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ON REVERSES OF THE GOLDEN–THOMPSON TYPE INEQUALITIES

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Abstract. In this paper we present some reverses of the Golden-Thompson type inequalities: Let H and K be Hermitian matrices such that $e^s e^H \preceq_{ols} e^K \preceq_{ols} e^t e^H$ for some scalars $s \leqslant t$, and $\alpha \in [0,1]$. Then for all $p > 0$ and $k = 1,2,\ldots,n$

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leqslant (\max\{S(e^{sp}),S(e^{tp})\})^{\frac{1}{p}} \lambda_k(e^{pH}\sharp_\alpha e^{pK})^{\frac{1}{p}},$$

where $A\sharp_\alpha B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}B^{\frac{1}{2}}A^{-\frac{1}{2}}\right)^\alpha A^{\frac{1}{2}}$ is α-geometric mean, $S(t)$ is the so called Specht ratio and $\preceq_{ols}$ is the so called Olson order. The same inequalities are also provided with other constants. The obtained inequalities improve some known results.

1. Introduction

In what follows, capital letters A,B,H and K stand for $n\times n$ matrices or bounded linear operators on an n-dimentional complex Hilbert space $(\mathscr{H},\langle\cdot\rangle)$. For a pair A,B of Hermitian matrices, we say $A \leqslant B$ if $B-A \geqslant 0$. Let A and B be two positive definite matrices. For each $\alpha \in [0,1]$, the weighted geometric mean $A\sharp_\alpha B$ of A and B in the sense of Kubo-Ando [9] is defined by

$$A\sharp_\alpha B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}B^{\frac{1}{2}}A^{-\frac{1}{2}}\right)^\alpha A^{\frac{1}{2}}.$$

Also for positive semidefinite matrices A and B, the weak log-majorization $A \prec_{wlog} B$ means that

$$\prod_{j=1}^{k}\lambda_j(A) \leqslant \prod_{j=1}^{k}\lambda_j(B), \qquad k = 1,2,\cdots,n,$$

where $\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant \lambda_n(A)$ are the eigenvalues of A listed in decreasing order. If equality holds when $k = n$, we have the log-majorization $A \prec_{log} B$. It is known that the weak log-majorization $A \prec_{wlog} B$ implies $\|A\|_u \leqslant \|B\|_u$ for any unitarily invariant norm $\|\cdot\|_u$, i.e. $\|UAV\|_u = \|A\|_u$ for all A and all unitaries U,V. See [2] for theory of majorization.

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In [14], Specht obtained an inequality for the arithmetic and geometric means of positive numbers: Let $x_1 \geqslant \ldots \geqslant x_n > 0$ and set $t = x_1/x_n$. Then

$$\frac{x_1 + \ldots + x_n}{n} \leqslant S(t)(x_1 \ldots x_n)^{\frac{1}{n}},$$

where

$$S(t) = \frac{(t-1)t^{1/(t-1)}}{e \log t} \quad (t \neq 1) \quad \text{and} \quad S(1) = 1 \tag{1}$$

is called the Specht ratio at t. Note that $\lim_{p \to 0} S(t^p)^{\frac{1}{p}} = 1$, $S(t^{-1}) = S(t) > 1$ for $t \neq 1$, $t > 0$ [5]. Specht's inequality is a ratio type reverse inequality of the classical arithmetic-geometric mean inequality. Using this nice ratio we can state our main result in Section 2.

The Golden-Thompson trace inequality, which is of importance in statistical mechanics and in the theory of random matrices, states that $Tre^{H+K} \leqslant Tre^H e^K$ for arbitrary Hermitian matrices H and K. This inequality has been complemented in several ways [1, 8]. Ando and Hiai in [1] proved that for every unitarily invariant norm $\|\cdot\|_u$ and $p > 0$

$$\|(e^{pH} \sharp_\alpha e^{pK})^{\frac{1}{p}}\|_u \leqslant \|e^{(1-\alpha)H + \alpha K}\|_u. \tag{2}$$

Seo in [12] found some upper bounds on $\|e^{(1-\alpha)H+\alpha K}\|_u$ in terms of scalar multiples of $\|(e^{pH} \sharp_\alpha e^{pK})^{\frac{1}{p}}\|_u$, which show reverse of the Golden-Thompson type inequality (2). In this paper we establish another reverses of this inequality, which improve and refine Seo's results. In fact the general sandwich condition $sA \leqslant B \leqslant tA$ for positive definite matrices, is the key for our statements. Also, the so called Olson order $\preceq_{ols}$ is used. For positive operators, $A \preceq_{ols} B$ if and only if $A^r \leqslant B^r$ for every $r \geqslant 1$ [11]. Our results are parallel to eigenvalue inequalities obtained in [3] and [7].

2. Reverse inequalities via Specht ratio

To study the Golden-Thompson inequality, Ando-Hiai in [1] developed the following log-majorizationes:

$$A^r \sharp_\alpha B^r \prec_{log} (A \sharp_\alpha B)^r, \qquad r \geqslant 1,$$

or equivalently

$$(A^p \sharp_\alpha B^p)^{\frac{1}{p}} \prec_{log} (A^q \sharp_\alpha B^q)^{\frac{1}{q}}, \qquad 0 < q \leqslant p.$$

There are some literatures [13] on the converse of these inequalities in terms of unitarily invariant norm $\|\cdot\|_u$. By the following lemmas, we obtain a new reverse of these inequalities in terms of eigenvalue inequalities.

LEMMA 1. *Let A and B be positive definite matrices such that $sA \leqslant B \leqslant tA$ for some scalars $0 < s \leqslant t$, and $\alpha \in [0,1]$. Then*

$$A^r \sharp_\alpha B^r \leqslant (\max\{S(s),S(t)\})^r (A\sharp_\alpha B)^r, \qquad 0 < r \leqslant 1, \tag{3}$$

where $S(t)$ is the Specht ratio defined as (1).

Proof. Let f be an operator monotone function on $[0,\infty)$. Then according to the proof of Theorem 1 in [6], we have

$$f(A)\sharp_\alpha f(B) \leqslant f(M(A\sharp_\alpha B)),$$

where $M = \max\{S(s),S(t)\}$. Putting $f(t) = t^r$ for $0 < r \leqslant 1$, we reach inequality (3). □

LEMMA 2. *Let A and B be positive definite matrices such that $sA \preceq_{ols} B \preceq_{ols} tA$ for some scalars $0 < s \leqslant t$, and $\alpha \in [0,1]$. Then*

$$\lambda_k(A\sharp_\alpha B)^r \leqslant \max\{S(s^r),S(t^r)\}\lambda_k(A^r \sharp_\alpha B^r), \qquad r \geqslant 1, \tag{4}$$

and hence,

$$\lambda_k(A^q \sharp_\alpha B^q)^{\frac{1}{q}} \leqslant (\max\{S(s^p),S(t^p)\})^{\frac{1}{p}} \lambda_k(A^p \sharp_\alpha B^p)^{\frac{1}{p}}, \quad 0 < q \leqslant p, \tag{5}$$

where $S(t)$ is the Specht ratio defined as (1) and $k = 1,2,\ldots,n$.

Proof. First note that the condition $sA \preceq_{ols} B \preceq_{ols} tA$ is equivalent to the condition $s^\nu A^\nu \leqslant B^\nu \leqslant t^\nu A^\nu$ for every $\nu \geqslant 1$. In particular, we have $sA \leqslant B \leqslant tA$ for $\nu = 1$. Also, for $r \geqslant 1$ we have $0 < \frac{1}{r} \leqslant 1$ and by (3)

$$A^{\frac{1}{r}} \sharp_\alpha B^{\frac{1}{r}} \leqslant (\max\{S(s),S(t)\})^{\frac{1}{r}} (A\sharp_\alpha B)^{\frac{1}{r}}. \tag{6}$$

On the other hand, from the condition $s^\nu A^\nu \leqslant B^\nu \leqslant t^\nu A^\nu$ for every $\nu \geqslant 1$ and letting $\nu = r$, we have

$$s^r A^r \leqslant B^r \leqslant t^r A^r.$$

Now if we let $X = A^r$, $Y = B^r$, $w = s^r$ and $z = t^r$, then

$$wX \leqslant Y \leqslant zX. \tag{7}$$

Using (6) under the condition (7), we have

$$X^{\frac{1}{r}} \sharp_\alpha Y^{\frac{1}{r}} \leqslant (\max\{S(w),S(z)\})^{\frac{1}{r}} (X\sharp_\alpha Y)^{\frac{1}{r}},$$

and this is the same as

$$A\sharp_\alpha B \leqslant (\max\{S(s^r),S(t^r)\})^{\frac{1}{r}} (A^r \sharp_\alpha B^r)^{\frac{1}{r}}.$$

Hence

$$\lambda_k(A\sharp_\alpha B) \leqslant (\max\{S(s^r),S(t^r)\})^{\frac{1}{r}}\lambda_k(A^r\sharp_\alpha B^r)^{\frac{1}{r}}.$$

By taking r-th power on both sides and using the spectral mapping theorem, we get the desired inequality (4). Note that from the minimax characterization of eigenvalues of a Hermitian matrix [2] it follows immediately that $A \leqslant B$ implies $\lambda_k(A) \leqslant \lambda_k(B)$ for each k. Similarly since $p/q \geqslant 1$, from inequality (4)

$$\lambda_k(A\sharp_\alpha B)^{\frac{p}{q}} \leqslant \max\{S(s^{\frac{p}{q}}),S(t^{\frac{p}{q}})\}\lambda_k(A^{\frac{p}{q}}\sharp_\alpha B^{\frac{p}{q}}). \tag{8}$$

Replacing A and B by A^q and B^q in (8), and using the sandwich condition $s^qA^q \leqslant B^q \leqslant t^qA^q$, we have

$$\lambda_k(A^q\sharp_\alpha B^q)^{\frac{p}{q}} \leqslant \max\{S(s^p),S(t^p)\}\lambda_k(A^p\sharp_\alpha B^p).$$

This completes the proof. □

Note that eigenvalue inequalities immediately imply log-majorization and unitarily invariant norm inequalities.

COROLLARY 1. *Let A and B be positive definite matrices such that $mI \leqslant A,B \leqslant MI$ for some scalars $0 < m \leqslant M$ with $h = M/m$, and let $\alpha \in [0,1]$. Then*

$$A^r\sharp_\alpha B^r \leqslant S(h)^r(A\sharp_\alpha B)^r, \qquad 0 < r \leqslant 1, \tag{9}$$

and hence

$$\lambda_k(A\sharp_\alpha B)^r \leqslant S(h^r)\lambda_k(A^r\sharp_\alpha B^r), \qquad r \geqslant 1, \tag{10}$$

$$\lambda_k(A^q\sharp_\alpha B^q)^{\frac{1}{q}} \leqslant S(h^p)^{\frac{1}{p}}\lambda_k(A^p\sharp_\alpha B^p)^{\frac{1}{p}}, \qquad 0 < q \leqslant p, \tag{11}$$

where $S(t)$ is the Specht ratio defined as (1) and $k = 1,2,\ldots,n$.

Proof. Since $mI \leqslant A,B \leqslant MI$ implies $\frac{m}{M}A \leqslant B \leqslant \frac{M}{m}A$, the inequality (9) is obtained by letting $s = m/M$, $t = M/m$ in Lemma 1. Also from $mI \leqslant A,B \leqslant MI$, we have $m^\nu I \leqslant A^\nu,B^\nu \leqslant M^\nu I$ for every $\nu \geqslant 1$, and so

$$\left(\frac{m}{M}\right)^\nu A^\nu \leqslant B^\nu \leqslant \left(\frac{M}{m}\right)^\nu A^\nu. \tag{12}$$

Using Lemma 2 under the condition (12), we reach inequalities (10) and (11). Note that $S(h) = S(\frac{1}{h})$ for every $h > 0$. □

REMARK 1. We remark that the matrix inequality (9) is more stronger than corresponding norm inequality obtained by Seo in [12, Corollary 3.2]. Also, inequality (11) is presented in [12, Lemma 3.1].

In the sequel we show a reverse of the Golden-Thompson type inequality (2), which is our main result.

THEOREM 1. *Let H and K be Hermitian matrices such that $e^s e^H \preceq_{ols} e^K \preceq_{ols} e^t e^H$ for some scalars $s \leqslant t$, and let $\alpha \in [0,1]$. Then for all $p > 0$,*

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leqslant (\max\{S(e^{sp}), S(e^{tp})\})^{\frac{1}{p}} \lambda_k(e^{pH} \sharp_\alpha e^{pK})^{\frac{1}{p}},$$

where $S(t)$ is the Specht ratio defined as (1) and $k = 1, 2, \ldots, n$.

Proof. Replacing A and B by e^H and e^K in the inequality (5) of Lemma 2, we can write

$$\lambda_k(e^{qH} \sharp_\alpha e^{qK})^{\frac{1}{q}} \leqslant (\max\{S(e^{sp}), S(e^{tp})\})^{\frac{1}{p}} \lambda_k(e^{pH} \sharp_\alpha e^{pK})^{\frac{1}{p}}, \qquad 0 < q \leqslant p.$$

By [8, Lemma 3.3], we have

$$e^{(1-\alpha)H+\alpha K} = \lim_{q \to 0} (e^{qH} \sharp_\alpha e^{qK})^{\frac{1}{q}},$$

and hence it follows that for each $p > 0$,

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leqslant (\max\{S(e^{sp}), S(e^{tp})\})^{\frac{1}{p}} \lambda_k(e^{pH} \sharp_\alpha e^{pK})^{\frac{1}{p}}. \quad \square$$

COROLLARY 2. *Let H and K be Hermitian matrices such that $e^s e^H \preceq_{ols} e^K \preceq_{ols} e^t e^H$ for some scalars $s \leqslant t$, and let $\alpha \in [0,1]$. Then for every unitarily invariant norm $\|\cdot\|_u$ and all $p > 0$,*

$$\|e^{(1-\alpha)H+\alpha K}\|_u \leqslant (\max\{S(e^{sp}), S(e^{tp})\})^{\frac{1}{p}} \|(e^{pH} \sharp_\alpha e^{pK})^{\frac{1}{p}}\|_u, \tag{13}$$

and the right-hand side of (13) converges to the left-hand side as $p \downarrow 0$. In particular,

$$\|e^{H+K}\|_u \leqslant \max\{S(e^{2s}), S(e^{2t})\} \|(e^{2H} \sharp e^{2K})\|_u.$$

COROLLARY 3. [12, Theorem 3.3–Theorem 3.4] *Let H and K be Hermitian matrices such that $mI \leqslant H, K \leqslant MI$ for some scalars $m \leqslant M$, and let $\alpha \in [0,1]$. Then for all $p > 0$,*

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leqslant S(e^{(M-m)p})^{\frac{1}{p}} \lambda_k(e^{pH} \sharp_\alpha e^{pK})^{\frac{1}{p}}, \qquad k = 1, 2, \ldots, n.$$

So, for every unitarily invariant norm $\|\cdot\|_u$

$$\|e^{(1-\alpha)H+\alpha K}\|_u \leqslant S(e^{(M-m)p})^{\frac{1}{p}} \|(e^{pH} \sharp_\alpha e^{pK})^{\frac{1}{p}}\|_u,$$

and the right-hand side of these inequalities converges to the left-hand side as $p \downarrow 0$.

Proof. From $mI \leqslant H, K \leqslant MI$, we have $e^{\nu m} \leqslant e^{\nu H}, e^{\nu K} \leqslant e^{\nu M}$ for every $\nu \geqslant 1$ and so we can derive $e^{m-M} e^H \preceq_{ols} e^K \preceq_{ols} e^{M-m} e^H$. Now the assertion is obtained by applying Theorem 1 and the fact that for every $t > 0$, $S(t) = S(\frac{1}{t})$. $\square$

3. Reverse inequalities via Kantorovich constant

A well-known matrix version of the Kantorovich inequality [10] asserts that if A and U are two matrices such that $0 < mI \leqslant A \leqslant MI$ and $UU^* = I$, then

$$UA^{-1}U^* \leqslant \frac{(m+M)^2}{4mM}(UAU^*)^{-1}. \tag{14}$$

Let $w > 0$. The generalized Kantorovich constant $K(w,\alpha)$ is defined by

$$K(w,\alpha) := \frac{(w^\alpha - w)}{(\alpha-1)(w-1)}\left(\frac{\alpha-1}{\alpha}\frac{w^\alpha-1}{w^\alpha-w}\right)^\alpha, \tag{15}$$

for any real number $\alpha \in \mathbb{R}$ [5]. In fact, $K(\frac{M}{m},-1) = K(\frac{M}{m},2)$ is the constant occurring in (14).

Now as a result of the following statement, we have another reverse Golden-Thompson type inequality which refines corresponding inequality in [12].

PROPOSITION 1. [7, Theorem 3] Let H and K be Hermitian matrices such that $e^s e^H \preceq_{ols} e^K \preceq_{ols} e^t e^H$ for some scalars $s \leqslant t$, and let $\alpha \in [0,1]$. Then

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leqslant K(e^{p(t-s)},\alpha)^{-\frac{1}{p}}\lambda_k(e^{pH}\sharp_\alpha e^{pK})^{\frac{1}{p}}, \qquad p > 0, \tag{16}$$

where $K(w,\alpha)$ is the generalized Kantorovich constant defined as (15).

THEOREM 2. *Let H and K be Hermitian matrices such that $mI \leqslant K,H \leqslant MI$ for some scalars $m \leqslant M$ and let $\alpha \in [0,1]$. Then for every $p > 0$,*

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leqslant K(e^{2p(M-m)},\alpha)^{-\frac{1}{p}}\lambda_k(e^{pH}\sharp_\alpha e^{pK})^{\frac{1}{p}}, \qquad k = 1,2,\ldots,n,$$

and the right-hand side of this inequality converges to the left-hand side as $p \downarrow 0$. In particular,

$$\lambda_k(e^{H+K}) \leqslant \frac{e^{2M}+e^{2m}}{2e^M e^m}\lambda_k(e^{2H}\sharp e^{2K}), \qquad k = 1,2,\ldots,n.$$

Proof. Since $mI \leqslant K,H \leqslant MI$ implies $e^{m-M}e^H \preceq_{ols} e^K \preceq_{ols} e^{M-m}e^H$, desired inequalities are obtained by letting $s = m-M$ and $t = M-m$ in Proposition 1. For the convergence, we know that $\frac{2w^{\frac{1}{4}}}{w^{\frac{1}{2}}+1} \leqslant K(w,\alpha) \leqslant 1$, for every $\alpha \in [0,1]$. So, for every $p > 0$

$$1 \leqslant K(w^p,\alpha)^{-\frac{1}{p}} \leqslant \left(\frac{2w^{\frac{p}{4}}}{w^{\frac{p}{2}}+1}\right)^{-\frac{1}{p}}.$$

A simple calculation shows that

$$\lim_{p\to 0} -\frac{1}{p}\log\left(\frac{2w^{\frac{p}{4}}}{w^{\frac{p}{2}}+1}\right) = \lim_{p\to 0}\frac{(w^{\frac{p}{2}}-1)\log(w)}{4(w^{\frac{p}{2}}+1)} = 0,$$

and hence $\lim_{p\to 0}\left(\frac{2w^{\frac{p}{4}}}{w^{\frac{p}{2}}+1}\right)^{-\frac{1}{p}} = 1$. Now by using the sandwich condition and letting $w = e^{2(M-m)}$, we have $\lim_{p\to 0} K(e^{2p(M-m)},\alpha)^{-\frac{1}{p}} = 1$. $\square$

REMARK 2. Under the assumptions of Theorem 2, Seo in [12, Theorem 4.2] proved that

$$\|e^{(1-\alpha)H+\alpha K}\|_u \leqslant K(e^{(M-m)},p)^{-\frac{\alpha}{p}} K(e^{2p(M-m)},\alpha)^{-\frac{1}{p}} \|(e^{pH}\sharp_\alpha e^{pK})^{\frac{1}{p}}\|_u, \qquad 0<p\leqslant 1,$$

and

$$\|e^{(1-\alpha)H+\alpha K}\|_u \leqslant K(e^{2p(M-m)},\alpha)^{-\frac{1}{p}} \|(e^{pH}\sharp_\alpha e^{pK})^{\frac{1}{p}}\|_u, \qquad p\geqslant 1.$$

But the first inequality of Theorem 2 shows that the sharper constant for all $p>0$ is $K(e^{2p(M-m)},\alpha)^{-\frac{1}{p}}$. Since for $0<p\leqslant 1$, $K(e^{(M-m)},p)^{-\frac{\alpha}{p}} \geqslant 1$ and hence

$$K(e^{2p(M-m)},\alpha)^{-\frac{1}{p}} \leqslant K(e^{(M-m)},p)^{-\frac{\alpha}{p}} K(e^{2p(M-m)},\alpha)^{-\frac{1}{p}}.$$

4. Some related results

It has been shown [6] that if $f:[0,\infty)\longrightarrow[0,\infty)$ is operator monotone function and $0<mI\leqslant A\leqslant B\leqslant MI\leqslant I$ with $h=\frac{M}{m}$, then for all $\alpha\in[0,1]$

$$f(A)\sharp_\alpha f(B) \leqslant \exp\left(\alpha(1-\alpha)\left(1-\frac{1}{h}\right)^2\right) f(A\sharp_\alpha B). \tag{17}$$

This new ratio has been introduced by Furuichi and Minculete in [4], which is different from Specht ratio and Kantorovich constant. By applying (17) for $f(t)=t^r$, $0<r\leqslant 1$ we have the following results similar to Lemma 1 and Lemma 2.

LEMMA 3. *Let A and B be positive definite matrices such that $0<mI\leqslant A\leqslant B\leqslant MI\leqslant I$ with $h=M/m$, and let $\alpha\in[0,1]$. Then*

$$A^r\sharp_\alpha B^r \leqslant \exp\left(r\alpha(1-\alpha)\left(1-\frac{1}{h}\right)^2\right)(A\sharp_\alpha B)^r, \qquad 0<r\leqslant 1.$$

LEMMA 4. *Let A and B be positive definite matrices such that $0<mI\preceq_{ols}A\preceq_{ols}B\preceq_{ols}MI\preceq_{ols}I$ with $h=M/m$, and let $\alpha\in[0,1]$. Then for all $k=1,2,\ldots,n$,*

$$\lambda_k(A\sharp_\alpha B)^r \leqslant \exp\left(\alpha(1-\alpha)\left(1-\frac{1}{h^r}\right)^2\right)\lambda_k(A^r\sharp_\alpha B^r), \qquad r\geqslant 1,$$

$$\lambda_k(A^q\sharp_\alpha B^q)^{\frac{1}{q}} \leqslant \exp\left(\frac{1}{p}\alpha(1-\alpha)\left(1-\frac{1}{h^p}\right)^2\right)\lambda_k(A^p\sharp_\alpha B^p)^{\frac{1}{p}}, \qquad 0<q\leqslant p. \tag{18}$$

THEOREM 3. *Let H and K be Hermitian matrices such that $e^m I \preceq_{ols} e^H \preceq_{ols} e^K \preceq_{ols} e^M I \preceq_{ols} I$ for some scalars $m \leqslant M$, and let $\alpha \in [0,1]$. Then for all $p > 0$ and $k = 1,2,\ldots,n$*

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leqslant \exp\left(\frac{1}{p}\alpha(1-\alpha)\left(1-\frac{1}{e^{p(M-m)}}\right)^2\right)\lambda_k(e^{pH}\sharp_\alpha e^{pK})^{\frac{1}{p}},$$

and so, for every unitarily invariant norm $\|\cdot\|_u$

$$\|e^{(1-\alpha)H+\alpha K}\|_u \leqslant \exp\left(\frac{1}{p}\alpha(1-\alpha)\left(1-\frac{1}{e^{p(M-m)}}\right)^2\right)\|(e^{pH}\sharp_\alpha e^{pK})^{\frac{1}{p}}\|_u.$$

Proof. The proof is similar to that of Theorem 1, by replacing A and B with e^H and e^K, and $h = e^{M-m}$ in the inequality (18). □

REMARK 3. Under the different conditions, the different coefficients are not comparable. But it is known that if we have a certain statement under the sandwich condition $0 < sA \leqslant B \leqslant tA$, then the same statement is also true under the condition $0 < mI \leqslant A,B \leqslant MI$ and $0 < mI \leqslant A \leqslant B \leqslant MI \leqslant I$. Hence, we can compare the following special cases:

(1) Comparison of the constants in Theorem 3 and in Theorem 2:

Let $e^m I \preceq_{ols} e^H \preceq_{ols} e^K \preceq_{ols} e^M I \preceq_{ols} I$. Operator monotony of $\log(t)$ leads to $mI \leqslant H \leqslant K \leqslant MI \leqslant I$, and so $mI \leqslant H,K \leqslant MI$. Now by applying Theorem 2 we have

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leqslant K(e^{2p(M-m)},\alpha)^{-\frac{1}{p}}\lambda_k(e^{pH}\sharp_\alpha e^{pK})^{\frac{1}{p}}, \qquad p > 0.$$

Also, by Theorem 3

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leqslant \exp\left(\frac{1}{p}\alpha(1-\alpha)\left(1-\frac{1}{e^{p(M-m)}}\right)^2\right)\lambda_k(e^{pH}\sharp_\alpha e^{pK})^{\frac{1}{p}}, \qquad p > 0.$$

Letting $h = e^{M-m} \geqslant 1$, the following numerical examples show that there is no ordering between these inequalities.

(i) Take $\alpha = \frac{1}{2}$, $p = \frac{1}{2}$ and $h = 2$, then we have

$$K(h^{2p},\alpha)^{-\frac{1}{p}} - \exp\left(\frac{1}{p}\alpha(1-\alpha)\left(1-\frac{1}{h^p}\right)^2\right) \simeq -0.0134963.$$

(ii) Take $\alpha = \frac{1}{2}$, $p = \frac{1}{2}$ and $h = 8$, then we have

$$K(h^{2p},\alpha)^{-\frac{1}{p}} - \exp\left(\frac{1}{p}\alpha(1-\alpha)\left(1-\frac{1}{h^p}\right)^2\right) \simeq 0.0631159.$$

(2) Comparison of the constants in Lemma 3 and in Lemma 1:

Let $0 < mI \leqslant A \leqslant B \leqslant MI \leqslant I$. Then the following sandwich condition is obtained

$$m \leqslant \frac{m}{M} \leqslant 1 \leqslant A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leqslant \frac{M}{m} \leqslant \frac{1}{m}.$$

Now by letting $s = 1$ and $t = \frac{M}{m} = h$ in Lemma 1, we get

$$A^r \sharp_\alpha B^r \leqslant S(h)^r (A \sharp_\alpha B)^r, \qquad 0 < r \leqslant 1. \tag{19}$$

Also, by Lemma 3

$$A^r \sharp_\alpha B^r \leqslant \exp\left(r\alpha(1-\alpha)\left(1 - \frac{1}{h}\right)^2 \right)(A \sharp_\alpha B)^r, \qquad 0 < r \leqslant 1. \tag{20}$$

It is shown in [4, Remark 2.4] that there is no ordering between coefficients of (19) and (20). Therefore, we may conclude evaluation of Lemma 3 and Lemma 1 are different.

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