

THE MOMENT OF MAXIMUM NORMED SUMS OF RANDOMLY WEIGHTED PAIRWISE NQD SEQUENCES

DONG ZHAO, XIAOQIN LI AND KEHAN WU

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Abstract. This paper investigates the moment of maximum normed sums of randomly weighted pairwise negative quadrant dependent (NQD) random variables. A sufficient condition to the moment of this stochastic process is obtained, which extends the existing results.

1. Introduction

The aim of this paper is to investigate the moment of maximum normed sums of randomly weighted pairwise negative quadrant dependent (NQD) random variables stochastically dominated by a random variable. Now, let's recall some definitions.

DEFINITION 1.1. Two random variables X and Y are said to be NQD if for all real numbers x and y ,

$$P(X < x, Y < y) \leq P(X < x)P(Y < y).$$

A sequence of random variables $\{X_n, n \geq 1\}$ is said to be pairwise NQD if X_i and X_j are NQD for any $i, j \in N^+$ and $i \neq j$.

DEFINITION 1.2. The sequence $\{X_n, n \geq 1\}$ is stochastically dominated by a random variable X if

$$\sup_{n \geq 1} P(|X_n| > t) \leq CP(|X| > t)$$

for some positive constant C and all $t \geq 0$.

The pairwise NQD is introduced by Lehmann [13] and it contains many dependence structures such as negatively associated (NA) sequences, negatively orthant dependent (NOD) sequences, negatively superadditive dependent (NSD) sequences, extended negatively dependent (END) sequences (see [5, 11, 19, 21] for the related definitions). Many authors pay attention to the research of pairwise NQD random variables. For example, Matula [15] established the strong law of large numbers for pairwise NQD

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sequences and the three series theorem for NA sequences; Wang et al. [23] studied the strong stability and Wu [25] investigated the maximal moment inequality for pairwise NQD sequences; Gan and Chen [8] studied some limit theorems and Sung [17, 18] studied the convergence in r -mean for pairwise NQD sequences; Wu and Jiang [28], Ko [12], Wu and Shen [29], Shen et al. [16], Wu and Guo [30] and Yang et al. [32] obtained some convergence results such as the strong law of large numbers, complete convergence and complete moment convergence for non-weighted or weighted pairwise NQD sequences; Yang and Hu [33] investigated the moving average process based on the pairwise NQD sequences, etc. In addition, for more works on the negative dependent sequences, one can refer to [4, 6, 7, 10, 20, 22, 26, 27, 31] and the references therein.

The conception of stochastic domination can be found in Adler and Rosalsky [1] and Adler et al. [2]. This assumption is very weak condition. For example, Hanson and Wright [9] and Wright [22] studied that a bound on tail probabilities for quadratic forms in independent random variables $\{X_n, n \geq 1\}$ by using the following condition: there exist $C > 0$ and $\gamma > 0$ such that $P(|X_n| \geq x) \leq C \int_x^\infty e^{-t^\gamma} dt$ for all $n \geq 1$ and all $x \geq 0$.

Chen and Gan [3] studied the moment of maximum normed sums of ρ -mixing random variables and obtained the results of (2.3) and (2.4) in the following Section 2. Yao and Lin [34] extended the results of Chen and Gan [3] to the case of randomly weighted martingale differences. Recently, Li et al. [14] studied the properties of randomly weighted pairwise NQD sequences and gave an application to the limit theory. So this paper will investigate the moment of maximum normed sums of randomly weighted pairwise NQD random variables and obtain a sufficient moment condition to (2.3) and (2.4). We extend the results of Chen and Gan [3] and Yao and Lin [34] to the case of randomly weighted pairwise NQD random variables. Throughout the paper, $I(A)$ is the indicator function of set A and C, C_1, C_2, \dots , denote some positive constants not depending on n .

LEMMA 1.1. (Lehmann [13]) *If random variables X and Y are NQD, then*

- (i) $EXY \leq EXEY$;
- (ii) $P(X > x, Y > y) \leq P(X > x)P(Y > y), \forall x, y \in \mathbb{R}$;
- (iii) *If f and g are both nondecreasing (or nonincreasing) functions, then $f(X)$ and $g(Y)$ are NQD.*

LEMMA 1.2. (Wu [25]) *Let $\{X_n, n \geq 1\}$ be a pairwise NQD sequence with $EX_n = 0$ and $EX_n^2 < \infty$ for all $n \geq 1$. Then*

$$E\left(\max_{1 \leq k \leq n} \left(\sum_{i=1}^k X_i\right)^2\right) \leq C \log^2 n \sum_{i=1}^n EX_i^2, \quad n \geq 1.$$

LEMMA 1.3. (Adler and Rosalsky [1] and Adler et al. [2]) *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which are stochastically dominated by a random variable X . Then, for any $p > 0$ and $b > 0$, the following two statements hold:*

$$\begin{aligned} E[|X_n|^p I(|X_n| \leq b)] &\leq C_1 \{E[|X|^p I(|X| \leq b)] + b^p P(|X| > b)\}, \\ E[|X_n|^p I(|X_n| > b)] &\leq C_2 E[|X|^p I(|X| > b)]. \end{aligned}$$

So we obtain that $E|X_n|^p \leq C_3 E|X|^p$ for all $n \geq 1$.

2. The main result and its proof

THEOREM 2.1. *Let $0 < r < 2$ and $0 < p < 2$. Assume that $\{X_n, n \geq 1\}$ is a mean zero sequence of pairwise NQD random variables which are stochastically dominated by a random variable X such that*

$$\left\{ \begin{array}{l} \text{for } p < r, \\ \text{for } p = r, \\ \text{for } p > r, \end{array} \right. \left\{ \begin{array}{l} E|X| < \infty, \text{ if } 0 < r < 1, \\ E[|X| \log^3(1 + |X|)] < \infty, \text{ if } r = 1, \\ E[|X|^r \log^2(1 + |X|)] < \infty, \text{ if } r > 1, \\ E|X| < \infty, \text{ if } 0 < r < 1, \\ E[|X| \log^4(1 + |X|)] < \infty, \text{ if } r = 1, \\ E[|X|^r \log^3(1 + |X|)] < \infty, \text{ if } r > 1, \\ E|X| < \infty, \text{ if } 0 < p < 1, \\ E[|X| \log(1 + |X|)] < \infty, \text{ if } p = 1, \\ E|X|^p < \infty, \text{ if } p > 1. \end{array} \right. \quad (2.1)$$

Suppose that $\{A_n, n \geq 1\}$ is an independent sequence of random variables which is also independent of the sequence $\{X_n, n \geq 1\}$. Let

$$\sum_{i=1}^n EA_i^2 = O(n). \quad (2.2)$$

Denote $S_n = \sum_{i=1}^n A_i X_i$, $n \geq 1$. Then it has

$$E\left(\sup_{n \geq 1} \left| \frac{S_n}{n^{1/r}} \right|^p\right) < \infty, \quad (2.3)$$

which implies

$$E\left(\sup_{n \geq 1} \left| \frac{X_n}{n^{1/r}} \right|^p\right) < \infty. \quad (2.4)$$

Proof. Combining Lemma 1.1 with Remark 3.1 of Li et al. [14], for all fixed n , we obtain that $\{A_i^+ X_i, 1 \leq i \leq n\}$, $\{A_i^- X_i, 1 \leq i \leq n\}$ are also pairwise NQD random variables. In view of $A_i X_i = A_i^+ X_i - A_i^- X_i$, without loss of generality, we assume that $A_i \geq 0$, a.s., in the proof. Since $S_n = \sum_{i=1}^n A_i X_i$, $n \geq 1$, it can be argued that

$$\begin{aligned} E\left(\sup_{n \geq 1} \left| \frac{S_n}{n^{1/r}} \right|^p\right) &\leq 2^{p/r} + \int_{2^{p/r}}^{\infty} \sum_{k=1}^{\infty} P\left(\max_{2^{k-1} \leq n < 2^k} \left| \frac{S_n}{n^{1/r}} \right| > t^{1/p}\right) dt \\ &\leq 2^{p/r} + 2^{p/r} \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} P\left(\max_{1 \leq n \leq 2^k} |S_n| > s^{1/p}\right) ds. \end{aligned} \quad (2.5)$$

For $s^{1/p} > 0$ and $1 \leq i \leq n$, denote

$$X_{si} = -s^{1/p}I(X_i < -s^{1/p}) + X_iI(|X_i| \leq s^{1/p}) + s^{1/p}I(X_i > s^{1/p}) \quad \text{and} \quad X_{si}^* = X_i - X_{si}.$$

For $1 \leq i \leq n$, it has

$$A_i X_i = [A_i X_{si} - E(A_i X_{si})] + A_i X_{si}^* + E(A_i X_{si}).$$

So it can be seen that

$$\begin{aligned} & \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} P\left(\max_{1 \leq n \leq 2^k} |S_n| > s^{1/p}\right) ds \\ \leq & \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} P\left(\max_{1 \leq n \leq 2^k} \left| \sum_{i=1}^n A_i X_{si}^* \right| > s^{1/p}/2\right) ds \\ & + \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} P\left(\max_{1 \leq n \leq 2^k} \left| \sum_{i=1}^n [A_i X_{si} - EA_i X_{si}] \right| > s^{1/p}/4\right) ds \\ & + \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} P\left(\max_{1 \leq n \leq 2^k} \left| \sum_{i=1}^n E(A_i X_{si}) \right| > s^{1/p}/4\right) ds \\ =: & H_1 + H_2 + H_3. \end{aligned} \tag{2.6}$$

For any $1 \leq q \leq 2$, we get by Hölder's inequality and (2.2) that

$$\sum_{i=1}^n E|A_i|^q \leq \left(\sum_{i=1}^n EA_i^2\right)^{q/2} \left(\sum_{i=1}^n 1\right)^{1-q/2} = O(n). \tag{2.7}$$

By the fact that $\{A_n, n \geq 1\}$ is independent of $\{X_n, n \geq 1\}$, we obtain by Markov's inequality, Lemma 1.3 and (2.7) with $q = 1$ that

$$\begin{aligned} H_1 & \leq 4 \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} s^{-1/p} \left(\sum_{i=1}^{2^k} E|A_i|E|X_i|I(|X_i| > s^{1/p})\right) ds \\ & \leq C_1 \sum_{k=1}^{\infty} 2^{k-kp/r} \sum_{m=k}^{\infty} \int_{2^{mp/r}}^{2^{(m+1)p/r}} s^{-1/p} E[|X|I(|X| > s^{1/p})] ds \\ & \leq C_2 \sum_{k=1}^{\infty} 2^{k-kp/r} \sum_{m=k}^{\infty} 2^{mp/r-m/r} E[|X|I(|X| > 2^{m/r})] \\ & = C_2 \sum_{m=1}^{\infty} 2^{mp/r-m/r} E[|X|I(|X| > 2^{m/r})] \sum_{k=1}^m 2^{k-kp/r} \\ & \leq \begin{cases} C_3 \sum_{m=1}^{\infty} 2^{m-m/r} E[|X|I(|X| > 2^{m/r})], & \text{if } p < r, \\ C_4 \sum_{m=1}^{\infty} m 2^{m-m/r} E[|X|I(|X| > 2^{m/r})], & \text{if } p = r, \\ C_5 \sum_{m=1}^{\infty} 2^{mp/r-m/r} E[|X|I(|X| > 2^{m/r})], & \text{if } p > r. \end{cases} \end{aligned} \tag{2.8}$$

First, we consider the case of $p < r$. If $0 < r < 1$, then

$$\begin{aligned} \sum_{m=1}^{\infty} 2^{m-m/r} E[|X|I(|X| > 2^{m/r})] &= \sum_{m=1}^{\infty} 2^{m-m/r} \sum_{k=m}^{\infty} E[|X|I(2^{k/r} < |X| \leq 2^{(k+1)/r})] \\ &= \sum_{k=1}^{\infty} E[|X|I(2^{k/r} < |X| \leq 2^{(k+1)/r})] \sum_{m=1}^k 2^{m(1-1/r)} \\ &\leq C_1 \sum_{k=1}^{\infty} E[|X|I(2^{k/r} < |X| \leq 2^{(k+1)/r})] \\ &\leq C_1 E|X|. \end{aligned}$$

If $r = 1$, then

$$\begin{aligned} \sum_{m=1}^{\infty} 2^{m-m/r} E[|X|I(|X| > 2^{m/r})] &= \sum_{m=1}^{\infty} E[|X|I(|X| > 2^m)] \\ &= \sum_{m=1}^{\infty} \sum_{k=m}^{\infty} E[|X|I(2^k < |X| \leq 2^{k+1})] \\ &= \sum_{k=1}^{\infty} k E[|X|I(2^k < |X| \leq 2^{k+1})] \\ &\leq C_1 \sum_{k=1}^{\infty} E[|X| \log(1 + |X|)I(2^k < |X| \leq 2^{k+1})] \\ &\leq C_1 E[|X| \log(1 + |X|)]. \end{aligned}$$

If $r > 1$, then

$$\begin{aligned} \sum_{m=1}^{\infty} 2^{m-m/r} E[|X|I(|X| > 2^{m/r})] &= \sum_{k=1}^{\infty} E[|X|I(2^{k/r} < |X| \leq 2^{(k+1)/r})] \sum_{m=1}^k 2^{m-m/r} \\ &\leq C_1 \sum_{k=1}^{\infty} E[|X|^r I(2^{k/r} < |X| \leq 2^{(k+1)/r})] \leq C_1 E|X|^r. \end{aligned}$$

Second, we consider the case $p = r$. Similarly, if $0 < r < 1$, then

$$\begin{aligned} \sum_{m=1}^{\infty} m 2^{m-m/r} E[|X|I(|X| > 2^{m/r})] &= \sum_{k=1}^{\infty} E[|X|I(2^{k/r} < |X| \leq 2^{(k+1)/r})] \sum_{m=1}^k m 2^{m(1-1/r)} \\ &\leq C_1 \sum_{k=1}^{\infty} E[|X|I(2^{k/r} < |X| \leq 2^{(k+1)/r})] \leq C_2 E|X|. \end{aligned}$$

If $r = 1$, then it has

$$\begin{aligned} \sum_{m=1}^{\infty} m 2^{m-m/r} E[|X|I(|X| > 2^{m/r})] &= \sum_{k=1}^{\infty} E[|X|I(2^k < |X| \leq 2^{k+1})] \sum_{m=1}^k m \\ &\leq C_2 \sum_{k=1}^{\infty} E[|X| \log^2(1 + |X|)I(2^k < |X| \leq 2^{k+1})] \\ &\leq C_2 E[|X| \log^2(1 + |X|)]. \end{aligned}$$

If $r > 1$, then

$$\begin{aligned} \sum_{m=1}^{\infty} m2^{m-m/r} E[|X|I(|X| > 2^{m/r})] &\leq \sum_{k=1}^{\infty} E[|X|I(2^{k/r} < |X| \leq 2^{(k+1)/r})] k \sum_{m=1}^k 2^{m-m/r} \\ &\leq C_1 \sum_{k=1}^{\infty} k2^{k-k/r} E[|X|I(2^{k/r} < |X| \leq 2^{(k+1)/r})] \\ &\leq C_2 E[|X|^r \log(1 + |X|)]. \end{aligned}$$

Third, it is time to consider the case $p > r$. If $0 < p < 1$, then

$$\begin{aligned} \sum_{m=1}^{\infty} 2^{mp/r-m/r} E[|X|I(|X| > 2^{m/r})] &= \sum_{k=1}^{\infty} E[|X|I(2^{k/r} < |X| \leq 2^{(k+1)/r})] \sum_{m=1}^k 2^{m(p-1)/r} \\ &\leq C_1 \sum_{k=1}^{\infty} E[|X|I(2^{k/r} < |X| \leq 2^{(k+1)/r})] \\ &\leq C_1 E|X|. \end{aligned}$$

If $p = 1$, we obtain

$$\begin{aligned} \sum_{m=1}^{\infty} 2^{mp/r-m/r} E[|X|I(|X| > 2^{m/r})] &= \sum_{m=1}^{\infty} \sum_{k=m}^{\infty} E[|X|I(2^{k/r} < |X| \leq 2^{(k+1)/r})] \\ &= \sum_{k=1}^{\infty} kE[|X|I(2^{k/r} < |X| \leq 2^{(k+1)/r})] \\ &\leq C_1 E|X| \log(1 + |X|). \end{aligned}$$

For $p > 1$, it has

$$\begin{aligned} \sum_{m=1}^{\infty} 2^{mp/r-m/r} E[|X|I(|X| > 2^{m/r})] &= \sum_{m=1}^{\infty} 2^{m(p-1)/r} \sum_{k=m}^{\infty} E[|X|I(2^{k/r} < |X| \leq 2^{(k+1)/r})] \\ &\leq C_1 \sum_{k=1}^{\infty} 2^{k(p-1)/r} E[|X|I(2^{k/r} < |X| \leq 2^{(k+1)/r})] \\ &\leq C_1 E|X|^p. \end{aligned}$$

Together with (2.8), we obtain that

$$H_1 \leq \begin{cases} C_1 \sum_{m=1}^{\infty} 2^{m-m/r} E[|X|I(|X| > 2^{m/r})], & \text{if } p < r, \\ C_2 \sum_{m=1}^{\infty} m2^{m-m/r} E[|X|I(|X| > 2^{m/r})], & \text{if } p = r, \\ C_3 \sum_{m=1}^{\infty} 2^{mp/r-m/r} E[|X|I(|X| > 2^{m/r})], & \text{if } p > r, \end{cases}$$

$$\leq \begin{cases} \text{for } p < r, & \begin{cases} C_4 E|X| < \infty, & \text{if } 0 < r < 1, \\ C_5 E[|X| \log(1 + |X|)] < \infty, & \text{if } r = 1, \\ C_6 E|X|^r < \infty, & \text{if } r > 1, \end{cases} \\ \text{for } p = r, & \begin{cases} C_7 E|X| < \infty, & \text{if } 0 < r < 1, \\ C_8 E[|X| \log^2(1 + |X|)] < \infty, & \text{if } r = 1, \\ C_9 E[|X|^r \log(1 + |X|)] < \infty, & \text{if } r > 1, \end{cases} \\ \text{for } p > r, & \begin{cases} C_{10} E|X| < \infty, & \text{if } 0 < p < 1, \\ C_{11} E[|X| \log(1 + |X|)] < \infty, & \text{if } p = 1, \\ C_{12} E|X|^p < \infty, & \text{if } p > 1. \end{cases} \end{cases} \quad (2.9)$$

From Lemma 1.1, it follows that $\{X_{si}, 1 \leq i \leq n\}$ are pairwise NQD random variables. Moreover, by Remark 3.1 of Li et al. [14], we obtain that $\{A_i X_{si} - E(A_i X_{si}), 1 \leq i \leq n\}$ are also pairwise NQD random variables with mean zero. Therefore, by Markov's inequality, C_r inequality, (2.2) and Lemmas 1.2 and 1.3, we establish that

$$\begin{aligned} H_2 &\leq C_1 \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} s^{-2/p} E \left\{ \max_{1 \leq n \leq 2^k} \left(\sum_{i=1}^n [A_i X_{si} - E(A_i X_{si})] \right)^2 \right\} ds \\ &\leq C_2 \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} s^{-2/p} k^2 \left(\sum_{i=1}^{2^k} E A_i^2 E X_{si}^2 \right) ds \\ &\leq C_3 \sum_{k=1}^{\infty} k^2 2^{-kp/r+k} \int_{2^{kp/r}}^{\infty} s^{-2/p} E[X^2 I(|X| \leq s^{1/p})] ds \\ &\quad + C_4 \sum_{k=1}^{\infty} k^2 2^{-kp/r+k} \int_{2^{kp/r}}^{\infty} P(|X| > s^{1/p}) ds \\ &=: C_3 H_{21} + C_4 H_{22}. \end{aligned} \quad (2.10)$$

For H_{21} , it can be found that

$$\begin{aligned} H_{21} &= \sum_{k=1}^{\infty} k^2 2^{-kp/r+k} \sum_{m=k}^{\infty} \int_{2^{mp/r}}^{2^{(m+1)p/r}} s^{-2/p} E[X^2 I(|X| \leq s^{1/p})] ds \\ &\leq \sum_{k=1}^{\infty} k^2 2^{-kp/r+k} \sum_{m=k}^{\infty} 2^{mp/r-2m/r} E[X^2 I(|X| \leq 2^{(m+1)/r})] \\ &= \sum_{m=1}^{\infty} 2^{m(p-2)/r} E[X^2 I(|X| \leq 2^{(m+1)/r})] \sum_{k=1}^m k^2 2^{k(1-p/r)}. \end{aligned}$$

For all $\alpha > 0$ and positive integer $k \geq 1$, it can be argued that

$$\sum_{n=m}^{\infty} \frac{n^k}{2^{\alpha n}} \leq C \frac{m^k}{2^{\alpha m}}, \quad (2.11)$$

where $m \geq 1$ and C is a positive constant not depending on m .

Now, we consider the case $p < r$. By $r < 2$ and (2.11), it follows

$$\begin{aligned}
& \sum_{m=1}^{\infty} 2^{m(p-2)/r} E[X^2 I(|X| \leq 2^{(m+1)/r})] \sum_{k=1}^m k^2 2^{k(1-p/r)} \\
& \leq C_1 \sum_{m=1}^{\infty} m^2 2^{m(r-2)/r} E[X^2 I(|X| \leq 2^{(m+1)/r})] \\
& = C_1 \sum_{m=1}^{\infty} m^2 2^{m(r-2)/r} E[X^2 I(|X| \leq 2^{1/r})] \\
& \quad + C_1 \sum_{m=1}^{\infty} m^2 2^{m(r-2)/r} \sum_{i=1}^m E[X^2 I(2^{i/r} < |X| \leq 2^{(i+1)/r})] \\
& \leq C_2 + C_1 \sum_{i=1}^{\infty} E[X^2 I(2^{i/r} < |X| \leq 2^{(i+1)/r})] \sum_{m=i}^{\infty} m^2 2^{m(r-2)/r} \\
& \leq C_2 + C_3 2^{(2-r)/r} \sum_{i=1}^{\infty} i^2 2^{(i+1)(r-2)/r} E[X^2 I(2^{i/r} < |X| \leq 2^{(i+1)/r})] \\
& \leq C_4 + C_5 E[|X|^r \log^2(1 + |X|)].
\end{aligned}$$

Next, we consider the $p = r$. By $r < 2$ and (2.11), we obtain that

$$\begin{aligned}
& \sum_{m=1}^{\infty} 2^{m(p-2)/r} E[X^2 I(|X| \leq 2^{(m+1)/r})] \sum_{k=1}^m k^2 2^{k(1-p/r)} \\
& \leq C_1 \sum_{m=1}^{\infty} m^3 2^{m(r-2)/r} E[X^2 I(|X| \leq 2^{(m+1)/r})] \\
& = C_1 \sum_{m=1}^{\infty} m^3 2^{m(r-2)/r} E[X^2 I(|X| \leq 2^{1/r})] \\
& \quad + C_1 \sum_{m=1}^{\infty} m^3 2^{m(r-2)/r} \sum_{i=1}^m E[X^2 I(2^{i/r} < |X| \leq 2^{(i+1)/r})] \\
& \leq C_2 + C_1 \sum_{i=1}^{\infty} E[X^2 I(2^{i/r} < |X| \leq 2^{(i+1)/r})] \sum_{m=i}^{\infty} m^3 2^{m(r-2)/r} \\
& \leq C_2 + C_3 2^{(2-r)/r} \sum_{i=1}^{\infty} i^3 2^{(i+1)(r-2)/r} E[X^2 I(2^{i/r} < |X| \leq 2^{(i+1)/r})] \\
& \leq C_4 + C_5 E[|X|^r \log^3(1 + |X|)].
\end{aligned}$$

Moreover, for the case $p > r$, we check by $p < 2$ that

$$\sum_{m=1}^{\infty} 2^{m(p-2)/r} E[|X|^2 I(|X| \leq 2^{(m+1)/r})] \leq CE|X|^p.$$

Consequently, it follows

$$\begin{aligned}
 H_{21} &\leq \begin{cases} C_1 \sum_{m=1}^{\infty} m^2 2^{m(r-2)/r} E[X^2 I(|X| \leq 2^{(m+1)/r})], & \text{if } p < r, \\ C_2 \sum_{m=1}^{\infty} m^3 2^{m(r-2)/r} E[X^2 I(|X| \leq 2^{(m+1)/r})], & \text{if } p = r, \\ C_3 \sum_{m=1}^{\infty} 2^{m(p-2)/r} E[X^2 I(|X| \leq 2^{(m+1)/r})], & \text{if } p > r, \end{cases} \\
 &\leq \begin{cases} C_4 + C_5 E[|X|^r \log^2(1 + |X|)] < \infty, & \text{if } p < r, \\ C_6 + C_7 E[|X|^r \log^3(1 + |X|)] < \infty, & \text{if } p = r, \\ C_8 E|X|^p < \infty, & \text{if } p > r. \end{cases} \tag{2.12}
 \end{aligned}$$

In addition, similar to the proofs of (2.8) and (2.9), we get that

$$\begin{aligned}
 H_{22} &\leq \sum_{k=1}^{\infty} k^2 2^{k-kp/r} \int_{2^{kp/r}}^{\infty} s^{-1/p} E[|X| I(|X| > s^{1/p})] ds \\
 &\leq C_2 \sum_{m=1}^{\infty} 2^{mp/r-m/r} E[|X| I(|X| > 2^{m/r})] \sum_{k=1}^m k^2 2^{k-kp/r} \\
 &\leq \begin{cases} C_3 \sum_{m=1}^{\infty} m^2 2^{m-m/r} E[|X| I(|X| > 2^{m/r})], & \text{if } p < r, \\ C_4 \sum_{m=1}^{\infty} m^3 2^{m-m/r} E[|X| I(|X| > 2^{m/r})], & \text{if } p = r, \\ C_5 \sum_{m=1}^{\infty} 2^{mp/r-m/r} E[|X| I(|X| > 2^{m/r})], & \text{if } p > r. \end{cases} \tag{2.13}
 \end{aligned}$$

Similarly, we consider the case $p < r$. If $0 < r < 1$, then

$$\begin{aligned}
 \sum_{m=1}^{\infty} m^2 2^{m-m/r} E[|X| I(|X| > 2^{m/r})] &= \sum_{k=1}^{\infty} E[|X| I(2^{k/r} < |X| \leq 2^{(k+1)/r})] \sum_{m=1}^k m^2 2^{m(1-1/r)} \\
 &\leq C_1 \sum_{k=1}^{\infty} E[|X| I(2^{k/r} < |X| \leq 2^{(k+1)/r})] \\
 &\leq C_1 E|X|.
 \end{aligned}$$

If $r = 1$, then

$$\begin{aligned}
 \sum_{m=1}^{\infty} m^2 2^{m-m/r} E[|X| I(|X| > 2^{m/r})] &= \sum_{k=1}^{\infty} E[|X| I(2^k < |X| \leq 2^{k+1})] \sum_{m=1}^k m^2 \\
 &\leq C_1 \sum_{k=1}^{\infty} E[|X| \log^3(1 + |X|) I(2^k < |X| \leq 2^{k+1})] \\
 &\leq C_1 E[|X| \log^3(1 + |X|)].
 \end{aligned}$$

If $r > 1$, then

$$\begin{aligned} \sum_{m=1}^{\infty} m^2 2^{m-m/r} E[|X|I(|X| > 2^{m/r})] &= \sum_{k=1}^{\infty} E[|X|I(2^{k/r} < |X| \leq 2^{(k+1)/r})] \sum_{m=1}^k m^2 2^{m-m/r} \\ &\leq C_1 \sum_{k=1}^{\infty} E[|X|^r \log^2(1 + |X|)I(2^{k/r} < |X| \leq 2^{(k+1)/r})] \\ &\leq C_1 E[|X|^r \log^2(1 + |X|)]. \end{aligned}$$

Second, we consider the case $p = r$. If $0 < r < 1$, then

$$\begin{aligned} \sum_{m=1}^{\infty} m^3 2^{m-m/r} E[|X|I(|X| > 2^{m/r})] &= \sum_{k=1}^{\infty} E[|X|I(2^{k/r} < |X| \leq 2^{(k+1)/r})] \sum_{m=1}^k m^3 2^{m(1-1/r)} \\ &\leq C_1 \sum_{k=1}^{\infty} E[|X|I(2^{k/r} < |X| \leq 2^{(k+1)/r})] \\ &\leq C_2 E|X|. \end{aligned}$$

If $r = 1$, then

$$\begin{aligned} \sum_{m=1}^{\infty} m^3 2^{m-m/r} E[|X|I(|X| > 2^{m/r})] &= \sum_{k=1}^{\infty} E[|X|I(2^k < |X| \leq 2^{k+1})] \sum_{m=1}^k m^3 \\ &\leq C_2 \sum_{k=1}^{\infty} E[|X| \log^4(1 + |X|)I(2^k < |X| \leq 2^{k+1})] \\ &\leq C_2 E[|X| \log^4(1 + |X|)]. \end{aligned}$$

If $r > 1$, it follows

$$\begin{aligned} \sum_{m=1}^{\infty} m^3 2^{m-m/r} E[|X|I(|X| > 2^{m/r})] &\leq \sum_{k=1}^{\infty} E[|X|I(2^{k/r} < |X| \leq 2^{(k+1)/r})] k^3 \sum_{m=1}^k 2^{m-m/r} \\ &\leq C_1 \sum_{k=1}^{\infty} k^3 2^{k-k/r} E[|X|I(2^{k/r} < |X| \leq 2^{(k+1)/r})] \\ &\leq C_2 E[|X|^r \log^3(1 + |X|)]. \end{aligned}$$

Third, we consider the case $p > r$. By (2.9), (2.13) and the inequalities above, it can be seen that

$$H_{22} \leq \begin{cases} C_1 \sum_{m=1}^{\infty} m^2 2^{m-m/r} E[|X|I(|X| > 2^{m/r})], & \text{if } p < r, \\ C_2 \sum_{m=1}^{\infty} m^3 2^{m-m/r} E[|X|I(|X| > 2^{m/r})], & \text{if } p = r, \\ C_3 \sum_{m=1}^{\infty} 2^{mp/r-m/r} E[|X|I(|X| > 2^{m/r})], & \text{if } p > r. \end{cases}$$

$$\leq \left\{ \begin{array}{l} \text{for } p < r, \\ \text{for } p = r, \\ \text{for } p > r, \end{array} \right\} \left\{ \begin{array}{l} C_4 E|X| < \infty, \text{ if } 0 < r < 1, \\ C_5 E[|X| \log^3(1 + |X|)] < \infty, \text{ if } r = 1, \\ C_6 E[|X|^r \log^2(1 + |X|)] < \infty, \text{ if } r > 1, \\ C_7 E|X| < \infty, \text{ if } 0 < r < 1, \\ C_8 E[|X| \log^4(1 + |X|)] < \infty, \text{ if } r = 1, \\ C_9 E[|X|^r \log^3(1 + |X|)] < \infty, \text{ if } r > 1, \\ C_{10} E|X| < \infty, \text{ if } 0 < p < 1, \\ C_{11} E[|X| \log(1 + |X|)] < \infty, \text{ if } p = 1, \\ C_{12} E|X|^p < \infty, \text{ if } p > 1. \end{array} \right. \quad (2.14)$$

Furthermore, by the independence and $EX_i = 0$, it has $E(A_i X_i) = 0, 1 \leq i \leq n$. So it follows

$$\begin{aligned} & \max_{1 \leq n \leq 2^k} \left| \sum_{i=1}^n E(A_i X_{si}) \right| \\ &= \max_{1 \leq n \leq 2^k} \left| \sum_{i=1}^n E\{A_i[-s^{1/p}I(X_i < -s^{1/p}) - X_i I(|X_i| > s^{1/p}) + s^{1/p}I(X_i > s^{1/p})]\} \right| \\ &\leq 2 \sum_{i=1}^{2^k} E|A_i| E[|X_i| I(|X_i| > s^{1/p})]. \end{aligned}$$

Obviously, by the proofs of (2.8), (2.9), we can establish that

$$\begin{aligned} H_3 &\leq C_1 \sum_{k=1}^{\infty} 2^{k-kp/r} \int_{2^{kp/r}}^{\infty} s^{-1/p} E[|X| I(|X| > s^{1/p})] ds \\ &\leq \left\{ \begin{array}{l} \text{for } p < r, \\ \text{for } p = r, \\ \text{for } p > r, \end{array} \right\} \left\{ \begin{array}{l} C_1 E|X| < \infty, \text{ if } 0 < r < 1, \\ C_2 E[|X| \log(1 + |X|)] < \infty, \text{ if } r = 1, \\ C_3 E|X|^r < \infty, \text{ if } r > 1, \\ C_4 E|X| < \infty, \text{ if } 0 < r < 1, \\ C_5 E[|X| \log^2(1 + |X|)] < \infty, \text{ if } r = 1, \\ C_6 E[|X|^r \log(1 + |X|)] < \infty, \text{ if } r > 1, \\ C_7 E|X| < \infty, \text{ if } 0 < p < 1, \\ C_8 E[|X| \log(1 + |X|)] < \infty, \text{ if } p = 1, \\ C_9 E|X|^p < \infty, \text{ if } p > 1. \end{array} \right. \quad (2.15) \end{aligned}$$

Therefore, (2.3) follows from (2.1), (2.5), (2.6), (2.9)–(2.12), (2.14) and (2.15) immediately. Applying (2.3), we establish (2.4) finally. \square

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Dong Zhao
Department of Basic Science
Huaibei Vocational and Technical College
Huaibei, 235000, China
e-mail: zhaodong1973@aliyun.com

Xiaoqin Li
School of Mathematical Sciences
Anhui University
Hefei 230601, China
e-mail: lixiaoqin1983@163.com

Kehan Wu
Training Center of Anhui Provincial Electric Power Company
Hefei, 230000, China
e-mail: wukehan@yeah.net