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A MORE ACCURATE MULTIDIMENSIONAL HARDY–HILBERT TYPE INEQUALITY WITH A GENERAL HOMOGENEOUS KERNEL

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Abstract. In this paper, by the use of the weight coefficients, the transfer formula, Hermite-Hadamard's inequality and the technique of real analysis, a more accurate multidimensional Hardy-Hilbert-type inequality with a general homogeneous kernel and a best possible constant factor is given, which is an extension of some published results. Moreover, the equivalent forms, the operator expressions and some particular examples are considered.

1. Introduction

If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_m,b_n\geqslant 0$, $a=\{a_m\}_{m=1}^{\infty}\in l^p$, $b=\{b_n\}_{n=1}^{\infty}\in l^q$, $||a||_p=(\sum_{m=1}^{\infty}a_m^p)^{\frac{1}{p}}>0$, $||b||_q>0$, then we have the following well known Hardy-Hilbert's inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{m+n}<\frac{\pi}{\sin(\pi/p)}||a||_p||b||_q, \tag{1}$$

and the following more accurate Hardy-Hilbert's inequality with the same best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [1], Theorem 315, Theorem 323):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{m+n-1}<\frac{\pi}{\sin(\pi/p)}||a||_p||b||_q, \tag{2}$$

Inequalities (1) and (2) are important in analysis and its applications (cf. [1], [2], [3]).

Assuming that $\{\mu_m\}_{m=1}^{\infty}$ and $\{\upsilon_n\}_{n=1}^{\infty}$ are positive sequences, such that

$$U_m=\sum_{i=1}^{m}\mu_i, V_n=\sum_{j=1}^{n}\upsilon_j \ (m,n\in\mathbf{N}=\{1,2,\ldots\}),$$

we have the following Hardy-Hilbert-type inequality (cf. [1], Theorem 321):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{U_m+V_n}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}\frac{a_m^p}{m^{p-1}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{b_n^q}{n^{q-1}}\right)^{\frac{1}{q}}. \tag{3}$$

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For $\mu_i = \upsilon_j = 1 \ (i, j \in \mathbf{N})$, inequality (3) reduces to (1).

In 2015, by using the transfer formula, Yang [4] gave the following multidimensional Hilbert-type inequality: For $i_0, j_0 \in \mathbf{N}$, $\alpha, \beta > 0$,

$$||x||_\alpha := \left(\sum_{k=1}^{i_0} |x^{(k)}|^\alpha \right)^{\frac{1}{\alpha}} \quad (x = (x^{(1)}, \ldots, x^{(i_0)}) \in \mathbf{R}^{i_0}),$$

$$||y||_\beta := \left(\sum_{k=1}^{j_0} |y^{(k)}|^\beta \right)^{\frac{1}{\beta}} \quad (y = (y^{(1)}, \ldots, y^{(j_0)}) \in \mathbf{R}^{j_0}),$$

$0 < \lambda_1 \leqslant i_0$, $0 < \lambda_2 \leqslant j_0$, $\lambda_1 + \lambda_2 = \lambda$, $a_m, b_n \geqslant 0$, we have

$$\sum_n \sum_m \frac{1}{||m||_\alpha^\lambda + ||n||_\beta^\lambda} a_m b_n$$
$$< K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} \left[\sum_m ||m||_\alpha^{p(i_0 - \lambda_1) - i_0} a_m^p \right]^{\frac{1}{p}} \left[\sum_n ||n||_\beta^{q(j_0 - \lambda_2) - j_0} b_n^q \right]^{\frac{1}{q}}, \tag{4}$$

where, $\sum_m = \sum_{m_{i_0}=1}^\infty \cdots \sum_{m_1=1}^\infty$, $\sum_n = \sum_{n_{j_0}=1}^\infty \cdots \sum_{n_1=1}^\infty$, the series in the right hand side of (4) are positive values, and the best possible constant factor $K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$ is indicated by

$$K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0 - 1} \Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\beta^{i_0 - 1} \Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} \frac{\pi}{\lambda \sin(\frac{\pi \lambda_1}{\lambda})}.$$

For $i_0 = j_0 = \lambda = \alpha = \beta = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, inequality (4) reduces to (1). Some other results on this type of inequalities and multiple inequalities were provided by [5]–[25].

Recently, by using the weight coefficients, Yang [26] gave an extension of (3) as follows: For $\eta > 0$, $0 < \lambda_1 \leqslant 1$, $0 < \lambda_2 \leqslant 1$, $\lambda_1 + \lambda_2 = \lambda$, $a_m, b_n \geqslant 0$,

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{(U_m^\eta + V_n^\eta)^{\lambda/\eta}}$$
$$< \frac{1}{\eta} B\left(\frac{\lambda_1}{\eta}, \frac{\lambda_2}{\eta} \right) \left[\sum_{m=1}^\infty \frac{U_m^{p(1-\lambda_1)-1} a_m^p}{m^{p-1}} \right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty \frac{V_n^{q(1-\lambda_2)-1} b_n^q}{n^{q-1}} \right]^{\frac{1}{q}}, \tag{5}$$

where, the constant factor $\frac{1}{\eta} B(\frac{\lambda_1}{\eta}, \frac{\lambda_2}{\eta})$ is the best possible (the series in the right hand side of (5) are positive values). Another results on Hardy-Hilbert-type inequalities and Hilbert-type inequalities were given by [27]–[37].

In this paper, by the use of the weight coefficients, the transfer formula, Hermite-Hadamard's inequality and the technique of real analysis, a more accurate multidimensional Hardy-Hilbert's inequality with a general homogeneous kernel and a best possible constant factor is given, which is an extension of (4) and (5). Moreover, the equivalent forms, the operator expressions and some particular examples are considered.

2. Some lemmas

If $\mu_i^{(k)} > 0,\ 0 \leqslant \widetilde{\mu}_i^{(k)} \leqslant \frac{1}{2}\mu_i^{(k)}$ $(k = 1,\dots,i_0;\ i = 1,\dots,m)$, $\upsilon_j^{(l)} > 0,\ 0 \leqslant \widetilde{\upsilon}_j^{(l)} \leqslant \frac{1}{2}\upsilon_j^{(l)}$ $(l = 1,\dots,j_0;\ j{=}1,\dots,n)$, then we set

$$U_m^{(k)} := \sum_{i=1}^{m} \mu_i^{(k)}, \quad \widetilde{U}_m^{(k)} := U_m^{(k)} - \widetilde{\mu}_m^{(k)} \quad (k = 1,\dots,i_0),$$
$$V_n^{(l)} := \sum_{j=1}^{n} \upsilon_j^{(l)}, \quad \widetilde{V}_n^{(l)} := V_n^{(l)} - \widetilde{\upsilon}_n^{(l)} \quad (l = 1,\dots,j_0),$$

$$\begin{aligned} U_m &:= (U_m^{(1)},\dots,U_m^{(i_0)}), \quad \widetilde{\mu}_m := (\widetilde{\mu}_m^{(1)},\dots,\widetilde{\mu}_m^{(i_0)}),\\ \widetilde{U}_m &:= (\widetilde{U}_m^{(1)},\dots,\ \widetilde{U}_m^{(i_0)}) = U_m - \widetilde{\mu}_m,\\ V_n &:= (V_n^{(1)},\dots,V_n^{(j_0)}), \quad \widetilde{\upsilon}_n := (\widetilde{\upsilon}_n^{(1)},\dots,\widetilde{\upsilon}_n^{(j_0)}),\\ \widetilde{V}_n &:= (\widetilde{V}_n^{(1)},\dots,\widetilde{V}_n^{(j_0)}) = V_n - \widetilde{\upsilon}_n \quad (m,n \in \mathbf{N}). \end{aligned} \tag{6}$$

We also set functions $\mu_k(t) := \mu_m^{(k)},\ t \in (m - \frac{1}{2}, m + \frac{1}{2}]$ $(m \in \mathbf{N})$; $\upsilon_l(t) := \upsilon_n^{(l)},\ t \in (n - \frac{1}{2}, n + \frac{1}{2}]$ $(n \in \mathbf{N})$, and

$$U_k(x) := \int_{\frac{1}{2}}^{x} \mu_k(t)dt \quad (k = 1,\dots,i_0),$$
$$V_l(y) := \int_{\frac{1}{2}}^{y} \upsilon_l(t)dt \quad (l = 1,\dots,j_0), \tag{7}$$
$$U(x) := (U_1(x),\dots,U_{i_0}(x)),$$
$$V(y) := (V_1(y),\dots,V_{j_0}(y)) \quad \left(x, y \geqslant \frac{1}{2}\right). \tag{8}$$

It follows that

$$\begin{aligned} U_k(m) &= \int_{\frac{1}{2}}^{m} \mu_k(t)dt = \int_{\frac{1}{2}}^{m+\frac{1}{2}} \mu_k(t)dt - \frac{1}{2}\mu_m^{(k)}\\ &\leqslant \widetilde{U}_m^{(k)} \leqslant U_k\left(m + \frac{1}{2}\right) \quad (k = 1,\dots,i_0;\ m \in \mathbf{N}),\\ V_l(n) &\leqslant \widetilde{V}_n^{(l)} \leqslant V_l\left(n + \frac{1}{2}\right) \quad (l = 1,\dots,j_0;\ n \in \mathbf{N}), \end{aligned}$$

and for $x \in (m - \frac{1}{2}, m + \frac{1}{2})$, $U_k'(x) = \mu_k(x) = \mu_m^{(k)}$ $(k = 1,\dots,i_0;\ m \in \mathbf{N})$; for $y \in (n - \frac{1}{2}, n + \frac{1}{2})$, $V_l'(y) = \upsilon_l(y) = \upsilon_n^{(l)}$ $(l = 1,\dots,j_0;\ n \in \mathbf{N})$.

LEMMA 1. (cf. [31]) *Suppose that* $g(t)\ (> 0)$ *is strictly decreasing and strictly convex in* $(\frac{1}{2}, \infty)$, *satisfying* $\int_{\frac{1}{2}}^{\infty} g(t)dt \in \mathbf{R}_+$. *We have the following Hermite-Hadamard's inequality*

$$\int_{n}^{n+1} g(t)dt < g(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} g(t)dt\ (n \in \mathbf{N}), \tag{9}$$

and then

$$\int_1^\infty g(t)dt < \sum_{n=1}^\infty g(n) < \int_{\frac{1}{2}}^\infty g(t)dt. \tag{10}$$

LEMMA 2. *If* $i_0 \in \mathbf{N},\ \alpha, M > 0,\ \Psi(u)$ *is a non-negative measurable function in* $(0,1]$, *and*

$$D_M := \left\{ x = (x_1, \ldots, x_{i_0}) \in \mathbf{R}_+^{i_0}; u = \sum_{i=1}^{i_0} \left(\frac{x_i}{M}\right)^\alpha \leqslant 1 \right\}, \tag{11}$$

then we have the following transfer formula (cf. [5]*):*

$$\int \cdots \int_{D_M} \Psi\left(\sum_{i=1}^{i_0}\left(\frac{x_i}{M}\right)^\alpha\right) dx_1 \cdots dx_s = \frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})} \int_0^1 \Psi(u) u^{\frac{i_0}{\alpha}-1} du. \tag{12}$$

LEMMA 3. *For* $i_0, j_0 \in \mathbf{N},\ \alpha, \beta, \varepsilon > 0,\ \mu_m^{(k)} \geqslant \mu_{m+1}^{(k)}\ \ (m \in \mathbf{N};\ k = 1, \ldots, i_0),$ $\upsilon_n^{(l)} \geqslant \upsilon_{n+1}^{(l)}\ (n \in \mathbf{N};\ l = 1, \ldots, i_0),\ c = \min_{1 \leqslant i \leqslant i_0, 1 \leqslant j \leqslant j_0}\{\mu_1^{(i)}, \upsilon_1^{(j)}\} (> 0)$, *we have*

$$\sum_m ||\widetilde{U}_m||_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} \leqslant \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon c^\varepsilon i_0^{\varepsilon/\alpha} \alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} + O(1), \tag{13}$$

$$\sum_n ||\widetilde{V}_n||_\beta^{-j_0-\varepsilon} \prod_{k=1}^{j_0} \upsilon_n^{(k)} \leqslant \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\varepsilon c^\varepsilon j_0^{\varepsilon/\beta} \beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \widetilde{O}(1)\ (\varepsilon \to 0^+). \tag{14}$$

Proof. For $M > c i_0^{1/\alpha}$, we set

$$\Psi(u) = \begin{cases} 0, & 0 < u < \frac{c^\alpha i_0}{M^\alpha}, \\ \frac{1}{(Mu^{1/\alpha})^{i_0+\varepsilon}}, & \frac{c^\alpha i_0}{M^\alpha} \leqslant u \leqslant 1. \end{cases}$$

By (12), it follows that

$$\begin{aligned} \int_{\{x \in \mathbf{R}_+^{i_0}; x_i \geqslant c\}} \frac{dx}{||x||_\alpha^{i_0+\varepsilon}} &= \lim_{M\to\infty} \int \cdots \int_{D_M} \Psi\left(\sum_{i=1}^{i_0}\left(\frac{x_i}{M}\right)^\alpha\right) dx_1 \cdots dx_{i_0} \\ &= \lim_{M\to\infty} \frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})} \int_{c^\alpha i_0/M^\alpha}^1 \frac{u^{\frac{i_0}{\alpha}-1}}{(Mu^{1/\alpha})^{i_0+\varepsilon}} du \\ &= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon c^\varepsilon i_0^{\varepsilon/\alpha} \alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}. \end{aligned}$$

Then by (10) and the above result, in view of $U_k(m) \leqslant \widetilde{U}_m^{(k)}$, we find

$$\begin{aligned}
0 &< \sum_{\{m\in\mathbf{N}^{i_0};m_i\geqslant 2\}} ||\widetilde{U}_m||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)} \\
&\leqslant \sum_{\{m\in\mathbf{N}^{i_0};m_i\geqslant 2\}} \int_{\{x\in\mathbf{R}_+^{i_0};m_i-\frac{1}{2}\leqslant x_i<m_i+\frac{1}{2}\}} ||U(m)||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)}dx \\
&< \sum_{\{m\in\mathbf{N}^{i_0};m_i\geqslant 2\}} \int_{\{x\in\mathbf{R}_+^{i_0};m_i-\frac{1}{2}\leqslant x_i<m_i+\frac{1}{2}\}} ||U(x)||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_k(x)dx \\
&= \int_{\{x\in\mathbf{R}_+^{i_0};x_i\geqslant\frac{3}{2}\}} ||U(x)||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_k(x)dx \\
&\overset{v=U(x)}{=} \int_{\{v\in\mathbf{R}_+^{i_0};v_i\geqslant\mu_1^{(i)}\}} ||v||_\alpha^{-i_0-\varepsilon}dv \leqslant \int_{\{v\in\mathbf{R}_+^{i_0};v_i\geqslant c\}} ||v||_\alpha^{-i_0-\varepsilon}dv \\
&= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon c^\varepsilon i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}.
\end{aligned}$$

For $i_0=1$, $0<\sum_{\{m\in\mathbf{N}^{i_0};m_i=1\}}||\widetilde{U}_m||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)}\leqslant(\mu_1^{(1)})^{-\varepsilon}<\infty$; for $i_0\geqslant 2$, we set

$$H_i := \sum_{\{m\in\mathbf{N}^{i_0};m_i=1\}} ||\widetilde{U}_m||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)}(i=1,\ldots,i_0).$$

Without lose of generality, we estimate H_{i_0} as follows:

$$\begin{aligned}
H_{i_0} &\leqslant \sum_{\{m\in\mathbf{N}^{i_0};m_{i_0}=1\}} ||U(m)||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)} \\
&= \mu_1^{(i_0)}\sum_{m\in\mathbf{N}^{i_0-1}} \frac{\prod_{k=1}^{i_0-1}\mu_m^{(k)}}{[\sum_{i=1}^{i_0-1}U_i^\alpha(m)+(\frac{1}{2}\mu_1^{(i_0)})^\alpha]^{\frac{1}{\alpha}(i_0+\varepsilon)}} \\
&< \sum_{m\in\mathbf{N}^{i_0-1}}\int_{\{x\in\mathbf{R}_+^{i_0-1};m_i-\frac{1}{2}\leqslant x_i<m_i+\frac{1}{2}\}} \frac{\mu_1^{(i_0)}\prod_{k=1}^{i_0-1}\mu_k(x)dx}{[\sum_{i=1}^{i_0-1}U_i^\alpha(x)+(\frac{1}{2}\mu_1^{(i_0)})^\alpha]^{\frac{1}{\alpha}(i_0+\varepsilon)}} \\
&= \mu_1^{(i_0)}\int_{\{x\in\mathbf{R}^{i_0-1};x_i\geqslant\frac{1}{2}\}} \frac{\prod_{k=1}^{i_0-1}\mu_k(x)}{[\sum_{i=1}^{i_0-1}U_i^\alpha(x)+(\frac{1}{2}\mu_1^{(i_0)})^\alpha]^{\frac{1}{\alpha}(i_0+\varepsilon)}}dx \\
&\overset{v=U(x)}{=} \mu_1^{(i_0)}\int_{\{v\in\mathbf{R}^{i_0-1}\}} \frac{1}{[M^\alpha\sum_{i=1}^{i_0-1}(\frac{v_i}{M})^\alpha+(\frac{1}{2}\mu_1^{(i_0)})^\alpha]^{\frac{1}{\alpha}(i_0+\varepsilon)}}dv.
\end{aligned}$$

By (12), we find

$$H_{i_0} \leqslant \mu_1^{(i_0)} \lim_{M\to\infty} \frac{M^{i_0-1}\Gamma^{i_0-1}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0-1}{\alpha})} \int_0^1 \frac{u^{\frac{i_0-1}{\alpha}-1}}{[M^\alpha u + (\frac{1}{2}\mu_1^{(i_0)})^\alpha]^{\frac{1}{\alpha}(i_0+\varepsilon)}} du$$

$$\overset{t=\frac{M^\alpha u}{(\frac{1}{2}\mu_1^{(i_0)})^\alpha}}{=} \frac{1}{(\frac{1}{2})^{1+\varepsilon}(\mu_1^{(i_0)})^\varepsilon} \frac{\Gamma^{i_0-1}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0-1}{\alpha})} \int_0^1 \frac{t^{\frac{i_0-1}{\alpha}-1}dt}{(t+1)^{\frac{1}{\alpha}(i_0+\varepsilon)}}$$

$$= \frac{1}{(\frac{1}{2})^{1+\varepsilon}(\mu_1^{(i_0)})^\varepsilon} \frac{\Gamma^{i_0-1}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0-1}{\alpha})} B\Big(\frac{i_0-1}{\alpha}, \frac{1+\varepsilon}{\alpha}\Big) < \infty,$$

namely, $H_{i_0} = O_{i_0}(1)$. Then we have

$$\sum_m ||\widetilde{U}_m||_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} \leqslant \sum_{\{m\in\mathbf{N}^{i_0};m_i\geqslant 2\}} ||\widetilde{U}_m||_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} + \sum_{i=1}^{i_0} H_i$$

$$\leqslant \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon c^\varepsilon i_0^{\varepsilon/\alpha} \alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} + \sum_{i=1}^{i_0} O_i(1)\ (\varepsilon \to 0^+),$$

and then (13) follows. In the same way, we have (14). □

DEFINITION 1. If $0 < \alpha, \beta \leqslant 1,\ \lambda_1, \lambda_2 \in \mathbf{R},\ \lambda_1 + \lambda_2 = \lambda,\ k_\lambda(x,y)$ is a positive continuous homogeneous function in $\mathbf{R}^2$, satisfying $k_\lambda(vx, vy) = v^{-\lambda} k_\lambda(x,y) > 0$,

$$\frac{\partial}{\partial x} k_\lambda(x,y) < 0, \quad \frac{\partial}{\partial y} k_\lambda(x,y) < 0,$$
$$\frac{\partial^2}{\partial x^2} k_\lambda(x,y) > 0, \quad \frac{\partial^2}{\partial y^2} k_\lambda(x,y) > 0 \quad (v,x,y > 0),$$

and

$$k(\lambda_1) := \int_0^\infty k_\lambda(u,1) u^{\lambda_1 - 1} du \in \mathbf{R}_+,$$

then we define two weight coefficients $w(\lambda_1, n)$ and $W(\lambda_2, m)$ as follows:

$$w(\lambda_1, n) := \sum_m k_\lambda(||\widetilde{U}_m||_\alpha, ||\widetilde{V}_n||_\beta) \frac{||\widetilde{V}_n||_\beta^{\lambda_2}}{||\widetilde{U}_m||_\alpha^{i_0-\lambda_1}} \prod_{k=1}^{i_0} \mu_m^{(k)}, \tag{15}$$

$$W(\lambda_2, m) := \sum_n k_\lambda(||\widetilde{U}_m||_\alpha, ||\widetilde{V}_n||_\beta) \frac{||\widetilde{U}_m||_\alpha^{\lambda_1}}{||\widetilde{V}_n||_\beta^{j_0-\lambda_2}} \prod_{l=1}^{j_0} \upsilon_n^{(l)}. \tag{16}$$

NOTE 1. With regards to the assumptions of Definition 1, (i) for $\lambda_1 \leqslant i_0, \lambda_2 \leqslant j_0$, we still can find that

$$\frac{\partial}{\partial x}\Big(k_\lambda(x,y)\frac{1}{x^{i_0-\lambda_1}}\Big), \quad \frac{\partial}{\partial y}\Big(k_\lambda(x,y)\frac{1}{y^{j_0-\lambda_2}}\Big) < 0;$$
$$\frac{\partial^2}{\partial x^2}\Big(k_\lambda(x,y)\frac{1}{x^{i_0-\lambda_1}}\Big), \quad \frac{\partial^2}{\partial y^2}\Big(k_\lambda(x,y)\frac{1}{y^{j_0-\lambda_2}}\Big) > 0 \quad (x,y > 0).$$

(ii) If $(-1)^i h^{(i)}(t) > 0$ $(t > 0;\ i = 0,1,2)$, $b > 0$, $0 < \alpha \leqslant 1$, then we have

$$\begin{aligned}
\frac{d}{dx}h((b+x^{\alpha})^{\frac{1}{\alpha}}) &= h'((b+x^{\alpha})^{\frac{1}{\alpha}})(b+x^{\alpha})^{\frac{1}{\alpha}-1}x^{\alpha-1} < 0,\\
\frac{d^2}{dx^2}h((b+x^{\alpha})^{\frac{1}{\alpha}}) &= h''((b+x^{\alpha})^{\frac{1}{\alpha}})(b+x^{\alpha})^{\frac{2}{\alpha}-2}x^{2\alpha-2}\\
&\quad +(1-\alpha)h'((b+x^{\alpha})^{\frac{1}{\alpha}})(b+x^{\alpha})^{\frac{1}{\alpha}-2}x^{2\alpha-2}\\
&\quad +(\alpha-1)h'((b+x^{\alpha})^{\frac{1}{\alpha}})(b+x^{\alpha})^{\frac{1}{\alpha}-1}x^{\alpha-2}\\
&= h''((b+x^{\alpha})^{\frac{1}{\alpha}})(b+x^{\alpha})^{\frac{2}{\alpha}-2}x^{2\alpha-2}\\
&\quad +b(\alpha-1)h'((b+x^{\alpha})^{\frac{1}{\alpha}})(b+x^{\alpha})^{\frac{1}{\alpha}-2}x^{\alpha-2} > 0 \quad (x>0).
\end{aligned}$$

Hence, by the assumptions and (9), for $m_i - \frac{1}{2} < x_i < m_i + \frac{1}{2}$ $(i = 1,\dots,i_0;\ m \in \mathbf{N})$, we have $\prod_{k=1}^{i_0} \mu_m^{(k)} = \prod_{k=1}^{i_0} \mu_k(x)$ and

$$\begin{aligned}
&k_{\lambda}(||U(m)||_{\alpha}, ||\widetilde{V}_n||_{\beta})||U(m)||_{\alpha}^{\lambda_1-i_0}\prod_{k=1}^{i_0}\mu_m^{(k)}\\
&< \int_{\{x\in\mathbf{R}_+^{i_0};m_i-\frac{1}{2}<x_i<m_i-\frac{1}{2}\}} k_{\lambda}(||U(x)||_{\alpha}, ||\widetilde{V}_n||_{\beta})||U(x)||_{\alpha}^{\lambda_1-i_0}\prod_{k=1}^{i_0}\mu_k(x)dx.
\end{aligned}$$

LEMMA 4. *With regards to the assumptions of Definition 1, then (i) we have*

$$w(\lambda_1,n) < K_{\beta}(\lambda_1)\ (\text{for } \lambda_1 \leqslant i_0; n \in \mathbf{N}^{j_0}), \tag{17}$$

$$W(\lambda_2,m) < K_{\alpha}(\lambda_1)\ \ (\text{for } \ \lambda_2 \leqslant j_0;\ m \in \mathbf{N}^{i_0}), \tag{18}$$

where,

$$K_{\beta}(\lambda_1) = \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}k(\lambda_1), K_{\alpha}(\lambda_1) = \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}k(\lambda_1); \tag{19}$$

(ii) for $\mu_m^{(k)} \geqslant \mu_{m+1}^{(k)}$ $(m \in \mathbf{N})$, $\upsilon_n^{(l)} \geqslant \upsilon_{n+1}^{(l)}$ $(n \in \mathbf{N})$, $U_\infty^{(k)} = V_\infty^{(l)} = \infty$ $(k = 1,\dots,i_0,\ l = 1,\dots,j_0)$, we have

$$0 < K_{\alpha}(\lambda_1)(1-\theta_{\lambda}(n)) < w(\lambda_1,n)\ \ (\text{for } \ \lambda_1 \leqslant i_0;\ n \in \mathbf{N}^{j_0}), \tag{20}$$

where, for $b := \max_{1\leqslant k\leqslant i_0}\{\mu_1^{(k)}\}$ (> 0),

$$\theta_{\lambda}(n) := \frac{1}{k(\lambda_1)}\int_0^{b i_0^{1/\alpha}/||\widetilde{V}_n||_{\beta}} k_{\lambda}(t,1)t^{\lambda_1-1}dt \in (0,1). \tag{21}$$

Proof. (i) Since $||\widetilde{U}_m||_{\alpha} \geqslant ||U(m)||_{\alpha}$, by (10), (12) and Note 1(ii), for $\lambda_1 \leqslant i_0$, it follows that

$$\begin{aligned}
w(\lambda_1,n) &= \sum_m k_\lambda(||\widetilde{U}_m||_\alpha,||\widetilde{V}_n||_\beta)\frac{||\widetilde{V}_n||_\beta^{\lambda_2}}{||\widetilde{U}_m||_\alpha^{i_0-\lambda_1}}\prod_{k=1}^{i_0}\mu_m^{(k)} \\
&\leqslant \sum_m k_\lambda(||U(m)||_\alpha,||\widetilde{V}_n||_\beta)\frac{||\widetilde{V}_n||_\beta^{\lambda_2}}{||U(m)||_\alpha^{i_0-\lambda_1}}\prod_{k=1}^{i_0}\mu_m^{(k)} \\
&< \sum_m \int_{\{x\in\mathbf{R}_+^{i_0};m_i-\frac{1}{2}<x_i\leqslant m_i+\frac{1}{2}\}} k_\lambda(||U(x)||_\alpha,||\widetilde{V}_n||_\beta)\frac{|\widetilde{V}_n||_\beta^{\lambda_2}}{||U(x)||_\alpha^{i_0-\lambda_1}}\prod_{k=1}^{i_0}\mu_k(x)dx \\
&= \int_{\{x\in\mathbf{R}_+^{i_0};x_i>\frac{1}{2}\}} k_\lambda(||U(x)||_\alpha,||\widetilde{V}_n||_\beta)||U(x)||_\alpha^{\lambda_1-i_0}||\widetilde{V}_n||_\beta^{\lambda_2}\prod_{k=1}^{i_0}\mu_k(x)dx \\
&\overset{v=U(x)}{=} \int_{\mathbf{R}_+^{i_0}} k_\lambda(||v||_\alpha,||\widetilde{V}_n||_\beta)||v||_\alpha^{\lambda_1-i_0}||\widetilde{V}_n||_\beta^{\lambda_2}dv \\
&= \lim_{M\to\infty}\int_{\mathbf{D}_M} k_\lambda(M[\sum_{i=1}^{i_0}(\frac{v_i}{M})^\alpha]^{\frac{1}{\alpha}},||\widetilde{V}_n||_\beta)M^{\lambda_1-i_0}[\sum_{i=1}^{j_0}(\frac{v_i}{M})^\alpha]^{\frac{\lambda_1-i_0}{\alpha}}||\widetilde{V}_n||_\beta^{\lambda_2}dv \\
&= \lim_{M\to\infty}\frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})}\int_0^1 k_\lambda(Mu^{1/\alpha},||\widetilde{V}_n||_\beta)M^{\lambda_1-i_0}u^{(\lambda_1-i_0)/\alpha}||\widetilde{V}_n||_\beta^{\lambda_2}u^{\frac{i_0}{\alpha}-1}du \\
&= \lim_{M\to\infty}\frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})}\int_0^1 k_\lambda(Mu^{1/\alpha},||\widetilde{V}_n||_\beta)(Mu^{1/\alpha})^{\lambda_1-i_0}||\widetilde{V}_n||_\beta^{\lambda_2}u^{\frac{i_0}{\alpha}-1}du \\
&\overset{t=\frac{Mu^{1/\alpha}}{||\widetilde{V}_n||_\beta}}{=} \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\int_0^\infty k_\lambda(t,1)t^{\lambda_1-1}dt = \frac{\Gamma^{i_0}(\frac{1}{\alpha})k_\lambda(\lambda_1)}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} = K_\alpha(\lambda_1).
\end{aligned}$$

Hence, we have (17). In the same way, $\lambda_2 \leqslant j_0,$ we have (18).

(ii) Since for $m_i \leqslant x_i < m_i+\frac{1}{2}$, $\mu_m^{(k)} \geqslant \mu_{m+1}^{(k)} = \mu^{(k)}(x+\frac{1}{2})$; for $m_i+\frac{1}{2} \leqslant x_i < m_i+1$, $\mu_m^{(k)} = \mu^{(k)}(x+\frac{1}{2})$, by (10) and in the same way, for $b = \max_{1\leqslant k\leqslant i_0}\{\mu_1^{(k)}\}\ (>0)$, $\lambda_1 \leqslant i_0,$ we have

$$\begin{aligned}
w(\lambda_1,n) &\geqslant \sum_m k_\lambda\left(\left|\left|U\left(m+\frac{1}{2}\right)\right|\right|_\alpha,||\widetilde{V}_n||_\beta\right)\left|\left|U\left(m+\frac{1}{2}\right)\right|\right|_\alpha^{\lambda_1-i_0}||\widetilde{V}_n||_\beta^{\lambda_2}\prod_{k=1}^{i_0}\mu_m^{(k)} \\
&> \sum_m \int_{\{x\in\mathbf{R}_+^{i_0};m_i\leqslant x_i<m_i+1\}} k_\lambda\left(\left|\left|U\left(x+\frac{1}{2}\right)\right|\right|_\alpha,||\widetilde{V}_n||_\beta\right)\left|\left|U\left(x+\frac{1}{2}\right)\right|\right|_\alpha^{\lambda_1-i_0} \\
&\quad \times||\widetilde{V}_n||_\beta^{\lambda_2}\prod_{k=1}^{i_0}\mu_k\left(x+\frac{1}{2}\right)dx \\
&= \int_{[1,\infty)^{i_0}} k_\lambda\left(\left|\left|U\left(x+\frac{1}{2}\right)\right|\right|_\alpha,||\widetilde{V}_n||_\beta\right)\left|\left|U\left(x+\frac{1}{2}\right)\right|\right|_\alpha^{\lambda_1-i_0}||\widetilde{V}_n||_\beta^{\lambda_2}\prod_{k=1}^{i_0}\mu_k\left(x+\frac{1}{2}\right)dx \\
&\geqslant \int_{[b,\infty)^{i_0}} k_\lambda(||v||_\alpha,||\widetilde{V}_n||_\beta)||v||_\alpha^{\lambda_1-i_0}||\widetilde{V}_n||_\beta^{\lambda_2}dv.
\end{aligned}$$

For $M > bi_0^{1/\alpha}$, we set

$$\Psi(u) = \begin{cases} 0, \quad 0 < u < \frac{b^\alpha i_0}{M^\alpha}, \\ k_\lambda(Mu^{1/\alpha}, ||\widetilde{V}_n||_\beta)(Mu^{1/\alpha})^{\lambda_1 - i_0}||\widetilde{V}_n||_\beta^{\lambda_2}, \quad \frac{b^\alpha i_0}{M^\alpha} \leqslant u \leqslant 1. \end{cases}$$

By (12), it follows that

$$\begin{aligned} &\int_{\{x\in\mathbf{R}_+^{i_0}; x_i \geqslant b\}} k_\lambda(||x||_\alpha, ||\widetilde{V}_n||_\beta)||x||_\alpha^{\lambda_1 - i_0}||\widetilde{V}_n||_\beta^{\lambda_2} dx \\ = \quad & \lim_{M\to\infty} \int \cdots \int_{D_M} \Psi\left(\sum_{i=1}^{i_0}\left(\frac{x_i}{M}\right)^\alpha\right) dx_1 \cdots dx_{i_0} \\ = \quad & \lim_{M\to\infty} \frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})} \int_{b^\alpha i_0/M^\alpha}^1 k_\lambda(Mu^{\frac{1}{\alpha}}, ||\widetilde{V}_n||_\beta)(Mu^{\frac{1}{\alpha}})^{\lambda_1 - i_0}||\widetilde{V}_n||_\beta^{\lambda_2} u^{\frac{i_0}{\alpha}-1} du \\ \overset{t=\frac{Mu^{1/\alpha}}{||\widetilde{V}_n||_\beta}}{=} \quad & \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \int_{bi_0^{1/\alpha}/||\widetilde{V}_n||_\beta}^\infty k_\lambda(t,1)t^{\lambda_1 - 1} dt. \end{aligned}$$

Hence, in view of (21), we have

$$w(\lambda_1, n) > \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \int_{bi_0^{1/\alpha}/||\widetilde{V}_n||_\beta}^\infty k_\lambda(t,1)t^{\lambda_1-1} dt = K_\alpha(\lambda_1)(1 - \theta_\lambda(n)),$$

and then (20) follows. □

NOTE 2. If there exist constants $a \geqslant 0$ and $\delta > -\lambda_1$, satisfying $\lim_{t\to 0^+} \frac{k_\lambda(t,1)}{t^\delta} = a$, then, there exists a constant $M > 0$, such that $\frac{k_\lambda(t,1)}{t^\delta} \leqslant M$ $(t \in (0, bi_0^{1/\alpha}/||\widetilde{V}_1||_\beta])$, and

$$0 < \theta_\lambda(n) \leqslant \frac{M}{k(\lambda_1)} \int_0^{bi_0^{1/\alpha}/||\widetilde{V}_n||_\beta} t^{\lambda_1+\delta-1} dt = \frac{1}{(\lambda_1+\delta)k(\lambda_1)} \left(\frac{bi_0^{1/\alpha}}{||\widetilde{V}_n||_\beta}\right)^{\lambda_1+\delta},$$

namely, $\theta_\lambda(n) = O\Big(\frac{1}{||\widetilde{V}_n||_\beta^{\lambda_1+\delta}}\Big)$ $(\delta > -\lambda_1)$.

3. Main results

Setting functions

$$\widetilde{\Phi}(m) := \frac{||\widetilde{U}_m||_\alpha^{p(i_0-\lambda_1)-i_0}}{(\prod_{k=1}^{i_0} \mu_m^{(k)})^{p-1}} \quad (m \in \mathbf{N}^{i_0}),$$

$$\widetilde{\Psi}(n) := \frac{||\widetilde{V}_n||_\beta||_\beta^{q(j_0-\lambda_2)-j_0}}{(\prod_{l=1}^{j_0} \upsilon_n^{(l)})^{q-1}} \quad (n \in \mathbf{N}^{j_0}),$$

and the following normed spaces

$$l_{p,\widetilde{\Phi}} := \left\{ a = \{a_m\}; ||a||_{p,\widetilde{\Phi}} := \{\sum_m \widetilde{\Phi}(m)|a_m|^p\}^{\frac{1}{p}} < \infty \right\},$$

$$l_{q,\widetilde{\Psi}} := \left\{ b = \{b_n\}; ||b||_{q,\widetilde{\Psi}} := \{\sum_n \widetilde{\Psi}(n)|b_n|^q\}^{\frac{1}{q}} < \infty \right\},$$

$$l_{p,\widetilde{\Psi}^{1-p}} := \left\{ c = \{c_n\}; ||c||_{p,\widetilde{\Psi}^{1-p}} := \{\sum_n \widetilde{\Psi}^{1-p}(n)|c_n|^p\}^{\frac{1}{p}} < \infty \right\},$$

we have

THEOREM 1. *With regards to the assumptions of Definition 1, if* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda_1 \leqslant i_0$, $\lambda_2 \leqslant j_0$, *then for* $a_m, b_n \geqslant 0$, $a = \{a_m\} \in l_{p,\widetilde{\Phi}}$, $b = \{b_n\} \in l_{q,\widetilde{\Psi}}$, $||a||_{p,\widetilde{\Phi}}$, $||b||_{q,\widetilde{\Psi}} > 0$, *we have the following equivalent inequalities*

$$I := \sum_n \sum_m k_\lambda(||\widetilde{U}_m||_\alpha, ||\widetilde{V}_n||_\beta) a_m b_n < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||a||_{p,\widetilde{\Phi}} ||b||_{q,\widetilde{\Psi}}, \tag{22}$$

$$J := \left\{ \sum_n \frac{\prod_{k=1}^{j_0} \upsilon_n^{(k)}}{||\widetilde{V}_n||_\beta^{j_0 - p\lambda_2}} \left[\sum_m k_\lambda(||\widetilde{U}_m||_\alpha, ||\widetilde{V}_n||_\beta) a_m \right]^p \right\}^{\frac{1}{p}}$$
$$< K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||a||_{p,\widetilde{\Phi}}. \tag{23}$$

where,

$$K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\beta^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} k(\lambda_1). \tag{24}$$

Proof. By Hölder's inequality with weight (cf. [38]), we have

$$I = \sum_n \sum_m k_\lambda(||\widetilde{U}_m||_\alpha, ||\widetilde{V}_n||_\beta) \left[\frac{||\widetilde{U}_m||_\alpha^{\frac{i_0-\lambda_1}{q}}}{||\widetilde{V}_n||_\beta^{\frac{j_0-\lambda_2}{p}}} \frac{(\prod_{l=1}^{j_0} \upsilon_n^{(l)})^{\frac{1}{p}} a_m}{(\prod_{k=1}^{i_0} \mu_m^{(k)})^{\frac{1}{q}}} \right]$$
$$\times \left[\frac{||\widetilde{V}_n||_\beta^{\frac{j_0-\lambda_2}{p}}}{||\widetilde{U}_m||_\alpha^{\frac{i_0-\lambda_1}{q}}} \frac{(\prod_{k=1}^{i_0} \mu_m^{(k)})^{\frac{1}{q}}}{(\prod_{l=1}^{j_0} \upsilon_n^{(l)})^{\frac{1}{p}}} b_n \right]$$
$$\leqslant \left[\sum_m W(\lambda_2, m) \frac{||\widetilde{U}_m||_\alpha^{p(i_0-\lambda_1)-i_0} a_m^p}{(\prod_{k=1}^{i_0} \mu_m^{(k)})^{p-1}} \right]^{\frac{1}{p}} \left[\sum_n w(\lambda_1, n) \frac{||\widetilde{V}_n||_\beta^{q(j_0-\lambda_2)-j_0} b_n^q}{(\prod_{l=1}^{j_0} \upsilon_n^{(l)})} \right]^{\frac{1}{q}}.$$

Then by (17) and (18), we have (22). We set

$$b_n := \frac{\prod_{l=1}^{j_0} \upsilon_n^{(l)}}{||\widetilde{V}_n||_\beta^{j_0-p\lambda_2}} \left[\sum_m k_\lambda(||\widetilde{U}_m||_\alpha, ||\widetilde{V}_n||_\beta) a_m \right]^{p-1}, \quad n \in \mathbf{N}^{j_0}.$$

Then we have $J = ||b||_{q,\widetilde{\Psi}}^{q-1}$. Since the right hand side of (23) is finite, it follows that $J < \infty$. If $J = 0$, then (23) is trivially valid; if $J > 0$, then by (22), we have

$$||b||_{q,\widetilde{\Psi}}^{q} = J^p = I < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||a||_{p,\widetilde{\Phi}} ||b||_{q,\widetilde{\Psi}},$$
$$||b||_{q,\widetilde{\Psi}}^{q-1} = J < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||a||_{p,\widetilde{\Phi}},$$

namely, (23) follows. On the other hand, assuming that (23) is valid, by Hölder's inequality (cf. [38]), we have

$$I = \sum_n \left[\frac{(\prod_{l=1}^{j_0} \upsilon_n^{(l)})^{1/p}}{||\widetilde{V}_n||_\beta^{(j_0/p)-\lambda_2}} \sum_m k_\lambda(||\widetilde{U}_m||_\alpha, ||\widetilde{V}_n||_\beta) a_m \right] \left[\frac{||\widetilde{V}_n||_\beta^{(j_0/p)-\lambda_2}}{(\prod_{l=1}^{j_0} \upsilon_n^{(l)})^{1/p}} b_n \right] \leqslant J ||b||_{q,\widetilde{\Psi}}. \tag{25}$$

Then by (23), we have (22), which is equivalent to (23). □

THEOREM 2. *With regards to the assumptions of Theorem 1, if* $\mu_m^{(k)} \geqslant \mu_{m+1}^{(k)}$ $(m \in \mathbf{N})$, $\upsilon_n^{(l)} \geqslant \upsilon_{n+1}^{(l)}$ $(n \in \mathbf{N})$, $U_\infty^{(k)} = V_\infty^{(l)} = \infty$ $(k = 1, \ldots, i_0,\ l = 1, \ldots, j_0)$, *there exist constants* $a \geqslant 0$ *and* $\delta > -\lambda_1$, *satisfying* $\lim_{t \to 0^+} \frac{k_\lambda(t,1)}{t^\delta} = a$, *then the constant factor* $K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)$ *in (22) and (23) is the best possible.*

Proof. For $\varepsilon > 0$, $\widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p} (< i_0)$, $\widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}$, we set

$$\widetilde{a} = \{\widetilde{a}_m\}, \widetilde{a}_m := ||\widetilde{U}_m||_\alpha^{-i_0 + \widetilde{\lambda}_1} \prod_{k=1}^{i_0} \mu_m^{(k)} \ (m \in \mathbf{N}^{i_0}),$$
$$\widetilde{b} = \{\widetilde{b}_n\}, \widetilde{b}_n := ||\widetilde{V}_n||_\beta^{-j_0 + \widetilde{\lambda}_2 - \varepsilon} \prod_{l=1}^{j_0} \upsilon_n^{(l)} \ (n \in \mathbf{N}^{j_0}).$$

Then by (13) and (14), we obtain

$$\begin{aligned}
||\widetilde{a}||_{p,\widetilde{\Phi}} ||\widetilde{b}||_{q,\widetilde{\Phi}} &= \left[\sum_m \frac{||\widetilde{U}_m||_\alpha^{p(i_0-\lambda_1)-i_0} \widetilde{a}_m^p}{(\prod_{k=1}^{i_0} \mu_m^{(k)})^{p-1}} \right]^{\frac{1}{p}} \left[\sum_n \frac{||\widetilde{V}_n||_\beta^{q(j_0-\lambda_2)-j_0} \widetilde{b}_n^q}{(\prod_{l=1}^{j_0} \upsilon_n^{(l)})^{q-1}} \right]^{\frac{1}{q}} \\
&= \left(\sum_m ||\widetilde{U}_m||_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} \right)^{\frac{1}{p}} \left(\sum_n ||\widetilde{V}_n||_\beta^{-j_0-\varepsilon} \prod_{l=1}^{j_0} \upsilon_n^{(j)} \right)^{\frac{1}{q}} \\
&\leqslant \frac{1}{\varepsilon} \left(\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{c^\varepsilon i_0^{\varepsilon/\alpha} \alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} + \varepsilon O(1) \right)^{\frac{1}{p}} \left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{c^\varepsilon j_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + \varepsilon \widetilde{O}(1) \right)^{\frac{1}{q}}.
\end{aligned}$$

By (20) and (21), we find

$$\begin{aligned}\widetilde{I} &:= \sum_n \left[\sum_m k_\lambda(||\widetilde{U}_m||_\alpha, ||\widetilde{V}_n||_\beta)\widetilde{a}_m\right]\widetilde{b}_n \\ &= \sum_n w(\widetilde{\lambda}_1, n)||\widetilde{V}_n||_\beta^{-j_0-\varepsilon}\prod_{l=1}^{j_0}\upsilon_n^{(l)} \\ &> K_\alpha(\widetilde{\lambda}_1)\sum_n\left(1 - O\left(\frac{1}{||\widetilde{V}_n||_\beta^{\lambda_1+\delta}}\right)\right)||\widetilde{V}_n||_\beta^{-j_0-\varepsilon}\prod_{l=1}^{j_0}\upsilon_n^{(l)} \\ &= K_\alpha(\widetilde{\lambda}_1)\left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\varepsilon c^\varepsilon j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \widetilde{O}(1) - O_1(1)\right).\end{aligned}$$

If there exists a constant $K \leqslant K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$, such that (22) is valid when replacing $K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$ by K, then we have $\varepsilon\widetilde{I} < \varepsilon K||\widetilde{a}||_{p,\widetilde{\Phi}}||\widetilde{b}||_{q,\widetilde{\Phi}}$, namely,

$$\begin{aligned}&K_\alpha\left(\lambda_1 - \frac{\varepsilon}{p}\right)\left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{c^\varepsilon j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \varepsilon\widetilde{O}(1) - \varepsilon O_1(1)\right) \\ &< K\left(\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{c^\varepsilon i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} + \varepsilon O(1)\right)^{\frac{1}{p}}\left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{c^\varepsilon j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \varepsilon\widetilde{O}(1)\right)^{\frac{1}{q}}.\end{aligned}$$

For $\varepsilon \to 0^+$, since $k(\lambda_1 - \frac{\varepsilon}{p}) \to k(\lambda_1)$ (cf. [3]), we find

$$\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\frac{\Gamma^{i_0}(\frac{1}{\alpha})k(\lambda_1)}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \leqslant K\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{p}}\left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{q}},$$

and then $K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1) \leqslant K$. Hence, $K = K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$ is the best possible constant factor of (22). The constant factor in (23) is still the best possible. Otherwise, we would reach a contradiction by (25) that the constant factor in (22) is not the best possible. □

4. Operator expressions and examples

With regards to the assumptions of Theorem 2, in view of

$$\begin{aligned}c_n &:= \frac{\prod_{k=1}^{j_0}\upsilon_n^{(k)}}{||\widetilde{V}_n||_\beta^{j_0-p\lambda_2}}\left[\sum_m k_\lambda(||\widetilde{U}_m||_\alpha, ||\widetilde{V}_n||_\beta)a_m\right]^{p-1}, \quad n \in \mathbf{N}^{j_0} \\ c &= \{c_n\}, ||c||_{p,\widetilde{\Psi}^{1-p}} = J < K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)||a||_{p,\widetilde{\Phi}} < \infty,\end{aligned}$$

we can set the following definition:

DEFINITION 2. Define a multidimensional Hardy-Hilbert-type operator $T: l_{p,\widetilde{\Phi}} \to l_{p,\widetilde{\Psi}^{1-p}}$ as follows: For any $a \in l_{p,\widetilde{\Phi}}$, there exists a unique representation $Ta = c \in l_{p,\widetilde{\Psi}^{1-p}}$, satisfying

$$Ta(n) := \sum_m k_\lambda(||\widetilde{U}_m||_\alpha, ||\widetilde{V}_n||_\beta) a_m \ (n \in \mathbf{N}^{j_0}). \tag{26}$$

For $b \in l_{q,\widetilde{\Psi}}$, we define the following formal inner product of Ta and b as follows:

$$(Ta, b) := \sum_n \left[\sum_m k_\lambda(||\widetilde{U}_m||_\alpha, ||\widetilde{V}_n||_\beta) a_m \right] b_n. \tag{27}$$

Then by Theorem 1, we have the following equivalent inequalities:

$$(Ta, b) < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||a||_{p,\widetilde{\Phi}} ||b||_{q,\widetilde{\Psi}}, \tag{28}$$

$$||Ta||_{p,\widetilde{\Psi}^{1-p}} < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||a||_{p,\widetilde{\Phi}}. \tag{29}$$

It follows that T is bounded with

$$||T|| := \sup_{a(\neq\theta) \in l_{p,\widetilde{\Phi}}} \frac{||Ta||_{p,\widetilde{\Psi}^{1-p}}}{||a||_{p,\widetilde{\Phi}}} \leqslant K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1). \tag{30}$$

Since by Theorem 2, the constant factor $K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)$ in (29) is the best possible, we have

$$||T|| = K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} k(\lambda_1). \tag{31}$$

EXAMPLE 1. Setting $k_\lambda(x,y) = \frac{1}{x^\lambda + y^\lambda}$ $(0 < \lambda \leqslant 1)$, we can find

$$\frac{\partial}{\partial x} k_\lambda(x,y) < 0, \quad \frac{\partial}{\partial y} k_\lambda(x,y) < 0,$$
$$\frac{\partial^2}{\partial x^2} k_\lambda(x,y) > 0, \quad \frac{\partial^2}{\partial y^2} k_\lambda(x,y) > 0 \quad (x, y > 0),$$

and for $0 < \lambda_1 < 1$ $(\leqslant i_0)$, $0 < \lambda_2 < 1$ $(\leqslant j_0)$,

$$k(\lambda_1) = \int_0^\infty \frac{u^{\lambda_1 - 1}}{u^\lambda + 1} du = \frac{1}{\lambda} \int_0^\infty \frac{1}{v+1} v^{\frac{\lambda_1}{\lambda} - 1} dv = \frac{\pi}{\lambda \sin(\frac{\pi \lambda_1}{\lambda})} \in \mathbf{R}_+.$$

We set $\delta = 0 (> -\lambda_1)$ and $a = 1$, in view of (31), we have

$$||T|| = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} \frac{\pi}{\lambda \sin(\frac{\pi \lambda_1}{\lambda})}.$$

EXAMPLE 2. Setting $k_\lambda(x,y) = \frac{\ln(x/y)}{x^\lambda - y^\lambda}$ $(0 < \lambda \leqslant 1)$, we can find (cf. [3], Example 2.2.2)

$$\frac{\partial}{\partial x}k_\lambda(x,y) < 0, \quad \frac{\partial}{\partial y}k_\lambda(x,y) < 0,$$
$$\frac{\partial^2}{\partial x^2}k_\lambda(x,y) > 0, \quad \frac{\partial^2}{\partial y^2}k_\lambda(x,y) > 0 \quad (x,y > 0),$$

and for $0 < \lambda_1 < 1 \ (\leqslant i_0)$, $0 < \lambda_2 < 1 \ (\leqslant j_0)$,

$$k(\lambda_1) = \int_0^\infty \frac{\ln u}{u^\lambda - 1}u^{\lambda_1 - 1}du = \frac{1}{\lambda^2}\int_0^\infty \frac{\ln v}{v-1}v^{\frac{\lambda_1}{\lambda}-1}dv = \left[\frac{\pi}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})}\right]^2 \in \mathbf{R}_+.$$

We set $\delta = 0 (> -\lambda_1)$ and $a = 1$, in view of (31), we have

$$||T|| = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}}\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}}\left[\frac{\pi}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})}\right]^2.$$

EXAMPLE 3. Setting $k_\lambda(x,y) = \frac{1}{(x+y)^\lambda}$ $(\lambda > 0)$, then we find

$$\frac{\partial}{\partial x}k_\lambda(x,y) < 0, \quad \frac{\partial}{\partial y}k_\lambda(x,y) < 0,$$
$$\frac{\partial^2}{\partial x^2}k_\lambda(x,y) > 0, \quad \frac{\partial^2}{\partial y^2}k_\lambda(x,y) > 0 \quad (x,y > 0),$$

and for $0 < \lambda_1 \leqslant i_0$, $0 < \lambda_2 \leqslant j_0$,

$$k(\lambda_1) = \int_0^\infty \frac{u^{\lambda_1-1}}{(u+1)^\lambda}du = B(\lambda_1,\lambda_2) \in \mathbf{R}_+.$$

We set $\delta = 0 (> -\lambda_1)$ and $a = 1$, in view of (31), we have

$$||T|| = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}}\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}}B(\lambda_1,\lambda_2).$$

REMARK 1. For $\widetilde{\mu}_i^{(k)} = 0$ $(k = 1,\ldots,i_0;\ i = 1,\ldots,m)$, $\widetilde{\upsilon}_j^{(l)} = 0$ $(l = 1,\ldots,j_0;$ $j=1,\ldots,n)$, setting

$$\Phi(m) := \frac{||U_m||_\alpha^{p(i_0-\lambda_1)-i_0}}{(\prod_{k=1}^{i_0}\mu_m^{(k)})^{p-1}}, \quad \Psi(n) := \frac{||V_n||_\beta^{q(j_0-\lambda_2)-j_0}}{(\prod_{l=1}^{j_0}\upsilon_n^{(l)})^{q-1}} \quad (m \in \mathbf{N}^{i_0},\ n \in \mathbf{N}^{j_0}),$$

(22) and (23) reduce the following equivalent inequalities with the same best possible constant factor $K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$:

$$\sum_n\sum_m k_\lambda(||U_m||_\alpha,||V_n||_\beta)a_mb_n < K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)||a||_{p,\Phi}||b||_{q,\Psi}, \tag{32}$$

$$\left\{\sum_n \frac{\prod_{k=1}^{j_0}\upsilon_n^{(k)}}{||V_n||_\beta^{j_0-p\lambda_2}}\left[\sum_m k_\lambda(||U_m||_\alpha,||V_n||_\beta)a_m\right]^p\right\}^{\frac{1}{p}} < K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)||a||_{p,\Phi}. \tag{33}$$

Hence, (22) and (23) are more accurate extensions of (32) and (33).

In particular, for $k_\lambda(x,y) = \frac{1}{x^\lambda + y^\lambda}$ $(0 < \lambda_1 \leqslant i_0,\ 0 < \lambda_2 \leqslant j_0)$, $\mu_i = \upsilon_j = 1$ $(i, j \in \mathbf{N})$, (32) reduces to (4); for $k_\lambda(x,y) = \frac{1}{(x^\eta + y^\eta)^{\lambda/\eta}}$ $(0 < \eta \leqslant 1,\ 0 < \lambda_1 \leqslant 1 = i_0,\ 0 < \lambda_2 \leqslant 1 = j_0)$, (32) reduces to (5).

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