

CONSIDERATIONS ABOUT THE SEVERAL INEQUALITIES IN AN INNER PRODUCT SPACE

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Abstract. The aim of this paper is to show new results concerning the Cauchy-Schwarz inequality in an inner product space. We find an improvement of Buzano's inequality and Richard's inequality, which are extensions of the Cauchy-Schwarz inequality.

1. Introduction

In a presentation Niculescu [10] makes a radiography of the inequalities that have played an important role in the Theory of Inequalities. The Cauchy-Schwarz Inequality is one of them. In 1821 Cauchy [4] shows the following identity:

$$\left(\sum_{i=1}^n a_i^2\right) \left(\sum_{i=1}^n b_i^2\right) = \left(\sum_{i=1}^n a_i b_i\right)^2 + \sum_{1 \leq i < j \leq n} (a_i b_j - a_j b_i)^2. \quad (1)$$

In fact this is *Lagrange's identity*, because Lagrange in 1773 proved the identity

$$\left(\sum_{i=1}^3 a_i^2\right) \left(\sum_{i=1}^3 b_i^2\right) = \left(\sum_{i=1}^3 a_i b_i\right)^2 + \sum_{1 \leq i < j \leq 3} (a_i b_j - a_j b_i)^2.$$

A consequence of Lagrange's identity is the famous Cauchy-Schwarz inequality which states: if $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ are two n -tuples of real numbers, then

$$\sqrt{(a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2)} \geq |a_1 b_1 + \dots + a_n b_n|, \quad (2)$$

with equality holding if and only if $\mathbf{a} = \lambda \mathbf{b}$. This result is called the *Cauchy-Schwarz-Buniakowski inequality* or simply the *Cauchy inequality*.

Many refinements for Cauchy-Schwarz-Buniakowski inequality can be found in literature (see [2], [4], [5] and [11]). In particular, we mention one of them: Ostrowski [11], in 1952, proved the following: if $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$ and

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$\mathbf{z} = (z_1, \dots, z_n)$ are n -tuples of real numbers such that \mathbf{x} and \mathbf{y} are not proportional and

$$\sum_{k=1}^n y_k z_k = 0, \quad \text{and} \quad \sum_{k=1}^n x_k z_k = 1, \quad \text{then} \tag{3}$$

$$\sum_{k=1}^n y_k^2 \not\leq \sum_{k=1}^n z_k^2 \leq \sum_{k=1}^n x_k^2 \sum_{k=1}^n y_k^2 - \left(\sum_{k=1}^n x_k y_k \right)^2.$$

For all $x, y \in X$ in an inner product space $X = (X, \langle \cdot, \cdot \rangle)$ over the field of complex numbers \mathbb{C} or real numbers \mathbb{R} , then we have the Cauchy-Schwarz inequality, given by the following:

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|. \tag{4}$$

The Cauchy-Schwarz inequality can be written, as in Aldaz [1] and Niculescu [10], in terms of the angular distance between two vectors, thus

$$\langle x, y \rangle = \|x\| \|y\| \left(1 - \frac{1}{2} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 \right), \tag{5}$$

for all nonzero vectors $x, y \in X$.

Buzano [3] showed an extension of the Cauchy-Schwarz inequality, given by the following:

$$|\langle a, x \rangle \langle x, b \rangle| \leq \frac{1}{2} \|x\|^2 (|\langle a, b \rangle| + \|a\| \cdot \|b\|). \tag{6}$$

for any $x, a, b \in X$.

It is easy to see that for $a = b$, inequality (6) becomes the Cauchy-Schwarz inequality.

Another inequality which included the Buzano inequality is mentioned by Precupanu [14] and Dragomir [6]:

$$\frac{1}{2} \|x\|^2 (|\langle a, b \rangle| - \|a\| \cdot \|b\|) \leq |\langle a, x \rangle \langle x, b \rangle| \leq \frac{1}{2} \|x\|^2 (|\langle a, b \rangle| + \|a\| \cdot \|b\|), \tag{7}$$

for any $x, a, b \in X$. In [7] Gavrea showed an extension of Buzano’s inequality in inner product space. For real inner spaces, Richard [15], found the following stronger inequality

$$\left| \langle a, x \rangle \langle x, b \rangle - \frac{1}{2} \|x\|^2 \langle a, b \rangle \right| \leq \frac{1}{2} \|x\|^2 \|a\| \cdot \|b\|, \tag{8}$$

for any $x, a, b \in X$.

In [13], Popa and Raşa showed that, for any $x, a, b \in X$, the inequality

$$\left| \operatorname{Re} \left(\langle a, x \rangle \langle x, b \rangle - \frac{1}{2} \|x\|^2 \langle a, b \rangle \right) \right| \leq \frac{1}{2} \|x\|^2 \sqrt{\|a\|^2 \cdot \|b\|^2 - (\operatorname{Im} \langle a, b \rangle)^2}, \tag{9}$$

holds.

Dragomir [5] presented the following refinement of the Richard inequality:

$$\left| \langle a, b \rangle \|x\|^2 - \alpha \langle a, x \rangle \langle x, b \rangle \right| \leq \max \{1, |1 - \alpha|\} \|a\| \cdot \|b\| \cdot \|x\|^2, \tag{10}$$

for all vectors x, a, b in an inner product space X and $\alpha \in \mathbb{C}$.

This inequality was found in another way by Khosravi *et al.* [8].

In [9], Lupu and Schwarz proved the following inequality:

$$\left| \|a\|^2 \langle b, c \rangle \right| + \left| \|b\|^2 \langle c, a \rangle \right| + \left| \|c\|^2 \langle a, b \rangle \right| \leq \|a\|^2 \|b\|^2 \|c\|^2 + 2 |\langle a, b \rangle \langle b, c \rangle \langle c, a \rangle|, \tag{11}$$

for any vectors $x, a, b \in X$.

These inequalities are applied to the theory of Hilbert \mathbb{C}^* -modules over non-commutative \mathbb{C}^* -algebras, see Aldaz [1], Pečarić and Rajić [12] and Dragomir [5], [6].

2. Main results

For beginning we prove two lemmas:

LEMMA 1. *In an inner product space X over the field of complex numbers \mathbb{C} , we have*

$$\|x + \alpha y\|^2 = \left| \alpha \|y\| + \frac{\langle x, y \rangle}{\|y\|} \right|^2 + \left\| x - \frac{\langle x, y \rangle}{\|y\|^2} y \right\|^2, \tag{12}$$

for all $x, y \in X$, $y \neq 0$, and for every $\alpha \in \mathbb{C}$.

Proof. By several calculations, we deduce the following:

$$\begin{aligned} \|x + ay\|^2 &= \langle x + \alpha y, x + \alpha y \rangle = \|x\|^2 + \bar{\alpha} \langle x, y \rangle + \alpha \overline{\langle x, y \rangle} + |\alpha|^2 \|y\|^2 \\ &= \left(\alpha \|y\| + \frac{\langle x, y \rangle}{\|y\|} \right) \left(\bar{\alpha} \|y\| + \frac{\overline{\langle x, y \rangle}}{\|y\|} \right) + \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2} \\ &= \left| \alpha \|y\| + \frac{\langle x, y \rangle}{\|y\|} \right|^2 + \left\| x - \frac{\langle x, y \rangle}{\|y\|^2} y \right\|^2, \end{aligned}$$

because, we have $\left\| x - \frac{\langle x, y \rangle}{\|y\|^2} y \right\|^2 = \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2}$. \square

REMARK 1. Let $x, e \in X$ with $\|e\| = 1$. If we take $y = e$ and $\alpha = -\lambda$ in relation (12), then we obtain $\|x - \lambda e\|^2 = |\lambda - \langle x, e \rangle|^2 + \|x - \langle x, e \rangle e\|^2$. Consequently, we deduce $\|x - \langle x, e \rangle e\|^2 = \inf_{\lambda \in \mathbb{C}} \|x - \lambda e\|^2$ which is a result found in [9].

LEMMA 2. *In an inner product space X over the field of complex numbers \mathbb{C} , we have*

$$\left\| \langle a, x \rangle x - \frac{1}{2} \|x\|^2 a \right\| = \frac{1}{2} \|x\|^2 \|a\|, \tag{13}$$

for all $x, a \in X$.

Proof. For $x = 0$ the equality is true. For $x \neq 0$ inequality (13) becomes $\left\| a - 2 \frac{\langle a, x \rangle}{\|x\|^2} x \right\| = \|a\|$. If we take in equality (12) $\alpha = -2$, $y = \frac{\langle a, x \rangle}{\|x\|^2} x$, then by simple calculations, we deduce the following:

$$\left\| a - 2 \frac{\langle a, x \rangle}{\|x\|^2} x \right\|^2 = \left| -2 \frac{\langle a, x \rangle}{\|x\|} + \frac{|\langle a, x \rangle|}{\|x\|} \right|^2 + \left\| a - \frac{\langle a, x \rangle}{\|x\|^2} x \right\|^2 = \|a\|^2.$$

Consequently, inequality (13) is true. \square

REMARK 2. A simple proof of Richard's inequality can be given by combining the Cauchy-Schwarz inequality and relation (13), thus:

$$\begin{aligned} \left| \langle a, x \rangle \langle x, b \rangle - \frac{1}{2} \|x\|^2 \langle a, b \rangle \right| &= \left| \left\langle \langle a, x \rangle x - \frac{1}{2} \|x\|^2 a, b \right\rangle \right| \\ &\leq \left\| \langle a, x \rangle x - \frac{1}{2} \|x\|^2 a \right\| \|b\| \\ &= \frac{1}{2} \|x\|^2 \|a\| \|b\|. \end{aligned}$$

THEOREM 1. In an inner product space X over the field of complex numbers \mathbb{C} , we have

$$|\alpha|^2 \|y\|^2 \|z\|^2 + \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \geq 2 \operatorname{Re} \left(\bar{\alpha} \left(\langle x, y \rangle \langle y, z \rangle - \langle x, z \rangle \|y\|^2 \right) \right), \quad (14)$$

for all $x, y, z \in X$, and for every $\alpha \in \mathbb{C}$.

Proof. For $y = 0$, inequality (14) is true. In the situation $y \neq 0$, using Lemma 1, we obtain the following calculations:

$$\begin{aligned} &\left\| x + \alpha z - \frac{\langle x, y \rangle}{\|y\|^2} y \right\|^2 \\ &= \|x + \alpha z\|^2 - \frac{2 \operatorname{Re} (\bar{\alpha} \langle x, y \rangle \langle y, z \rangle)}{\|y\|^2} - \frac{|\langle x, y \rangle|^2}{\|y\|^2} \\ &= \|x\|^2 + \bar{\alpha} \langle x, z \rangle + \alpha \overline{\langle x, z \rangle} + |\alpha|^2 \|z\|^2 - \frac{2 \operatorname{Re} (\bar{\alpha} \langle x, y \rangle \langle y, z \rangle)}{\|y\|^2} - \frac{|\langle x, y \rangle|^2}{\|y\|^2} \\ &= |\alpha|^2 \|z\|^2 + 2 \operatorname{Re} (\bar{\alpha} \langle x, z \rangle) - \frac{2 \operatorname{Re} (\bar{\alpha} \langle x, y \rangle \langle y, z \rangle)}{\|y\|^2} + \frac{\|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2}{\|y\|^2} \\ &= \frac{1}{\|y\|^2} \left[|\alpha|^2 \|y\|^2 \|z\|^2 - 2 \left(\operatorname{Re} (\bar{\alpha} \langle x, y \rangle \langle y, z \rangle) - \operatorname{Re} (\bar{\alpha} \langle x, z \rangle) \|y\|^2 \right) \right. \\ &\quad \left. + \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \right] \geq 0 \end{aligned}$$

Consequently, we deduce the inequality of the statement. \square

COROLARY 1. *In an inner product space X over the field of real numbers \mathbb{R} , we have*

$$\frac{\|y\|^2}{\|z\|^2} \left(\frac{\langle x, y \rangle \langle y, z \rangle}{\|y\|^2} - \langle x, z \rangle \right)^2 \leq \|x\|^2 \|y\|^2 - \langle x, y \rangle^2, \tag{15}$$

for all $x, y, z \in X$, $y \neq 0$, $z \neq 0$.

Proof I. If $y \neq 0$, $z \neq 0$, then we apply Theorem 1 for $\alpha \in \mathbb{R}$, and we have

$$\|y\|^2 \|z\|^2 \alpha^2 - 2\alpha (\langle x, y \rangle \langle y, z \rangle - \langle x, z \rangle \|y\|^2) + \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \geq 0,$$

for all $x, y, z \in X$, and for every $\alpha \in \mathbb{R}$. Since $\|y\|^2 \|z\|^2 > 0$, then the discriminant is negative, i.e.,

$$\Delta = (\langle x, y \rangle \langle y, z \rangle - \langle x, z \rangle \|y\|^2)^2 - \|y\|^2 \|z\|^2 (\|x\|^2 \|y\|^2 - \langle x, y \rangle^2) \leq 0.$$

Therefore, we prove the statement. \square

Proof II. For $\alpha = \langle x, y \rangle \langle y, z \rangle - \langle x, z \rangle \|y\|^2$ in relation (14), we have

$$\|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \geq (2 - \|y\|^2 \|z\|^2) |\langle x, y \rangle \langle y, z \rangle - \langle x, z \rangle \|y\|^2|^2.$$

for $x = 0$, inequality (15) is true. In the situation $x \neq 0$, $y \neq 0$, if we replace in the above relation x and y by $\frac{x}{\|x\|}$ and $\frac{y}{\|y\|}$, then we deduce the statement. \square

REMARK 3. If we take $\langle x, z \rangle = 1$ and $\langle y, z \rangle = 0$, in inequality (15), then we find the inequality of Ostrowski for inner product spaces over the field of real numbers,

$$\|y\|^2 / \|z\|^2 \leq \|x\|^2 \|y\|^2 - \langle x, y \rangle^2, \tag{16}$$

for all $x, y, z \in X$, $y \neq 0$, $z \neq 0$.

It is easy to see that for $x, y, z \in \mathbb{R}^n$ we obtain inequality (3).

THEOREM 2. *In an inner product space X over the field real or complex numbers, for any nonzero vectors $x, a, b \in X$, we have*

$$\frac{1}{2} \|x\|^2 \|a\| \cdot \|b\| - \left| \langle a, x \rangle \langle x, b \rangle - \frac{1}{2} \|x\|^2 \langle a, b \rangle \right| \geq \frac{A}{\|x\|^2 \|a\| \cdot \|b\|} \geq 0, \tag{17}$$

where

$$A = \left(|\langle a, x \rangle| (\|x\|^2 \|b\|^2 - |\langle x, b \rangle|^2) - \frac{1}{2} \|x\|^2 (\|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2) \right)^2.$$

Proof. The equality $\left\| x - \frac{\langle x, y \rangle}{\|y\|^2} y \right\|^2 = \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2}$ implies

$$\left\| \|y\| x - \frac{\langle x, y \rangle}{\|y\|} y \right\|^2 = \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2.$$

Hence, using the Cauchy-Schwarz inequality, we have the relation

$$\|x\| \cdot \|y\| - |\langle x, y \rangle| = \frac{\left\| \left\| y \left\| x - \frac{\langle x, y \rangle}{\|y\|} y \right\| \right\|^2}{\|x\| \cdot \|y\| + |\langle x, y \rangle|} \geq \frac{\left\| \left\| y \left\| x - \frac{\langle x, y \rangle}{\|y\|} y \right\| \right\|^2}{2\|x\| \cdot \|y\|}.$$

In this inequality, we make the substitutions $x \rightarrow \langle a, x \rangle x - \frac{1}{2} \|x\|^2 a$ and $y \rightarrow b$ in the above inequality and using the equality (13), $\left\| \langle a, x \rangle x - \frac{1}{2} \|x\|^2 a \right\| = \frac{1}{2} \|x\|^2 a$, implies

$$\frac{1}{2} \|x\|^2 \|a\| \cdot \|b\| - \left| \langle a, x \rangle \langle x, b \rangle - \frac{1}{2} \|x\|^2 \langle a, b \rangle \right| \geq \frac{\|u - v\|^2}{\|x\|^2 \|a\| \cdot \|b\|},$$

where $u = \langle a, x \rangle \left(\|b\| x - \frac{\langle a, x \rangle \langle x, b \rangle}{\|b\|} b \right)$ and $v = -\frac{1}{2} \|x\|^2 \left(\|b\| a - \frac{\langle a, b \rangle}{\|b\|} b \right)$.

It follows that $\|u\| = |\langle a, x \rangle| \left(\|x\|^2 \|b\|^2 - |\langle x, b \rangle|^2 \right)$ and

$$\|u\| = \frac{1}{2} \|x\|^2 \left(\|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2 \right).$$

Since $\|u - v\|^2 \geq \|u\| - \|v\|^2$, $u, v \in X$, we obtain the inequality of the statement. \square

REMARK 4. a) For real or complex inner spaces, inequality (17) represents an improvement of Richard's inequality, given thus:

$$\left| \langle a, x \rangle \langle x, b \rangle - \frac{1}{2} \|x\|^2 \langle a, b \rangle \right| \leq \frac{1}{2} \|x\|^2 \|a\| \cdot \|b\| - \frac{A}{\|x\|^2 \|a\| \cdot \|b\|},$$

where

$$A = \left(|\langle a, x \rangle| \left(\|x\|^2 \|b\|^2 - |\langle x, b \rangle|^2 \right) - \frac{1}{2} \|x\|^2 \left(\|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2 \right) \right)^2.$$

b) Also, using above inequality, and from the continuity property of the modulus, i.e., $|\alpha - \beta| \geq \left| |\alpha| - |\beta| \right|$, $\alpha, \beta \in \mathbb{C}$, we deduce the inequality

$$\begin{aligned} & \frac{1}{2} \|x\|^2 (|\langle a, b \rangle| - \|a\| \cdot \|b\|) + \frac{A}{\|x\|^2 \|a\| \cdot \|b\|} \\ & \leq |\langle a, x \rangle \langle x, b \rangle| \\ & \leq \frac{1}{2} \|x\|^2 (|\langle a, b \rangle| + \|a\| \cdot \|b\|) - \frac{A}{\|x\|^2 \|a\| \cdot \|b\|} \end{aligned}$$

which is in fact a refinement of Buzano's inequality.

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