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INEQUALITIES VIA HARMONIC CONVEX FUNCTIONS: CONFORMABLE FRACTIONAL CALCULUS APPROACH

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Abstract. The main objective of this paper is to establish some conformable fractional estimates of Hermite-Hadamard type integral inequalities via harmonic convex functions.

1. Introduction and preliminaries

A set $\mathscr{C} \subset \mathbb{R}$ is said to be convex, if

$$(1-\lambda)u+\lambda v \in \mathscr{C}, \quad \forall u,v \in \mathscr{C}, \lambda \in [0,1].$$

A function $f : \mathscr{C} \to \mathbb{R}$ is said to be convex, if

$$f((1-\lambda)u+\lambda v) \leqslant (1-\lambda)f(u)+tf(v), \quad \forall u,v \in \mathscr{C}, \lambda \in [0,1].$$

In recent years we have noticed that theory of convexity developed rapidly. Consequently several new generalizations of convex functions have been proposed in the literature, for example, see [2].

Harmonic convex sets are defined as:

DEFINITION 1.1. ([17]) A set $\mathscr{H} \subset \mathbb{R}_+ \setminus \{0\}$ is said to be harmonic convex, if

$$\frac{uv}{\lambda u+(1-\lambda)v} \in \mathscr{H} \quad \forall u,v \in \mathscr{H}, \lambda \in [0,1].$$

Recently Iscan [5] introduced the notion of harmonic convex functions.

DEFINITION 1.2. ([5]) A function $f : \mathscr{H} \to \mathbb{R}$ is said to be harmonic convex function, if

$$f\left(\frac{uv}{\lambda u+(1-\lambda)v}\right) \leqslant (1-\lambda)f(u)+\lambda f(v), \quad \forall u,v \in K, \lambda \in [0,1].$$

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Using the inequality $HM \leqslant AM$, it is known that [5] that $f:(0,\infty)\to\mathbb{R}$, $f(u)=u$, then f is a harmonic convex function. One of the main reason behind the development of theory of convexity is its close linkage with theory of inequalities. Numerous inequalities have been obtained via convex functions, for example see [3, 4, 14]. The famous result of Hermite and Hadamard for convex functions which provides us a necessary and sufficient condition for a function to be convex is Hermite-Hadamard's inequality. This inequality reads as follows:

THEOREM 1.1. *Let $f:\mathscr{I}=[a,b]\to\mathbb{R}$ be a convex function, then*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(u)\mathrm{d}u \leqslant \frac{f(a)+f(b)}{2}.$$

Iscan [5] gave a new refinement of Hermite-Hadamard's inequality via harmonic convex functions.

THEOREM 1.2. ([5]) *Let $f:\mathscr{I}=[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a harmonic convex function, then*

$$f\left(\frac{2ab}{a+b}\right) \leqslant \frac{ab}{b-a}\int_a^b f(u)\mathrm{d}u \leqslant \frac{f(a)+f(b)}{2}.$$

For some recent studies on harmonic convex functions and related inequalities, see [9, 11, 12]. The great impact of fractional calculus in pure and applied sciences can not be denied. Resultantly many researchers used the techniques of fractional calculus intensively to get the new refinements of the previously known results. Motivated by this Sarikaya et al. [13] were the first to gave a new extension of Hermite-Hadamard's inequality. Then Iscan et al. [6] obtained the fractional version of Hermite-Hadamard's inequality via harmonic convex functions. In [8] authors have given a new well-behaved fractional derivative called as conformable fractional derivative. In [1] author has defined the left and right conformable fractional integrals of order $\alpha > 0$. Very recently Set et al. [15] have given a new generalization of Hermite-Hadamard's inequality via conformable fractional integrals. For some more details and studies on inequalities via conformable fractional integrals, see [15, 16].

The motivation of this article is to obtain some conformable fractional estimates of Hermite-Hadamard type inequalities via harmonic convex functions. Before we move towards the main results of this article let us have a brief review of the previously known concepts and results. These preliminaries will be helpful in obtaining the main results.

The following preliminary concepts and results will be helpful in obtaining our main results.

The well-known Gamma and Beta functions are defined respectively as:

$$\Gamma(\alpha) = \int_0^\infty e^{-\lambda}\lambda^{\alpha-1}d\lambda,$$

$$\mathscr{B}(a,b)=\int_0^1 \lambda^{a-1}(1-\lambda)^{b-1}\mathrm{d}\lambda=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},\quad a,b>0.$$

The incomplete Beta function is defined as:

$$\mathscr{B}_u(a,b)=\int_0^u \lambda^{a-1}(1-\lambda)^{b-1}\mathrm{d}\lambda,\quad u\in[0,1].$$

Note that for $u=1$ the incomplete Beta functions reduce to classical Beta function.

The integral form of the hypergeometric function is given as

$${}_2F_1(u,v;c;z)=\frac{1}{\mathscr{B}(v,c-v)}\int_0^1 \lambda^{v-1}(1-\lambda)^{c-v-1}(1-z\lambda)^{-u}\mathrm{d}\lambda$$

for $|z|<1$, $\Re(c)>\Re(v)>0$. For more information, see [7].

Riemann-Liouville integrals are defined as follows:

DEFINITION 1.3. ([10]) Let $f\in L_1[a,b]$. Then Riemann-Liouville integrals $J_{a^+}^{\alpha}f$ and $J_{b^-}^{\alpha}f$ of order $\alpha>0$ with $a\geqslant 0$ are defined by

$$J_{a^+}^{\alpha}f(u)=\frac{1}{\Gamma(\alpha)}\int_a^u (u-\lambda)^{\alpha-1}f(\lambda)\mathrm{d}\lambda,\quad u>a,$$

and

$$J_{b^-}^{\alpha}f(u)=\frac{1}{\Gamma(\alpha)}\int_u^b (\lambda-u)^{\alpha-1}f(\lambda)\mathrm{d}\lambda,\quad u<b,$$

Abdeljawad [1] has given the definition of left and right conformable fractional integrals of any order $\alpha>0$ as:

DEFINITION 1.4. ([1]) Let $\alpha\in(n,n+1]$ and $\beta=\alpha-n$. Then the left and right conformable fractional integrals starting at a of order α as defined by

$$J_{\alpha}^{a}f(\lambda)=\frac{1}{n!}\int_a^{\lambda}(\lambda-u)^n(u-a)^{\beta-1}f(u)\mathrm{d}u,$$

and

$$J_{\alpha}^{b}f(\lambda)=\frac{1}{n!}\int_{\lambda}^{b}(u-\lambda)^n(b-u)^{\beta-1}f(u)\mathrm{d}u.$$

Note that if $\alpha=n+1$ then $\beta=1$ where $n=0,1,2,\ldots$.

Also

$$\mathscr{B}(a,b)=\mathscr{B}_{\lambda}(a,b)+\mathscr{B}_{1-\lambda}(a,b).$$

$$\mathscr{B}_u(a+1,b)=\frac{aB_u(a,b)-(\frac{1}{2})^{a+b}}{a+b};$$

$$\mathscr{B}_u(a,b+1)=\frac{bB_u(a,b)-(\frac{1}{2})^{a+b}}{a+b}.$$

2. Results and discussions

In this section, we derive our main results.

THEOREM 2.1. *Let* $f : \mathscr{I} = [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ *be a harmonic convex function such that* $f \in L_1[a,b]$, *then*

$$f\left(\frac{2ab}{a+b}\right) \leqslant \frac{2\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J_{\alpha}^{\frac{1}{a}}(f\circ g)\left(\frac{1}{b}\right)+J_{\alpha}^{\frac{1}{b}}(f\circ g)\left(\frac{1}{a}\right)\right] \leqslant \frac{f(a)+f(b)}{2}.$$

Proof. Since it is known that f is harmonic convex function, then, for $u,v \in [a,b]$, we have

$$f\left(\frac{2uv}{u+v}\right) \leqslant \frac{f(u)+f(v)}{2}.$$

Let $u = \frac{ab}{\lambda a+(1-\lambda)b}$ and $v = \frac{ab}{(1-\lambda)a+\lambda b}$, then

$$2f\left(\frac{2ab}{a+b}\right) \leqslant f\left(\frac{ab}{\lambda a+(1-\lambda)b}\right)+f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right).$$

Multiplying both sides of above inequality with $\frac{1}{n!}\lambda^n(1-\lambda)^{\alpha-n-1}$ and then integrating it with respect to λ on $[0,1]$, we have

$$\begin{aligned}
&\frac{2}{n!}f\left(\frac{2ab}{a+b}\right)\int_0^1 \lambda^n(1-\lambda)^{\alpha-n-1}\mathrm{d}\lambda \\
&\leqslant \frac{1}{n!}\int_0^1 \lambda^n(1-\lambda)^{\alpha-n-1} f\left(\frac{ab}{\lambda a+(1-\lambda)b}\right)\mathrm{d}\lambda \\
&\qquad + \frac{1}{n!}\int_0^1 \lambda^n(1-\lambda)^{\alpha-n-1} f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)\mathrm{d}\lambda \\
&= I_1 + I_2.
\end{aligned} \tag{2.1}$$

Now using the change of variable technique, we have

$$I_1 = \frac{1}{n!}\int_0^1 \lambda^n(1-\lambda)^{\alpha-n-1} f\left(\frac{ab}{\lambda a+(1-\lambda)b}\right)\mathrm{d}\lambda = \left(\frac{ab}{b-a}\right)^{\alpha} J_{\alpha}^{\frac{1}{b}}(f\circ g)\left(\frac{1}{a}\right), \tag{2.2}$$

where $g(u) = \frac{1}{u}$.

Also

$$I_2 = \frac{1}{n!}\int_0^1 \lambda^n(1-\lambda)^{\alpha-n-1} f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)\mathrm{d}\lambda = \left(\frac{ab}{b-a}\right)^{\alpha} J_{\alpha}^{\frac{1}{a}}(f\circ g)\left(\frac{1}{b}\right), \tag{2.3}$$

where $g(u) = \frac{1}{u}$.

Utilizing (2.2) and (2.3) in (2.1), we get

$$\frac{2\Gamma(\alpha-n)}{\Gamma(\alpha+1)} f\left(\frac{2ab}{a+b}\right) \leqslant \left(\frac{ab}{b-a}\right)^{\alpha} \left[J_{\alpha}^{\frac{1}{a}}(f\circ g)\left(\frac{1}{b}\right) + J_{\alpha}^{\frac{1}{b}}(f\circ g)\left(\frac{1}{a}\right)\right]. \tag{2.4}$$

Now we prove the other side of the inequality. Since it is known that f is harmonic convex function, then

$$f\left(\frac{ab}{\lambda a+(1-\lambda)b}\right) + f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right) \leqslant f(a)+f(b).$$

Multiplying both sides of above inequality with $\frac{1}{n!}\lambda^n(1-\lambda)^{\alpha-n-1}$ and then integrating it with respect to λ on $[0,1]$, we have

$$\left(\frac{ab}{b-a}\right)^{\alpha} \left[J_{\alpha}^{\frac{1}{a}}(f\circ g)\left(\frac{1}{b}\right) + J_{\alpha}^{\frac{1}{b}}(f\circ g)\left(\frac{1}{a}\right)\right] \leqslant \{f(a)+f(b)\}\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)}. \tag{2.5}$$

On summation of inequalities (2.4) and (2.5) and suitable rearrangement completes the proof. □

We now derive a new conformable fractional integral identity which will be helpful in obtaining our next results.

LEMMA 2.1. *Let $f : \mathscr{I} = [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on $\mathscr{I}^{\circ}$. If $f' \in L[a,b]$, then, the following equality holds:*

$$\Delta_f(a,b;\alpha;\mathscr{B};J) = \frac{ab(b-a)}{2}\int_0^1 \left\{\frac{\mathscr{B}_{\lambda}(n+1,\alpha-n) - \mathscr{B}_{1-\lambda}(n+1,\alpha-n)}{A_{\lambda}^2}\right\} f'\left(\frac{ab}{A_{\lambda}}\right) \mathrm{d}\lambda,$$

where

$$\Delta_f(a,b;\alpha;\mathscr{B};J) = \mathscr{B}(n+1,\alpha-n)\left(\frac{f(a)+f(b)}{2}\right) - \frac{n!}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J_{\alpha}^{\frac{1}{b}}(f\circ g)\left(\frac{1}{a}\right) + J_{\alpha}^{\frac{1}{a}}(f\circ g)\left(\frac{1}{b}\right)\right],$$

and $A_{\lambda} = \lambda a + (1-\lambda)b$.

Proof. Let

$$\begin{aligned} I &= \int_0^1 \left\{\frac{\mathscr{B}_{\lambda}(n+1,\alpha-n) - \mathscr{B}_{1-\lambda}(n+1,\alpha-n)}{A_{\lambda}^2}\right\} f'\left(\frac{ab}{A_{\lambda}}\right) \mathrm{d}\lambda \\ &= \int_0^1 \left\{\frac{\mathscr{B}_{\lambda}(n+1,\alpha-n)}{A_{\lambda}^2}\right\} f'\left(\frac{ab}{A_{\lambda}}\right) \mathrm{d}\lambda - \int_0^1 \left\{\frac{\mathscr{B}_{1-\lambda}(n+1,\alpha-n)}{A_{\lambda}^2}\right\} f'\left(\frac{ab}{A_{\lambda}}\right) \mathrm{d}\lambda \\ &= I_1 - I_2. \end{aligned} \tag{2.6}$$

Now

$$\begin{aligned}
&\int_0^1 \left\{ \frac{\mathscr{B}_\lambda(n+1,\alpha-n)}{A_\lambda^2} \right\} f'\left(\frac{ab}{A_\lambda}\right) \mathrm{d}\lambda \\
&= \int_0^1 \frac{1}{A_\lambda^2} \left(\int_0^\lambda u^n(1-u)^{\alpha-n-1} \mathrm{d}u \right) f'\left(\frac{ab}{A_\lambda}\right) \mathrm{d}\lambda \\
&= \frac{1}{ab(b-a)} \mathscr{B}(n+1,\alpha-n) f(b) - \frac{1}{ab(b-a)} \int_0^1 \lambda^n (1-\lambda)^{\alpha-n-1} f\left(\frac{ab}{A_\lambda}\right) \mathrm{d}\lambda \\
&= \frac{1}{ab(b-a)} \mathscr{B}(n+1,\alpha-n) f(b) - \frac{n!}{ab(b-a)} \left(\frac{ab}{b-a}\right)^\alpha J_\alpha^{\frac{1}{b}} (f\circ g)\left(\frac{1}{a}\right).
\end{aligned} \tag{2.7}$$

Also

$$\begin{aligned}
&\int_0^1 \left\{ \frac{\mathscr{B}_{1-\lambda}(n+1,\alpha-n)}{A_\lambda^2} \right\} f'\left(\frac{ab}{A_\lambda}\right) \mathrm{d}\lambda \\
&= \int_0^1 \frac{1}{A_\lambda^2} \left(\int_0^{1-\lambda} u^n(1-u)^{\alpha-n-1} \mathrm{d}u \right) f'\left(\frac{ab}{A_\lambda}\right) \mathrm{d}\lambda \\
&= -\frac{1}{ab(b-a)} \mathscr{B}(n+1,\alpha-n) f(a) + \frac{1}{ab(b-a)} \int_0^1 (1-\lambda)^n \lambda^{\alpha-n-1} f\left(\frac{ab}{A_\lambda}\right) \mathrm{d}\lambda \\
&= -\frac{1}{ab(b-a)} \mathscr{B}(n+1,\alpha-n) f(a) + \frac{n!}{ab(b-a)} \left(\frac{ab}{b-a}\right)^\alpha J_\alpha^{\frac{1}{a}} (f\circ g)\left(\frac{1}{b}\right).
\end{aligned} \tag{2.8}$$

On summation of inequalities (2.6), (2.7) and (2.8) and then multiplying by $\frac{ab(b-a)}{2}$ completes the proof. □

Now with the help of Lemma 2.1, we derive our coming results.

THEOREM 2.2. *Let $f : \mathscr{I} = [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on $\mathscr{I}^\circ$ such that $f' \in L[a,b]$. If $|f'|^q$ where $q \geqslant 1$ is harmonic convex function, then*

$$\left|\Delta_f(a,b;\alpha;\mathscr{B};J)\right| \leqslant \frac{ab(b-a)}{2} \psi^{1-\frac{1}{q}} \left(\psi_1 |f'(a)|^q + \psi_2 |f'(b)|^q\right)^{\frac{1}{q}},$$

where

$$\psi := \mathscr{B}(n+1,\alpha-n+1) - \mathscr{B}(n+2,\alpha-n),$$

$$\psi_1 := \frac{b^{-2}}{2} {}_2\mathscr{F}_1\left(2,1;3;1-\frac{a}{b}\right),$$

$$\psi_2 := \frac{b^{-2}}{2}\,{}_2\mathscr{F}_1\Big(2,2;3;1-\frac{a}{b}\Big),$$

$\mathscr{B}_\lambda(a,b)$ is incomplete Beta functions respectively and $\alpha \in (n,n+1]$, where $n \in 0,1,2\ldots$.

Proof. Using Lemma 2.1, property of modulus, Holder's inequality and the fact that $|f'|^q$ is harmonic convex function, we have

$$\begin{aligned}
&\left|\Delta_f(a,b;\alpha;\mathscr{B};J)\right| \\
&= \left|\frac{ab(b-a)}{2}\int_0^1 \left\{\frac{\mathscr{B}_\lambda(n+1,\alpha-n)-\mathscr{B}_{1-\lambda}(n+1,\alpha-n)}{A_\lambda^2}\right\} f'\left(\frac{ab}{A_\lambda}\right)\mathrm{d}\lambda\right| \\
&\leqslant \frac{ab(b-a)}{2}\left(\int_0^1 \{\mathscr{B}_\lambda(n+1,\alpha-n)-\mathscr{B}_{1-\lambda}(n+1,\alpha-n)\}\,\mathrm{d}\lambda\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1 \frac{1}{A_\lambda^2}\left|f'\left(\frac{ab}{A_\lambda}\right)\right|^q \mathrm{d}\lambda\right)^{\frac{1}{q}} \\
&\leqslant \frac{ab(b-a)}{2}\left(\mathscr{B}(n+1,\alpha-n+1)-\mathscr{B}(n+2,\alpha-n)\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1 \frac{1}{A_\lambda^2}\Big[(1-\lambda)|f'(a)|^q+\lambda|f'(b)|^q\Big]\mathrm{d}\lambda\right)^{\frac{1}{q}} \\
&= \frac{ab(b-a)}{2}\left(\mathscr{B}(n+1,\alpha-n+1)-\mathscr{B}(n+2,\alpha-n)\right)^{1-\frac{1}{q}}\left(\psi_1|f'(a)|^q+\psi_2|f'(b)|^q\right)^{\frac{1}{q}},
\end{aligned}$$

where

$$\psi_1 := \int_0^1 \frac{1-\lambda}{A_\lambda^2}\mathrm{d}\lambda = \frac{b^{-2}}{2}\,{}_2\mathscr{F}_1\Big(2,1;3;1-\frac{a}{b}\Big),$$

and

$$\psi_2 := \int_0^1 \frac{\lambda}{A_\lambda^2}\mathrm{d}\lambda = \frac{b^{-2}}{2}\,{}_2\mathscr{F}_1\Big(2,2;3;1-\frac{a}{b}\Big).$$

This completes the proof. □

THEOREM 2.3. *et $f : \mathscr{I} = [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on $\mathscr{I}^\circ$ such that $f' \in L[a,b]$. If $|f'|^q$ where $q \geqslant 1$ is harmonic convex function, then*

$$\left|\Delta_f(a,b;\alpha;\mathscr{B};J)\right| \leqslant \frac{ab(b-a)}{2}\phi^{1-\frac{1}{q}}\left\{(\phi_1-\phi_2)|f'(a)|^q+(\phi_3-\phi_4)|f'(b)|^q\right\}^{\frac{1}{q}},$$

where

$$\phi := \frac{1}{b^2}\,{}_2\mathscr{F}_1\Big(2,1;2;1-\frac{a}{b}\Big),$$
$$\phi_1 = \frac{1}{2}\mathscr{B}(n+1,\alpha-n+2),$$
$$\phi_2 = \frac{1}{2}\mathscr{B}(\alpha-n,n+1)-\frac{1}{2}\mathscr{B}(\alpha-n,n+3),$$
$$\phi_3 = \frac{1}{2}\mathscr{B}(n+1,\alpha-n)-\frac{1}{2}\mathscr{B}(n+3,\alpha-n),$$
$$\phi_4 = \frac{1}{2}\mathscr{B}(\alpha-n+2,n+1),$$

$\mathscr{B}_\lambda(a,b)$ *is incomplete Beta functions respectively and* $\alpha\in(n,n+1]$, *where* $n\in 0,1,2\ldots$.

Proof. Using Lemma 2.1, property of modulus, Holder's inequality and the fact that $|f'|^q$ is harmonic convex function, we have

$$\begin{aligned}
&\big|\Delta_f(a,b;\alpha;\mathscr{B};J)\big| \\
&= \left|\frac{ab(b-a)}{2}\int_0^1\left\{\frac{\mathscr{B}_\lambda(n+1,\alpha-n)-\mathscr{B}_{1-\lambda}(n+1,\alpha-n)}{A_\lambda^2}\right\}f'\left(\frac{ab}{A_\lambda}\right)\mathrm{d}\lambda\right| \\
&\leqslant \frac{ab(b-a)}{2}\left(\int_0^1\frac{1}{A_\lambda^2}\mathrm{d}\lambda\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1\{\mathscr{B}_\lambda(n+1,\alpha-n)-\mathscr{B}_{1-\lambda}(n+1,\alpha-n)\}\left|f'\left(\frac{ab}{A_\lambda}\right)\right|^q\mathrm{d}\lambda\right)^{\frac{1}{q}} \\
&\leqslant \frac{ab(b-a)}{2}\left(\frac{1}{b^2}\,{}_2\mathscr{F}_1\Big(2,1;2;1-\frac{a}{b}\Big)\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1\{\mathscr{B}_\lambda(n+1,\alpha-n)-\mathscr{B}_{1-\lambda}(n+1,\alpha-n)\}\Big[(1-\lambda)|f'(a)|^q+\lambda|f'(b)|^q\Big]\mathrm{d}\lambda\right)^{\frac{1}{q}} \\
&= \frac{ab(b-a)}{2}\left(\frac{1}{b^2}\,{}_2\mathscr{F}_1\Big(2,1;2;1-\frac{a}{b}\Big)\right)^{1-\frac{1}{q}}\left\{(\phi_1-\phi_2)|f'(a)|^q+(\phi_3-\phi_4)|f'(b)|^q\right\}^{\frac{1}{q}},
\end{aligned}$$

where

$$\phi_1 := \int_0^1(1-\lambda)\mathscr{B}_\lambda(n+1,\alpha-n)\mathrm{d}\lambda = \frac{1}{2}\mathscr{B}(n+1,\alpha-n+2),$$

$$\phi_2 := \int_0^1(1-\lambda)\mathscr{B}_{1-\lambda}(n+1,\alpha-n)\mathrm{d}\lambda = \frac{1}{2}\mathscr{B}(\alpha-n,n+1)-\frac{1}{2}\mathscr{B}(\alpha-n,n+3),$$

$$\phi_3 = \int_0^1 \lambda \mathscr{B}_\lambda(n+1,\alpha-n)\mathrm{d}\lambda = \frac{1}{2}\mathscr{B}(n+1,\alpha-n) - \frac{1}{2}\mathscr{B}(n+3,\alpha-n),$$

and

$$\phi_4 = \int_0^1 \lambda \mathscr{B}_{1-\lambda}(n+1,\alpha-n)\mathrm{d}\lambda = \frac{1}{2}\mathscr{B}(\alpha-n+2,n+1).$$

This completes the proof. □

THEOREM 2.4. *et $f : \mathscr{I} = [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on $\mathscr{I}^\circ$ such that $f' \in L[a,b]$. If $|f'|^q$ where $q \geqslant 1$ is harmonic convex function, then*

$$\left|\Delta_f(a,b;\alpha;\mathscr{B};J)\right| \leqslant \frac{ab(b-a)}{2}\theta^{\frac{1}{p}}\left(\theta_1|f'(a)|^q + \theta_2|f'(b)|^q\right)^{\frac{1}{q}},$$

where

$$\theta := 2\int_0^{\frac{1}{2}}\left(\int_t^{1-t} x^n(1-x)^{\alpha-n-1}\mathrm{d}x\right)^p,$$

$$\theta_1 := \frac{b^{-2q}}{2}\,{}_2\mathscr{F}_1\Big(2q,1;3;1-\frac{a}{b}\Big),$$

$$\theta_2 := \frac{b^{-2q}}{2}\,{}_2\mathscr{F}_1\Big(2q,2;3;1-\frac{a}{b}\Big).$$

$\mathscr{B}_\lambda(a,b)$ is incomplete Beta functions respectively and $\alpha \in (n,n+1]$, where $n \in$ $0,1,2\ldots.$

Proof. Using Lemma 2.1, property of modulus, Holder's inequality and the fact that $|f'|^q$ is harmonic convex function, we have

$$\begin{aligned}
&\left|\Delta_f(a,b;\alpha;\mathscr{B};J)\right| \\
&= \left|\frac{ab(b-a)}{2}\int_0^1\left\{\frac{\mathscr{B}_\lambda(n+1,\alpha-n)-\mathscr{B}_{1-\lambda}(n+1,\alpha-n)}{A_\lambda^2}\right\}f'\left(\frac{ab}{A_\lambda}\right)\mathrm{d}\lambda\right| \\
&\leqslant \frac{ab(b-a)}{2}\left(\int_0^1|\mathscr{B}_\lambda(n+1,\alpha-n)-\mathscr{B}_{1-\lambda}(n+1,\alpha-n)|^p\,\mathrm{d}\lambda\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^1\frac{1}{A_\lambda^{2q}}\left|f'\left(\frac{ab}{A_\lambda}\right)\right|^q\mathrm{d}\lambda\right)^{\frac{1}{q}}
\end{aligned}$$

$$\leqslant \frac{ab(b-a)}{2}\left[2\int_0^{\frac{1}{2}}\left(\int_t^{1-t} x^n(1-x)^{\alpha-n-1}\mathrm{d}x\right)^p\right]^{\frac{1}{p}}\left(\int_0^1 \frac{1}{A_\lambda^{2q}}\left[(1-\lambda)|f'(a)|^q+\lambda|f'(b)|^q\right]\mathrm{d}\lambda\right)^{\frac{1}{q}}$$

$$= \frac{ab(b-a)}{2}\left[2\int_0^{\frac{1}{2}}\left(\int_t^{1-t} x^n(1-x)^{\alpha-n-1}\mathrm{d}x\right)^p\right]^{\frac{1}{p}}\left(\theta_1|f'(a)|^q+\theta_2|f'(b)|^q\right)^{\frac{1}{q}},$$

where

$$\theta_1 := \int_0^1 \frac{1-\lambda}{A_\lambda^{2q}}\mathrm{d}\lambda = \frac{b^{-2q}}{2}\,{}_2\mathscr{F}_1\left(2q,1;3;1-\frac{a}{b}\right),$$

and

$$\theta_2 := \int_0^1 \frac{\lambda}{A_\lambda^{2q}}\mathrm{d}\lambda = \frac{b^{-2q}}{2}\,{}_2\mathscr{F}_1\left(2q,2;3;1-\frac{a}{b}\right).$$

This completes the proof. □

Conclusion. We have derived new Hermite-Hadamard type inequalities via harmonic convex functions involving conformable fractional integrals. A new conformable fractional integral identity is also obtained. It is expected that ideas and techniques of this article may attract interested readers.

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REFERENCES

[1] T. ABDELJAWAD, *On conformable fractional calculus*, J. Comput. Appl. Math., **279** (2015), 57–66.
[2] G. CRISTESCU, L. LUPSA, *Non-connected Convexities and Applications*, Kluwer Academic Publishers, Dordrecht, Holland, 2002.
[3] S. S. DRAGOMIR, R. P. AGARWAL, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. **11** (1998), 91–95.
[4] S. S. DRAGOMIR, C. E. M. PEARCE, *Selected topics on Hermite-Hadamard inequalities and applications*, Victoria University, Australia, 2000.
[5] I. ISCAN, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacettepe J. Math. Stat. **43** (6), 935-942, (2014).
[6] I. ISCAN, S. WU, *Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals*, Appl. Math. Comput., **238**, 237–244, (2014).
[7] A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO, *Theory and applications of fractional differential equations*, Elsevier B. V., Amsterdam, Netherlands, (2006).
[8] R. KHALIL, M. AL HORANI, A. YOUSEF, M. SABABHEH, *A new definition of fractional derivative*, J. Computat. Appl. Math., **264** (2014), 65–70.

[9] M. V. MIHAI, M. A. NOOR, K. I. NOOR, M. U. AWAN, *Some integral inequalities for harmonic h-convex functions involving hypergeometric functions*, Appl. Math. Comput. **252**, 257–262, (2015).

[10] K. S. MILLER, B. ROSS, *An introduction to the fractional calculus and fractional differential equations*, New York: Wiley, 1993.

[11] M. A. NOOR, K. I. NOOR, M. U. AWAN, *Integral inequalities for coordinated harmonically convex functions*, Complex Var. Elliptic Equat. **60** (6), 776–786, (2015).

[12] M. A. NOOR, K. I. NOOR, M. U. AWAN, S. COSTACHE, *Some integral inequalities for harmonically h-convex functions*, U. P. B. Sci. Bull., Series A. **77** (1), 5–16, (2015).

[13] M. Z. SARIKAYA, E. SET, H. YALDIZ, N. BASAK, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Model., **57** (2013), 2403–2407.

[14] E. SET, *New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals*, Comput. Math. Appl., **63** (7), (2012), 1147–1154.

[15] E. SET, A. O. AKDEMIR, I. MUMCU, *The Hermite-Hadamard's inequality and its extentions for conformable fractioanal integrals of any order $\alpha > 0$*, preprint, (2016).

[16] E. SET, M. Z. SARIKAYA, A. GOZPINAR, *Some Hermite-Hadamard type inequalities for convex functions via conformable fractional integrals and related inequalities*, preprint, (2016).

[17] H.-N. SHI AND J. ZHANG, *Some new judgement theorems of Schur geometric and Schur harmonic convexities for a class of symmetric functions*, J. Inequal. Appl. **2013** (2013), 527.

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