1]$.

This result can be also found, with a different proof, in [3, p. 21].

**Theorem 3.5.** Under the same hypotheses of Theorem 3.1 we have

$$\lim_{n \to \infty} n \left[ n \left( \overline{B}_n^{-1} p_n(t) - p_n(t) \right) + \frac{t(1-t)}{2} p_n^2(t) \right]$$

$$= \frac{t(1-t)}{24} \left( 3t(1-t)p^4(t) + 4(1-2t)p^3(t) \right), \quad t \in [0,1],$$

(3.9)

uniformly on $[0,1]$.

**Proof.** Let $A_{i,k}(x)$ be defined by

$$(n)_k x^k - (nx)_k = \sum_{i=0}^k A_{i,k}(x) n^i.$$

As in the proof of Theorem 3.1, by a direct calculation we can prove that for each $j \geq 0$,

$$\lim_{n \to \infty} n \left[ n \left( \overline{B}_n^{-1} e_j(t) - e_j(t) \right) + \frac{t(1-t)}{2} e_j^2(t) \right]$$

$$= C_{j-1,j} A_{j-2,j-1}(t) + A_{j-2,j}(t) := Ve_j(t).$$

Then $V$ can be extended as a linear operator $V : \Pi \to \Pi$, and

$$\lim_{n \to \infty} n \left[ n \left( \overline{B}_n^{-1} q(t) - q(t) \right) + \frac{t(1-t)}{2} q^2(t) \right] = Vq, \quad q \in \Pi.$$

(3.10)
In Theorem 2.1 (ii) let \( P_n := \mathbb{B}_n \) and \( Q_n := \mathbb{B}_n^{-1} \). Then, according to (1.3) and (1.4),

\[
Aq(t) = \frac{t(1-t)}{2} q^{(2)}(t) \quad \text{and} \quad Uq(t) = \frac{t(1-t)}{24} \left( 3t(1-t)q^{(4)}(t) + 8(1-2t)q^{(3)}(t) - 12q^{(2)}(t) \right).
\]

Moreover, (3.8) shows that \( Bq(t) = -\frac{t(1-t)}{2} q^{(2)}(t) \). From (2.6) we deduce that \( Uq + Vq + ABq = 0 \), which yields

\[
Vq(t) = \frac{t(1-t)}{24} \left( 3t(1-t)q^{(4)}(t) + 4(1-2t)q^{(3)}(t) \right).
\]

Combined with (3.10), this leads to

\[
\lim_{n \to \infty} n \left[ n \left( \mathbb{B}_n^{-1} q(t) - q(t) \right) + \frac{t(1-t)}{2} q^{(2)}(t) \right] = \frac{t(1-t)}{24} \left( 3t(1-t)q^{(4)}(t) + 4(1-2t)q^{(3)}(t) \right), \quad q \in \prod. \quad (3.11)
\]

In order to conclude the proof we have only to apply the argument which was used in the final part of the proof of Theorem 3.1. □

**REMARK 3.6.** Another proof of (3.11) can be found in [10].

### 4. The operator \( F_n \)

Let \( F_n : C[0,1] \to \prod_n, F_n := \mathbb{B}_n^{-1} \circ B_n \). This operator was investigated in [3], [10], [11]. Now we are in a position to generalize a result from [3, p. 17].

**THEOREM 4.1.** Under the same hypotheses of Theorem 3.1 we have

\[
\lim_{n \to \infty} n^2 \left( F_n p_n(t) - p_n(t) \right) = \frac{t(1-t)}{2} p^{(2)}(t) - \frac{t(1-t)(1-2t)}{6} p^{(3)}(t), \quad (4.1)
\]

uniformly on \([0,1]\).

**Proof.** If \( p_n \in \Pi_m \), \( n \geq 1 \), and \( \lim_{n \to \infty} p_n = p \), then \( B_n p_n \in \Pi_m \) and \( \lim_{n \to \infty} B_n p_n = p \). Consequently, Theorem 3.5 shows that

\[
\lim_{n \to \infty} n \left[ n \left( \mathbb{B}_n^{-1} B_n p_n(t) - B_n p_n(t) \right) + \frac{t(1-t)}{2} (B_n p_n)^{(2)}(t) \right] = \frac{t(1-t)}{24} \left( 3t(1-t)p^{(4)}(t) + 4(1-2t)p^{(3)}(t) \right).
\]
This can be rewritten as
\[
\lim_{n \to \infty} \left\{ n^2 \left( F_n p_n(t) - p_n(t) \right) - n \left[ n \left( B_n p_n(t) - p_n(t) \right) - \frac{t(1-t)}{2} p_n^{(2)}(t) \right] \right. \\
\left. + \frac{t(1-t)}{2} n \left( B_n p_n(t) - p_n(t) \right)^{(2)} \right\} \\
= \frac{t(1-t)}{24} \left( 3t(1-t)p^{(4)}(t) + 4(1-2t)p^{(3)}(t) \right).
\]

Combining (1.1) and (1.2) with the coordinate-wise convergence of \((p_n)\) and \((B_n p_n)\) we get
\[
\lim_{n \to \infty} n \left[ n \left( B_n p_n(t) - p_n(t) \right) - \frac{t(1-t)}{2} p_n^{(2)}(t) \right] \\
= \frac{t(1-t)}{24} \left( 3t(1-t)p^{(4)}(t) + 4(1-2t)p^{(3)}(t) \right),
\]
\[
\lim_{n \to \infty} \left( B_n p_n(t) - p_n(t) \right)^{(2)} = \frac{t(1-t)}{2} p^{(2)}(t).
\]

Now (4.1) is a consequence of (4.2), (4.3) and (4.4). □

**Remark 4.2.** With notation as in Corollary 3.4, (4.1) yields
\[
\lim_{n \to \infty} n^2 \left( F_n p_k^{(n)}(t) - p_k^{(n)}(t) \right) = \frac{t(1-t)}{2} (p_k^*)^{(2)}(t) - \frac{t(1-t)(1-2t)}{6} (p_k^*)^{(3)}(t).
\]

This improves a result from [3, p. 21].

**Remark 4.3.** With the above methods it can be proved that if \(p_n \in \Pi_m, p_n \to p \in \Pi_m\), then
\[
\lim_{n \to \infty} n \left[ n \left( B_n^{-1} p_n(t) - p_n(t) \right) + \frac{t(1-t)}{2} p_n^{(2)}(t) \right] \\
= \frac{t(1-t)}{24} \left( 3t(1-t)p^{(4)}(t) + 8(1-2t)p^{(3)}(t) - 12p^{(2)}(t) \right),
\]
and
\[
\lim_{n \to \infty} n^2 \left( F_n^{-1} p_n(t) - p_n(t) \right) = \frac{t(1-t)(1-2t)}{6} p^{(3)}(t) - \frac{t(1-t)}{2} p^{(2)}(t).
\]

5. Inequalities for \(\overline{H}_n^{-1}\)

The operator \(\overline{H}_n^{-1}\) is not positive. Indeed, see [3], \(\overline{H}_n^{-1} e_2 = \frac{n+1}{n} e_2 - \frac{1}{n} e_1\), and \(\overline{H}_n^{-1} e_2(t) < 0\) for all \(t \in (0, \frac{1}{n+1})\).

However, \(\overline{H}_n^{-1}\) possesses the following property.
Proposition 5.1. Let $p \in \prod_m$ with $p^{(j)}(-1) \geq 0$, $j = 0, 1, \ldots, m$. Then

$$\left(\overline{p}_{n}^{-1} \right)^{(j)}(t) \geq 0, \ n \geq m, \ t \in [0, 1], \ j = 0, 1, \ldots, m. \quad (5.1)$$

Proof. We have $p(t) = \sum_{j=0}^{m} \frac{p^{(j)}(-1)}{j!}(t+1)^{j}$, which implies

$$p^{(j)}(t) \geq 0, \ t \geq -1, \ j = 0, 1, \ldots, m. \quad (5.2)$$

According to the mean value theorem for divided differences, there exists $t_{nj} \in \left[-\frac{j}{n}, 0\right]$ such that

$$\left[0, -\frac{1}{n}, \ldots, -\frac{j}{n}; p\right] = \frac{p^{(j)}(t_{nj})}{j!}.$$

Since $-1 \leq -\frac{m}{n} \leq -\frac{j}{n} \leq t_{nj}$, (5.2) shows that $p^{(j)}(t_{nj}) \geq 0$, and consequently $\left[0, -\frac{1}{n}, \ldots, -\frac{j}{n}; p\right] \geq 0$, $j = 0, 1, \ldots, n$. Now, according to (3.4),

$$\overline{p}_{n}^{-1} p(t) = \sum_{j=0}^{m} c_{jn} t^{j}$$

with $c_{jn} := \frac{(n)!}{n^{j}} \left[0, -\frac{1}{n}, \ldots, -\frac{j}{n}; p\right] \geq 0$, and this immediately implies (5.1). □

The inequality (5.3) below is a companion (and a consequence) of (3.8).

Theorem 5.2. Let $0 < a < \frac{1}{2}$ and $p \in \prod$ with $p^{(2)}(t) > 0$ for all $t \in [a, 1-a]$. Then there exists $n_{0} \geq 1$ such that

$$\overline{p}_{n}^{-1} p(t) \leq p(t), \ n \geq n_{0}, \ t \in [a, 1-a]. \quad (5.3)$$

Proof. Let $l(t) := \frac{t(1-t)}{2} p^{(2)}(t)$. Then

$$M := \min \left\{ l(t) \mid t \in [a, 1-a] \right\} > 0.$$

On the other hand, (3.8) shows that there exists $n_{0} \geq 1$ such that

$$-M \leq n \left(\overline{p}_{n}^{-1} p(t) - p(t)\right) + l(t) \leq M, \ n \geq n_{0}, \ t \in [0, 1].$$

This implies

$$n \left(\overline{p}_{n}^{-1} p(t) - p(t)\right) \leq M - l(t) \leq 0, \ n \geq 0, \ t \in [a, 1-a],$$

which leads to (5.3) and concludes the proof. □
REMARK 5.3. There is experimental evidence (see [3]) that in many cases the approximation provided by $F_n$ is better than that furnished by $B_n$. Theorem 5.2 may be useful in order to explain heuristically this phenomenon. Namely, take a convex function $f \in C[0,1]$; then $B_nf$ is convex and $B_nf \geq f$, i.e., $B_nf$ is above $f$. Theorem 5.2 shows that, roughly speaking, $B_{n^{-1}}B_nf$ is under $B_nf$, and so $F_nf$ is closer to $f$ than $B_nf$ is.

Problems of this kind are still under research.

REFERENCES


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