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MATRIX RICHARD INEQUALITY VIA THE GEOMETRIC MEAN

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Dedicated to the memory of the late Professor Takayuki Furuta

(*Communicated by J. I. Fujii*)

Abstract. In this paper, we show the matrix version of Richard inequality by virture of Cauchy-Schwartz type inequalities via the matrix geometric mean. As an application, we show a matrix Buzano inequality.

1. Introduction

In [1], Buzano showed the following extention of the Cauchy-Schwarz inequality in a complex inner product space $(H;\langle\cdot,\cdot\rangle)$:

$$|\langle a,x\rangle\langle x,b\rangle| \leqslant \frac{1}{2}(\| a \|\| b \| + |\langle a,b\rangle|) \| x \|^2 \tag{1.1}$$

for all $a,b,x \in H$. If $a=b$, then (1.1) just becomes the Cauchy-Schwarz inequality

$$|\langle a,x\rangle|^2 \leqslant \| a \|^2 \| x \|^2 . \tag{1.2}$$

Dragomir [2] pointed out that in the proof of Buzano inequality (1.1), the following inequality due to Richard [7] is an essential part:

$$\left| \frac{\langle a,x\rangle\langle x,b\rangle}{\| x \|^2} - \frac{\langle a,b\rangle}{2} \right| \leqslant \frac{\| a \|\| b \|}{2} \tag{1.3}$$

for all $a,b,x \in H$. In fact, it follows from the triangle inequality that (1.3) implies (1.1). Thus, we call (1.3) the Richard inequality.

Let $\mathbb{M}_{m\times n} = \mathbb{M}_{m\times n}(\mathbb{C})$ be the space of $m\times n$ complex matrices and $\mathbb{M}_n = \mathbb{M}_{n\times n}$, and denote the matrix absolute value of any $A \in \mathbb{M}_n$ by $|A| = (A^*A)^{1/2}$. For $A \in \mathbb{M}_n$, we write $A \geqslant 0$ if A is positive semidefinite and $A > 0$ if A is positive definite; that is, $x^*Ax > 0$ for all nonzero column vectors $x \in \mathbb{C}^n$. For two Hermitian matrices A and B in $\mathbb{M}_n$, we write $A \geqslant B$ if $A-B \geqslant 0$ and $A > B$ if $A-B > 0$.

In the previous paper [5], we presented Cauchy-Schwarz type inequalities for matrices of the same size in terms of the matrix geometric mean and the polar decomposition. As a continuation, in this paper, we show the matrix version of the Richard inequality (1.3) by virture of Cauchy-Schwartz type inequalities via the matrix geometric mean. As an application, we show a matrix version of the Buzano inequality (1.1).

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2. Results

First of all, we recall the matrix geometric mean: Let A and B be two positive semidefinite matrices in $\mathbb{M}_n$. Then the matrix geometric mean $A \# B$ is defined by

$$A \# B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2} \tag{2.1}$$

if A is positive definite, also see [6]. By monotonicity, we can uniquely extend the definition of $A \# B$ for all positive semidefinite matrices A and B by setting

$$A \# B = \lim_{\varepsilon \downarrow 0}(A+\varepsilon I) \# (B+\varepsilon I).$$

For the sake of convenience, we cite a useful lemma which we will use frequently in the below.

LEMMA 2.1. *Let A,B,C and D be positive semidefinite matrices.*

(1) *Consistency with scalars: If A and B commute, then $A \# B = A^{1/2}B^{1/2}$;*

(2) *Monotonicity: $A \leqslant C$ and $B \leqslant D \implies A \# B \leqslant C \# D$;*

(3) *Transformer equality: $T^*AT \# T^*BT = T^*(A \# B)T$ for nonsingular T;*

(4) *Symmetry: $A \# B = B \# A$;*

(5) *Arithmetic-Geometric mean inequality: $A \# B \leqslant \frac{A+B}{2}$.*

In [5], we presented matrix Cauchy-Scwarz inequalities that derived by the matrix geometric mean, also see [3]:

LEMMA 2.2. (Matrix Cauchy-Schwarz inequality) *Let $X,Y \in \mathbb{M}_{k\times n}$ and $Y^*X = U|Y^*X|$ be a polar decomposition of Y^*X, where U is unitary in $\mathbb{M}_n$. Then*

$$|Y^*X| \leqslant X^*X \# U^*Y^*YU \tag{2.2}$$

and

$$|X^*Y| \leqslant UX^*XU^* \# Y^*Y. \tag{2.3}$$

Under the assumption that $\ker X \subset \ker YU$ (*resp.* $\ker Y \subset \ker XU^*$), *the equality in* (2.2) *(resp.* (2.3)) *holds if and only if there exists $W \in \mathbb{M}_n$ such that $YU = XW$ (resp. $XU^* = YW$).*

Note that the matrix Cauchy-Schwarz inequality (2.2) is a natural extension of the Cauchy-Schwarz inequality (1.2). In fact, let x and y be column vectors in $\mathbb{C}^n$. Since $\langle x,y\rangle = e^{i\theta}|\langle x,y\rangle|$ for some real number $\theta \in \mathbb{R}$, it follows from Lemma 2.2 that

$$\begin{aligned}|\langle x,y\rangle| &\leqslant \langle x,x\rangle \# e^{-i\theta}\langle y,y\rangle e^{i\theta}\\ &= \langle x,x\rangle^{\frac{1}{2}}\langle y,y\rangle^{\frac{1}{2}}.\end{aligned}$$

Inspired by the idea in [4], we show the following generalization of the matrix Cauchy-Schwarz inequality (2.2):

THEOREM 2.3. *Let* $X, Y \in \mathbb{M}_{k\times n}$, *$P$ be an orthogonal projection in* $\mathbb{M}_k$, *and* $Y^*(2P-I)X = U|Y^*(2P-I)X|$ *a polar decomposition of* $Y^*(2P-I)X$, *where U is unitary in* $\mathbb{M}_n$. *Then*

$$\left|Y^*PX - \frac{1}{2}Y^*X\right| \leqslant \frac{1}{2}(X^*X\#U^*Y^*YU) \tag{2.4}$$

and

$$\left|X^*PY - \frac{1}{2}X^*Y\right| \leqslant \frac{1}{2}(UX^*XU^*\#Y^*Y). \tag{2.5}$$

Under the assumption that $\ker X \subset \ker YU$ (*resp.* $\ker Y \subset \ker XU^*$), *the equality holds in* (2.4) (*resp.* (2.5)) *if and only if there exists* $W \in \mathbb{M}_n$ *such that* $YU = (2P-I)XW$ (*resp.* $XU^* = (2P-I)YW$).

Proof. Since P is an orthogonal projection, $2P-I$ is self-adjoint unitary and by using the Matrix Cauchy-Schwarz inequality (2.2), it follows that

$$\begin{aligned} 2\left|Y^*PX - \frac{1}{2}Y^*X\right| &= |Y^*(2P-I)X| \\ &\leqslant X^*(2P-I)^*(2P-I)X\#U^*Y^*YU \\ &= X^*X\#U^*Y^*YU. \end{aligned}$$

Since $\ker\,(2P-I)X = \ker X \subset \ker YU$, the equality in (2.4) immediately follows from the equality condition in Lemma 2.2. □

REMARK 2.4. If we put $x = b$ in the Richard inequality (1.3), then we have the Cauchy-Schwarz inequality (1.2). Similarly, if $P = I$ in Theorem 2.3, then (2.4) just becomes the matrix Cauchy-Schwarz inequality (2.2).

To prove the following corollary, we need the following Thompson inequality [8] (p. 289 Theorem8.22): For any square matrices A and B in $\mathbb{M}_n$, there exist unitary matrices U and V in $\mathbb{M}_n$ such that $|A+B| \leqslant U^*|A|U + V^*|B|V$.

COROLLARY 2.5. *Let* $X, Y \in \mathbb{M}_{k\times n}$, *$P$ be an orthogonal projection in* $\mathbb{M}_k$, *and* $U \in \mathbb{M}_n$ *a unitary matrix in a polar decomposition of* $Y^*(2P-I)X$. *Then there exist unitary* $V, W \in \mathbb{M}_n$ *such that*

$$V^*|Y^*PX|V \leqslant \frac{1}{2}(X^*X\#U^*Y^*YU + W^*|Y^*X|W).$$

Proof. By Thompson inequality, there exist unitary matrices V and W such that

$$V^*|Y^*PX|V - \frac{1}{2}W^*|Y^*X|W \leqslant \left|Y^*PX - \frac{1}{2}Y^*X\right|.$$

By Theorem 2.3, we have

$$\left|Y^*PX - \frac{1}{2}Y^*X\right| \leqslant \frac{1}{2}(X^*X \# U^*Y^*YU),$$

and hence combining two inequality above, we have this corollary. □

For $Z \in \mathbb{M}_{k\times n}$, we denote a generalized inverse of Z by Z^-, i.e., Z^- satisfies $ZZ^-Z = Z$, and $Z(Z^*Z)^-Z^*$ is the orthogonal projection onto the column space of Z. Hence we have the following results by Theorem 2.3.

The following corollary is the matrix version of the Richard inequality (1.3):

COROLLARY 2.6. (Matrix Richard inequality) *Let $X,Y,Z \in \mathbb{M}_{k\times n}$ and $U \in \mathbb{M}_n$ a unitary matrix in a polar decomposition of $Y^*(2Z(Z^*Z)^-Z^*X - X)$. Then*

$$\left|Y^*Z(Z^*Z)^-Z^*X - \frac{1}{2}Y^*X\right| \leqslant \frac{1}{2}(X^*X \# U^*Y^*YU). \tag{2.6}$$

Under the assumption that $\ker X \subset \ker YU$, *the equality holds in* (2.6) *if and only if there exists $W \in \mathbb{M}_n$ such that $YU = (2Z(Z^*Z)^-Z^* - I)XW$.*

Proof. If we put $P = Z(Z^*Z)^-Z^*$ in Theorem 2.3, then we have this corollary. □

In addition, we show the matrix version of the Buzano inequality (1.1) too:

COROLLARY 2.7. (Matrix Buzano inequality) *Let $X,Y,Z \in \mathbb{M}_{k\times n}$ and $U \in \mathbb{M}_n$ be a unitary matrix in a polar decomposition of $Y^*(2Z(Z^*Z)^-Z^*X - X)$. Then there exist unitary matrices V and W in $\mathbb{M}_n$ such that*

$$V^*|Y^*Z(Z^*Z)^-Z^*X|V \leqslant \frac{1}{2}(X^*X \# U^*Y^*YU + W^*|Y^*X|W).$$

Proof. If we put $P = Z(Z^*Z)^-Z^*$ in Corollary 2.5, then we have this corollary. □

REMARK 2.8. If $n = 1$ in Corollary 2.6 (resp. Corollary 2.7), then it just becomes the Richard inequality (1.3) (resp. the Buzano inequality (1.1)).

Lastly we present the following corollary related to the matrix Richard inequality:

COROLLARY 2.9. *Let $X,Y,Z \in \mathbb{M}_{k\times n}$ and $U \in \mathbb{M}_n$ (resp. $V \in \mathbb{M}_n$) be a unitary matrix in a polar decomposition of $Y^*(Z(Z^*Z)^-Z^*X - X)$ in (1) (resp. $Y^*Z(Z^*Z)^-Z^*X$ in (2)). Then*

(1) $|Y^*Z(Z^*Z)^-Z^*X - Y^*X| \leqslant (X^*X - X^*Z(Z^*Z)^-Z^*X) \# U^*Y^*YU$;

(2) $|Y^*Z(Z^*Z)^-Z^*X| \leqslant X^*Z(Z^*Z)^-Z^*X \# V^*Y^*YV$.

Proof. By the matrix Cauchy-Schwarz inequality in Lemma 2.2, we have this corollary. □

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