Journal of Mathematical Inequalities
Volume 12, Number 1 (2018), 95–105
doi:10.7153/jmi-2018-12-07

ANOTHER LOOK AT VORONOVSKAJA TYPE FORMULAS

FADEL NASAIREH AND IOAN RAŞA

(*Communicated by J. Pečarić*)

Abstract. Voronovskaja type formulas are usually established for sequences of positive linear operators, i.e, operators P_n with $P_n f \geqslant 0$ whenever the function f is positive. The aim of this paper is twofold. First, we obtain some general formulas concerning compositions of operators on Banach spaces, without any assumption of positivity. Next, we establish Voronovskaja type formulas for operators which are manifestly nonpositive. Combining these two approaches we recover some known results and obtain new applications. The final section is devoted to inequalities accompanying our formulas.

1. Introduction

Voronovskaja type formulas are important tools in Approximation Theory. Rates of convergence and saturation properties for sequences of positive linear operators can be established by using such formulas. The prototypical Voronovskaja formula is related with the classical Bernstein operators $B_n : C[0,1] \to C[0,1]$,

$$B_n f(t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} f\left(\frac{k}{n}\right), \; f \in C[0,1], \; t \in [0,1],$$

and reads as follows:

$$\lim_{n\to\infty} n\Big(B_n f(t) - f(t)\Big) = \frac{t(1-t)}{2} f^{(2)}(t), \; f \in C^2[0,1], \tag{1.1}$$

uniformly on $[0,1]$. Moreover (see, e.g. [1]),

$$\begin{aligned} &\lim_{n\to\infty} n\left[n\Big(B_n f(t) - f(t)\Big) - \frac{t(1-t)}{2} f^{(2)}(t)\right] \\ &\quad = \frac{t(1-t)}{24}\Big(3t(1-t) f^{(4)}(t) + 4(1-2t) f^{(3)}(t)\Big), \; f \in C^4[0,1], \end{aligned} \tag{1.2}$$

uniformly on $[0,1]$.

Mathematics subject classification (2010): 41A36, 41A35, 26D20, 41A30.
Keywords and phrases: Positive operators, nonpositive operators, Voronovskaja type formulas, inequalities.

Consider also the Beta operators of Lupaş and Mühlbach, $\overline{\mathbb{B}}_n : C[0,1] \to C[0,1]$, defined by (see, e.g. [12], [13], [14], [4])

$$\overline{\mathbb{B}}_n f(x) = \begin{cases} f(0), & x=0, \\ \frac{1}{B(nx,n-nx)} \int_0^1 t^{nx-1}(1-t)^{n-1-nx} f(t)dt, & 0<x<1, \\ f(1), & x=1, \end{cases}$$

where B is Euler's Beta function.

In this case, see [9],

$$\lim_{n\to\infty} n\left(\overline{\mathbb{B}}_n f(t) - f(t)\right) = \frac{t(1-t)}{2} f^{(2)}(t),\ f \in C^2[0,1], \tag{1.3}$$

$$\lim_{n\to\infty} n\left[n\left(\overline{\mathbb{B}}_n f(t) - f(t)\right) - \frac{t(1-t)}{2} f^{(2)}(t)\right]$$
$$= \frac{t(1-t)}{24}\left(3t(1-t)f^{(4)}(t) + 8(1-2t)f^{(3)}(t) - 12f^{(2)}(t)\right),\ f \in C^4[0,1], \tag{1.4}$$

uniformly on $[0,1]$.

It is easy to compute the images of monomials under $\overline{\mathbb{B}}_n$; see (3.1) below. Convenient expressions for the images of monomials under $\overline{\mathbb{B}}_n^{-1}$ are also known, see [3]. Due to these circumstances, the operators $\overline{\mathbb{B}}_n$ are used in order to represent other classical operators by means of composition and decomposition; see [3], [10], [11] and the references given there.

In Section 2 we determine the Voronovskaja formula for the sequence $(P_n Q_n)_{n\geqslant 1}$ in terms of the corresponding formulas for the (quite general) sequences $(P_n)_{n\geqslant 1}$ and $(Q_n)_{n\geqslant 1}$. Some examples illustrate this general result.

At the beginning of Section 3 we present a new formula describing the action of $\overline{\mathbb{B}}_n^{-1}$ on polynomials.

Combining this formula with the results of Section 2, we give new proofs and extend some Voronovskaja type results for $\overline{\mathbb{B}}_n^{-1}$ obtained in [3] and [10].

Section 4 is devoted to the operator $F_n := \overline{\mathbb{B}}_n^{-1} \circ B_n$. It was investigated in [3], [8], [10], [11] in relation with the problem of decomposing B_n into simpler blocks. With the methods already used in Section 3, we extend and give new proofs to the Voronovskaja type formulas for F_n and F_n^{-1} obtained in [3].

Voronovskaja type formulas are usually accompanied by inequalities. For example, (1.1) is accompanied by the inequality $B_n f \geqslant f,\ n \geqslant 1,\ f \in C[0,1]$ convex.

In Section 5 we present a similar inequality for the operator $\overline{\mathbb{B}}_n^{-1}$.

Throughout the paper we consider the spaces $C^k[0,1]$, $k \geqslant 0$, endowed with the topology of uniform convergence.

2. A general result

Let X be a Banach space and $W \subset Z \subset Y$ linear subspaces of X. Let $A,B : Y \to X$; $U,V : Z \to X$; $S,T : W \to X$ be linear operators.

Consider also two sequences of linear operators $P_n : X \to X$, $Q_n : Y \to X$, $n \geqslant 1$, and suppose that each P_n is bounded.

THEOREM 2.1. *(i) Suppose that*

$$\lim_{n\to\infty} P_n x = x,\ x \in X, \tag{2.1}$$

$$\lim_{n\to\infty} n(P_n y - y) = Ay; \qquad \lim_{n\to\infty} n(Q_n y - y) = By,\ y \in Y. \tag{2.2}$$

Then

$$\lim_{n\to\infty} n(P_n Q_n y - y) = Ay + By,\ y \in Y. \tag{2.3}$$

(ii) In addition to (2.1) and (2.2), suppose that

$$Bz \in Y,\ z \in Z, \tag{2.4}$$

$$\lim_{n\to\infty} n\Big[n(P_n z - z) - Az\Big] = Uz; \quad \lim_{n\to\infty} n\Big[n(Q_n z - z) - Bz\Big] = Vz,\ z \in Z. \tag{2.5}$$

Then

$$\lim_{n\to\infty} n\Big[n(P_n Q_n z - z) - Az - Bz\Big] = Uz + Vz + ABz,\ z \in Z. \tag{2.6}$$

(iii) Let (2.1), (2.2), (2.4), (2.5) be satisfied. Moreover, suppose that for each $w \in W$ we have $Vw \in Y$, $Bw \in Z$, and

$$\lim_{n\to\infty} n\Big\{n\Big[n(P_n w - w) - Aw\Big] - Uw\Big\} = Sw,$$

$$\lim_{n\to\infty} n\Big\{n\Big[n(Q_n w - w) - Bw\Big] - Vw\Big\} = Tw.$$

Then, for all $w \in W$,

$$\lim_{n\to\infty} n\Big\{n\Big[n(P_n Q_n w - w) - Aw - Bw\Big] - Uw - Vw - ABw\Big\} = Sw + Tw + AVw + UBw.$$

Proof. (i) By using (2.1) and the Banach-Steinhaus Theorem we infer that the sequence $(\|P_n\|)_{n\geqslant 1}$ is bounded, i.e., there exists $M > 0$ such that $\|P_n\| \leqslant M$, $n \geqslant 1$. Let $y \in Y$. Then

$$n(P_n Q_n y - y) = n(P_n y - y) + P_n By + P_n\Big[n(Q_n y - y) - By\Big].$$

According to (2.2) and (2.1), the first two summands tend to Ay, respectively By, while

$$\Big\|P_n\Big[n(Q_n y - y) - By\Big]\Big\| \leqslant M\Big\|n(Q_n y - y) - By\Big\| \to 0.$$

This proves (2.3).

(ii) Let $z \in Z$. Then

$$n\Big[n\left(P_nQ_nz-z\right)-Az-Bz\Big]$$
$$=n\Big[n\left(P_nz-z\right)-Az\Big]+P_nVz+n\left(P_nBz-Bz\right)+P_n\Big\{n\Big[n\left(Q_nz-z\right)-Bz\Big]-Vz\Big\}.$$

According to (2.5), (2.1), (2.4) and (2.2), the first three summands tend, respectively, to Uz, Vz, ABz.

Moreover,

$$\left\|P_n\Big\{n\Big[n(Q_nz-z)-Bz\Big]-Vz\Big\}\right\|\leqslant M\left\|n\Big[n(Q_nz-z)-Bz\Big]-Vz\right\|\to 0.$$

This proves (2.6).

(iii) The proof of (iii) is similar and we omit the details. □

EXAMPLE 2.2. With notation from Section 1, let $P_n := B_n$ and $Q_n := \overline{\mathbb{B}}_n$. Then, according to (1.1)–(1.4), we have

$$Af(t)=Bf(t)=\frac{t(1-t)}{2}f^{(2)}(t),\ f\in C^2\left[0,1\right],\ t\in[0,1],$$
$$Uf(t)=\frac{t(1-t)}{24}\Big(3t(1-t)f^{(4)}(t)+4(1-2t)f^{(3)}(t)\Big),$$
$$Vf(t)=\frac{t(1-t)}{24}\Big(3t(1-t)f^{(4)}(t)+8(1-2t)f^{(3)}(t)-12f^{(2)}(t)\Big),\ f\in C^4\left[0,1\right],\ t\in[0,1].$$

Let $U_n := P_nQ_n = B_n\overline{\mathbb{B}}_n$; it is called the genuine Bernstein-Durrmeyer operator, see, e.g., [5], [6] and the references therein. By using Theorem 2.1 we get

$$\lim_{n\to\infty} n\Big(U_nf(t)-f(t)\Big)=t(1-t)f^{(2)}(t),\ f\in C^2\left[0,1\right],\ t\in\left[0,1\right],$$

$$\lim_{n\to\infty} n\Big[n\Big(U_nf(t)-f(t)\Big)-t(1-t)f^{(2)}(t)\Big]$$
$$=\frac{t(1-t)}{2}\Big(t(1-t)f^{(4)}(t)+2(1-2t)f^{(3)}(t)-2f^{(2)}(t)\Big),\ f\in C^4\left[0,1\right],\ t\in[0,1].$$

EXAMPLE 2.3. Let now $P_n := \overline{\mathbb{B}}_n$ and $Q_n := B_n$. Then $S_n := P_nQ_n = \overline{\mathbb{B}}_nB_n$ is a special Stancu operator; see, e.g., [7] and the references given there.

As in the previous example, we get

$$\lim_{n\to\infty} n\Big(S_nf(t)-f(t)\Big)=t(1-t)f^{(2)}(t),\ f\in C^2\left[0,1\right],\ t\in\left[0,1\right].$$

$$\lim_{n\to\infty} n\Big[n\Big(S_nf(t)-f(t)\Big)-t(1-t)f^{(2)}(t)\Big]$$
$$=\frac{t(1-t)}{2}\Big(t(1-t)f^{(4)}(t)+2(1-2t)f^{(3)}(t)-2f^{(2)}(t)\Big),\ f\in C^4\left[0,1\right],\ t\in\left[0,1\right].$$

EXAMPLE 2.4. Let $P_n := B_n$, $Q_n := B_{n+1}$. The operator $D_n := P_nQ_n = B_nB_{n+1}$ was considered in [5] and [7]. By using Theorem 2.1 we infer

$$\lim_{n\to\infty} n\Big(D_nf(t)-f(t)\Big)=t(1-t)f^{(2)}(t),\ f\in C^2[0,1],\ t\in[0,1],$$

$$\lim_{n\to\infty} n\Big[n\Big(D_nf(t)-f(t)\Big)-t(1-t)f^{(2)}(t)\Big]$$
$$=\frac{t(1-t)}{6}\Big(3t(1-t)f^{(4)}(t)+5(1-2t)f^{(3)}(t)-3f^{(2)}(t)\Big),\ f\in C^4[0,1],\ t\in[0,1].$$

3. The operator $\overline{\mathbb{B}}_n^{-1}$

Let $\prod_n$ be the space of all polynomial functions of degree at most n, defined on $\mathbb{R}$, and $\prod = \bigcup_{n\geqslant 0}\prod_n$. For $k\geqslant 0$ consider the monomial $e_k(t)=t^k$. Moreover, let $(a)_k := a(a+1)\dots(a+k-1)$ for $a\in\mathbb{R}$, $k\geqslant 1$, and $(a)_0 := 1$. It is easy to check that

$$\overline{\mathbb{B}}_n e_k(t)=\frac{(nt)_k}{(n)_k},\ n\geqslant 1,\ k\geqslant 0. \tag{3.1}$$

Let $[a_0,a_1,\dots,a_k;f]$ be the divided difference of the function f on the knots $a_0,a_1,\dots,a_k$.

The Lagrange interpolation polynomial associated with f and the knots $a_0,\dots,a_m$ has the Newton form

$$L_mf(t;a_0,\dots,a_m)=\sum_{k=0}^{m}[a_0,\dots,a_k;f](t-a_0)\dots(t-a_{k-1}).$$

If $p\in\prod_m$, then $p(t)=L_mp(t;a_0,\dots,a_m)$. In particular,

$$p(t)=L_mp\Big(t;0,-\frac{1}{n},\dots,-\frac{m}{n}\Big),$$

which leads to

$$p(t)=\sum_{k=0}^{m}\frac{(n)_k}{n^k}\left[0,-\frac{1}{n},\dots,-\frac{k}{n};p\right]\frac{(nt)_k}{(n)_k},\ p\in\Pi_m,\ n\geqslant 1. \tag{3.2}$$

From (3.1) and (3.2) we deduce

$$p(t)=\sum_{k=0}^{m}\frac{(n)_k}{n^k}\left[0,-\frac{1}{n},\dots,-\frac{k}{n};p\right]\overline{\mathbb{B}}_n e_k(t),\ p\in\Pi_m,\ n\geqslant 1. \tag{3.3}$$

By a slight abuse of notation, let us consider the operators $\overline{\mathbb{B}}_n^{-1}:\prod\to\prod$, $n\geqslant 1$. Then (3.3) can be written as

$$\overline{\mathbb{B}}_n^{-1}p(t)=\sum_{k=0}^{m}\frac{(n)_k}{n^k}\left[0,-\frac{1}{n},\dots,-\frac{k}{n};p\right]t^k,\ p\in\Pi_m,\ n\geqslant 1. \tag{3.4}$$

THEOREM 3.1. *Let $m \geqslant 0$ and $p_n \in \Pi_m$, $n \geqslant 1$. Suppose that the sequence (p_n) is uniformly convergent on $[0,1]$ to $p \in \Pi_m$. Then*

$$\lim_{n\to\infty} n\Big(\overline{\mathbb{B}}_n^{-1} p_n(t) - p_n(t)\Big) = -\frac{t(1-t)}{2} p^{(2)}(t),\ t \in [0,1]. \tag{3.5}$$

Proof. It is easy to verify that

$$\left[0, -\frac{1}{n}, \ldots, -\frac{k}{n}; e_j\right] = C_{k,j} n^{k-j}$$

for some real numbers $C_{k,j}$, with $C_{j,j} = 1$. Then, according to (3.4),

$$\begin{aligned}
\lim_{n\to\infty} n\Big(\overline{\mathbb{B}}_n^{-1} e_j(t) - e_j(t)\Big) &= \lim_{n\to\infty} n\Big(\sum_{k=0}^{j} \frac{(n)_k}{n^j} C_{k,j} t^k - t^j\Big) \\
&= \lim_{n\to\infty} \sum_{k=0}^{j-1} \frac{(n)_k}{n^{j-1}} C_{k,j} t^k + t^j \lim_{n\to\infty} n\Big(\frac{(n)_j}{n^j} - 1\Big) \\
&= C_{j-1,j} t^{j-1} + \frac{j(j-1)}{2} t^j.
\end{aligned}$$

Denote

$$Be_j := C_{j-1,j} e_{j-1} + \frac{j(j-1)}{2} e_j,\ j \geqslant 0. \tag{3.6}$$

Then B can be extended as a linear operator $B : \prod \to \prod$, and

$$\lim_{n\to\infty} n(\overline{\mathbb{B}}_n^{-1} q - q) = Bq,\ q \in \prod. \tag{3.7}$$

In Theorem 2.1 (i) let $P_n := \overline{\mathbb{B}}_n$ and $Q_n := \overline{\mathbb{B}}_n^{-1}$. Then, according to (1.3),

$$Aq(t) = \frac{t(1-t)}{2} q^{(2)}(t).$$

From (2.3) we deduce that $Aq + Bq = 0,\ q \in \prod$, so that (3.7) yields

$$\lim_{n\to\infty} n\Big(\overline{\mathbb{B}}_n^{-1} q(t) - q(t)\Big) = -\frac{t(1-t)}{2} q^{(2)}(t),\ q \in \prod. \tag{3.8}$$

Now let $p_n = \sum_{j=0}^{m} a_{nj} e_j \in \Pi_m$ be uniformly convergent on $[0,1]$ to $p = \sum_{j=0}^{m} a_j e_j \in \Pi_m$. Then $a_{nj} \to a_j$ and (3.8) leads to

$$\begin{aligned}
\lim_{n\to\infty} n\Big(\overline{\mathbb{B}}_n^{-1} p_n(t) - p_n(t)\Big) &= \lim_{n\to\infty} \sum_{j=0}^{m} a_{nj} n\Big(\overline{\mathbb{B}}_n^{-1} e_j(t) - e_j(t)\Big) \\
&= -\frac{t(1-t)}{2} \Big(\sum_{j=0}^{m} a_j e_j\Big)^{(2)}(t) = -\frac{t(1-t)}{2} p^{(2)}(t).
\end{aligned}$$

This proves (3.5). □

REMARK 3.2. After establishing (3.6), another continuation of the above proof can be based on the fact that $C_{j-1,j} = -\dfrac{j(j-1)}{2}$.

REMARK 3.3. (3.8) was obtained in [10] by using the eigenstructure of $\overline{\mathbb{B}}_n$ described in [3] and [4].

With notation from [2] and [3], let $p_k^{(n)}$ be the eigenpolynomials of $B_n : \Pi_n \to \Pi_n$, $0 \leqslant k \leqslant n$. Then $p_k^{(n)} \in \Pi_k$. According to [2, Th. 4.1], for each k there exists the limit $p_k^* := \lim_{n\to\infty} p_k^{(n)}$. From Theorem 3.1 we infer

COROLLARY 3.4. *For each $k \geqslant 0$,*

$$\lim_{n\to\infty} n\left(\overline{\mathbb{B}}_n^{-1} p_k^{(n)}(t) - p_k^{(n)}(t)\right) = -\frac{t(1-t)}{2}(p_k^*)^{(2)}(t),$$

uniformly on $[0,1]$.

This result can be also found, with a different proof, in [3, p. 21].

THEOREM 3.5. *Under the same hypotheses of Theorem 3.1 we have*

$$\lim_{n\to\infty} n\left[n\left(\overline{\mathbb{B}}_n^{-1} p_n(t) - p_n(t)\right) + \frac{t(1-t)}{2} p_n^{(2)}(t)\right]$$
$$= \frac{t(1-t)}{24}\left(3t(1-t)p^{(4)}(t) + 4(1-2t)p^{(3)}(t)\right),\ t \in [0,1], \quad (3.9)$$

uniformly on $[0,1]$.

Proof. Let $A_{i,k}(x)$ be defined by

$$(n)_k x^k - (nx)_k = \sum_{i=0}^{k} A_{i,k}(x) n^i.$$

As in the proof of Theorem 3.1, by a direct calculation we can prove that for each $j \geqslant 0$,

$$\lim_{n\to\infty} n\left[n\left(\overline{\mathbb{B}}_n^{-1} e_j(t) - e_j(t)\right) + \frac{t(1-t)}{2}(e_j)^{(2)}(t)\right]$$
$$= C_{j-1,j} A_{j-2,j-1}(t) + A_{j-2,j}(t) =: Ve_j(t).$$

Then V can be extended as a linear operator $V : \prod \to \prod$, and

$$\lim_{n\to\infty} n\left[n\left(\overline{\mathbb{B}}_n^{-1} q(t) - q(t)\right) + \frac{t(1-t)}{2} q^{(2)}(t)\right] = Vq,\ q \in \prod. \quad (3.10)$$

In Theorem 2.1 (ii) let $P_n := \overline{\mathbb{B}}_n$ and $Q_n := \overline{\mathbb{B}}_n^{-1}$. Then, according to (1.3) and (1.4),

$$\begin{aligned} Aq(t) &= \frac{t(1-t)}{2} q^{(2)}(t) \text{ and} \\ Uq(t) &= \frac{t(1-t)}{24}\Big(3t(1-t)q^{(4)}(t) + 8(1-2t)q^{(3)}(t) - 12q^{(2)}(t)\Big). \end{aligned}$$

Moreover, (3.8) shows that $Bq(t) = -\frac{t(1-t)}{2} q^{(2)}(t)$. From (2.6) we deduce that $Uq + Vq + ABq = 0$, which yields

$$Vq(t) = \frac{t(1-t)}{24}\Big(3t(1-t)q^{(4)}(t) + 4(1-2t)q^{(3)}(t)\Big).$$

Combined with (3.10), this leads to

$$\begin{aligned} \lim_{n\to\infty} n\Big[n\Big(\overline{\mathbb{B}}_n^{-1} q(t) - q(t)\Big) + \frac{t(1-t)}{2} q^{(2)}(t)\Big] \\ = \frac{t(1-t)}{24}\Big(3t(1-t)q^{(4)}(t) + 4(1-2t)q^{(3)}(t)\Big),\ q \in \prod. \quad (3.11) \end{aligned}$$

In order to conclude the proof we have only to apply the argument which was used in the final part of the proof of Theorem 3.1. □

REMARK 3.6. Another proof of (3.11) can be found in [10].

4. The operator F_n

Let $F_n : C[0,1] \to \prod_n$, $F_n := \overline{\mathbb{B}}_n^{-1} \circ B_n$. This operator was investigated in [3], [10], [11]. Now we are in a position to generalize a result from [3, p. 17].

THEOREM 4.1. *Under the same hypotheses of Theorem 3.1 we have*

$$\lim_{n\to\infty} n^2\Big(F_n p_n(t) - p_n(t)\Big) = \frac{t(1-t)}{2} p^{(2)}(t) - \frac{t(1-t)(1-2t)}{6} p^{(3)}(t), \quad (4.1)$$

uniformly on $[0,1]$.

Proof. If $p_n \in \Pi_m$, $n \geqslant 1$, and $\lim_{n\to\infty} p_n = p$, then $B_n p_n \in \Pi_m$ and $\lim_{n\to\infty} B_n p_n = p$. Consequently, Theorem 3.5 shows that

$$\begin{aligned} \lim_{n\to\infty} n\Big[n\Big(\overline{\mathbb{B}}_n^{-1} B_n p_n(t) - B_n p_n(t)\Big) + \frac{t(1-t)}{2}(B_n p_n)^{(2)}(t)\Big] \\ = \frac{t(1-t)}{24}\Big(3t(1-t)p^{(4)}(t) + 4(1-2t)p^{(3)}(t)\Big). \end{aligned}$$

This can be rewritten as

$$\lim_{n\to\infty}\Big\{n^2\Big(F_np_n(t)-p_n(t)\Big)-n\Big[n\Big(B_np_n(t)-p_n(t)\Big)-\frac{t(1-t)}{2}p_n^{(2)}(t)\Big] \\ +\frac{t(1-t)}{2}n\Big(B_np_n(t)-p_n(t)\Big)^{(2)}\Big\} \\ =\frac{t(1-t)}{24}\Big(3t(1-t)p^{(4)}(t)+4(1-2t)p^{(3)}(t)\Big). \tag{4.2}$$

Combining (1.1) and (1.2) with the coordinate-wise convergence of (p_n) and (B_np_n) we get

$$\lim_{n\to\infty} n\Big[n\Big(B_np_n(t)-p_n(t)\Big)-\frac{t(1-t)}{2}p_n^{(2)}(t)\Big] \\ =\frac{t(1-t)}{24}\Big(3t(1-t)p^{(4)}(t)+4(1-2t)p^{(3)}(t)\Big), \tag{4.3}$$

$$\lim_{n\to\infty} n\Big(B_np_n(t)-p_n(t)\Big)^{(2)}=\frac{t(1-t)}{2}p^{(2)}(t). \tag{4.4}$$

Now (4.1) is a consequence of (4.2), (4.3) and (4.4). □

REMARK 4.2. With notation as in Corollary 3.4, (4.1) yields

$$\lim_{n\to\infty} n^2\Big(F_np_k^{(n)}(t)-p_k^{(n)}(t)\Big)=\frac{t(1-t)}{2}(p_k^*)^{(2)}(t)-\frac{t(1-t)(1-2t)}{6}(p_k^*)^{(3)}(t).$$

This improves a result from [3 , p. 21].

REMARK 4.3. With the above methods it can be proved that if $p_n\in\Pi_m,\ p_n\to p\in\Pi_m$, then

$$\lim_{n\to\infty} n\Big[n\Big(B_n^{-1}p_n(t)-p_n(t)\Big)+\frac{t(1-t)}{2}p_n^{(2)}(t)\Big] \\ =\frac{t(1-t)}{24}\Big(3t(1-t)p^{(4)}(t)+8(1-2t)p^{(3)}(t)-12p^{(2)}(t)\Big),$$

and

$$\lim_{n\to\infty} n^2\Big(F_n^{-1}p_n(t)-p_n(t)\Big)=\frac{t(1-t)(1-2t)}{6}p^{(3)}(t)-\frac{t(1-t)}{2}p^{(2)}(t).$$

5. Inequalities for $\overline{\mathbb{B}}_n^{-1}$

The operator $\overline{\mathbb{B}}_n^{-1}$ is not positive. Indeed, see $[3]$, $\overline{\mathbb{B}}_n^{-1}e_2=\frac{n+1}{n}e_2-\frac{1}{n}e_1$, and $\overline{\mathbb{B}}_n^{-1}e_2(t)<0$ for all $t\in(0,\frac{1}{n+1})$.

However, $\overline{\mathbb{B}}_n^{-1}$ possesses the following property.

PROPOSITION 5.1. *Let* $p \in \prod_m$ *with* $p^{(j)}(-1) \geqslant 0,\ j = 0,1,\ldots,m.$ *Then*

$$\left(\overline{\mathbb{B}}_n^{-1} p\right)^{(j)}(t) \geqslant 0,\ n \geqslant m,\ t \in [0,1],\ j = 0,1,\ldots,m. \tag{5.1}$$

Proof. We have $p(t) = \sum_{j=0}^{m} \frac{p^{(j)}(-1)}{j!}(t+1)^j$, which implies

$$p^{(j)}(t) \geqslant 0,\ t \geqslant -1,\ j = 0,1,\ldots m. \tag{5.2}$$

According to the mean value theorem for divided differences, there exists $t_{nj} \in \left[-\frac{j}{n}, 0\right]$ such that

$$\left[0, -\frac{1}{n}, \ldots, -\frac{j}{n}; p\right] = \frac{p^{(j)}(t_{nj})}{j!}.$$

Since $-1 \leqslant -\frac{m}{n} \leqslant -\frac{j}{n} \leqslant t_{nj}$, (5.2) shows that $p^{(j)}(t_{nj}) \geqslant 0$, and consequently $\left[0, -\frac{1}{n}, \ldots, -\frac{j}{n}; p\right] \geqslant 0,\ j = 0,1,\ldots,n.$ Now, according to (3.4),

$$\overline{\mathbb{B}}_n^{-1} p(t) = \sum_{j=0}^{m} c_{jn} t^j$$

with $c_{jn} := \frac{(n)_j}{n^j}\left[0, -\frac{1}{n}, \ldots, -\frac{j}{n}; p\right] \geqslant 0$, and this immediately implies (5.1). □

The inequality (5.3) below is a companion (and a consequence) of (3.8).

THEOREM 5.2. *Let* $0 < a < \frac{1}{2}$ *and* $p \in \prod$ *with* $p^{(2)}(t) > 0$ *for all* $t \in [a, 1-a]$. *Then there exists* $n_0 \geqslant 1$ *such that*

$$\overline{\mathbb{B}}_n^{-1} p(t) \leqslant p(t),\ n \geqslant n_0,\ t \in [a, 1-a]. \tag{5.3}$$

Proof. Let $l(t) := \dfrac{t(1-t)}{2} p^{(2)}(t)$. Then

$$M := \min\left\{l(t) \mid t \in [a, 1-a]\right\} > 0.$$

On the other hand, (3.8) shows that there exists $n_0 \geqslant 1$ such that

$$-M \leqslant n\left(\overline{\mathbb{B}}_n^{-1} p(t) - p(t)\right) + l(t) \leqslant M,\ n \geqslant n_0,\ t \in [0,1].$$

This implies

$$n\left(\overline{\mathbb{B}}_n^{-1} p(t) - p(t)\right) \leqslant M - l(t) \leqslant 0,\ n \geqslant 0,\ t \in [a, 1-a],$$

which leads to (5.3) and concludes the proof. □

REMARK 5.3. There is experimental evidence (see [3]) that in many cases the approximation provided by F_n is better than that furnished by B_n. Theorem 5.2 may be useful in order to explain heuristically this phenomenon. Namely, take a convex function $f \in C[0,1]$; then $B_n f$ is convex and $B_n f \geqslant f$, i.e., $B_n f$ is above f. Theorem 5.2 shows that, roughly speaking, $\overline{\mathbb{B}}_n^{-1} B_n f$ is under $B_n f$, and so $F_n f$ is closer to f than $B_n f$ is.

Problems of this kind are still under research.

REFERENCES

[1] U. ABEL, M. IVAN, *Asymptotic expansion of the multivariate Bernstein polynomials on a simplex*, Approx. Theory Appl. **16** (2000), 85–93.

[2] SH. COOPER, SH. WALDRON, *The eigenstructure of the Bernstein operator*, J. Approx. Theory **105** (2000), 133–165.

[3] H. GONSKA, M. HEILMANN, A. LUPAŞ, I. RAŞA, *On the composition and decomposition of positive linear operators III: A non-trivial decomposition of the Bernstein operator*, `http://arxiv.org/abs/1204.2723`, August 30, 2012.

[4] H. GONSKA, M. HEILMANN, I. RAŞA, *Eigenstructure of the genuine Beta operators of Lupaş and Mühlbach*, Studia Univ. Babeş-Bolyai Math. **61** (2016), 383–388.

[5] H. GONSKA, D. KACSÓ, I. RAŞA, *On genuine Bernstein-Durrmeyer operators*, Results Math. **50** (2007), 213–225.

[6] H. GONSKA, D. KACSÓ, I. RAŞA, *The genuine Bernstein-Durrmeyer revisited*, Results Math. **62** (2012), 295–310.

[7] H. GONSKA, P. PIŢUL, I. RAŞA, *On differences of positive linear operators*, Carpathian J. Math. **22** (2006), 65–78.

[8] H. GONSKA, I. RAŞA, *On the composition and decomposition of positive linear operators (V)*, Results Math. DOI 10.1007/s00025-06-0618-8.

[9] H. GONSKA, I. RAŞA, E.-D. STĂNILĂ, *Beta operators with Jacobi weights, In: Constructive theory of functions*, Sozopol 2013 (K. Ivanov, G. Nikolov and R. Uluchev, Eds.), 99–112. Prof. Marin Drinov Academic Publishing House, Sofia, 2014.

[10] M. HEILMANN, F. NASAIREH, I. RAŞA, *Beta and related operators revisited*, Preprint 2016.

[11] M. HEILMANN, I. RAŞA, *On the decomposition of Bernstein operators*, Numer. Funct. Anal. Optimiz., **36** (2015), 72–85.

[12] A. LUPAŞ, *Die Folge der Betaoperatoren*, Dissertation, Universität Stuttgart (1972).

[13] G. MÜHLBACH, *Rekursionsformeln für die zentralen Momente der Pólya- und der Beta-Verteilung*, Metrika **19** (1972), 171–177.

[14] G. MÜHLBACH, *Verallgemeinerungen der Bernstein- und der Lagrangepolynome. Bemerkungen zu einer Klasse linearer Polynomoperatoren von D. D. Stancu*, Rev. Roumaine Math. Pures Appl. **15** (1970), 1235–1252.

(Received December 21, 2016)

Fadel Nasaireh
Technical University of Cluj-Napoca
Department of Mathematics
Memorandumului Street 28, 400114, Cluj-Napoca, Romania
e-mail: `fadelnasierh@gmail.com`

Ioan Raşa
Technical University of Cluj-Napoca
Department of Mathematics
Memorandumului Street 28, 400114, Cluj-Napoca, Romania
e-mail: `ioan.rasa@math.utcluj.ro`