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CAUCHY AND PÓLYA–SZEGÖ TYPE INEQUALITIES INVOLVING TWO LINEAR ISOTONIC FUNCTIONALS

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Abstract. We consider inequalities which involve two linear isotonic functionals. We give two variants of the Cauchy inequality and few Pólya-Szegö type inequalities in which functions with variable bounds occurred. With help of these inequalities we are able to obtain a new bound for the Chebyshev difference and give some particular cases. Connections of the presented results with earlier results involving fractional integrals are also pointed out.

1. Introduction and preliminaries

The well-known classical Cauchy inequality for two sequences states that if $(a_1,\ldots,a_s)$ and $(b_1,\ldots,b_s)$ are two real s-tuples, then

$$\left(\sum_{k=1}^{s} a_k b_k\right)^2 \leqslant \sum_{k=1}^{s} a_k^2 \sum_{k=1}^{s} b_k^2. \tag{1}$$

The above-mentioned inequality is also known as the Cauchy-Schwarz or the Cauchy-Buniakowski-Schwarz inequality or, shortly, the CBS inequality. There are lots of different versions or/and generalizations of the above inequality. For example, we get the Cauchy inequality for integrals, for positive functionals, for C^*-valued sesquilinear forms etc. This type of inequality has been obtained in different settings such as an inner product space, a C^*-algebra, a semi-inner product C^*-module, a semi-inner product H^*-module etc. We refer reader to papers [5, 6, 8, 9] and references therein.

A research, which has been done parallelly by the investigation of the Cauchy inequality, was connected with the reversed inequalities. Let us point out the so-called Pólya-Szegö inequality which was obtained at 1925 in [14, pp. 71–72].

THEOREM 1. *Let $(a_1,\ldots,a_s)$ and $(b_1,\ldots,b_s)$ be two real s-tuples. If*

$$0 < m \leqslant a_i \leqslant M, \quad 0 < n \leqslant b_i \leqslant N, \quad i \in \{1,2,\ldots,s\}, \tag{2}$$

then

$$\sum_{k=1}^{s} a_k^2 \sum_{k=1}^{s} b_k^2 \leqslant \frac{(mn+MN)^2}{4mnMN}\left(\sum_{k=1}^{s} a_k b_k\right)^2. \tag{3}$$

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The weighted version of the above-mentioned inequality is known as the Greub-Reinboldt inequality. When condition (2) is replaced by the following

$$0 < m \leqslant \frac{a_i}{b_i} \leqslant M \quad \text{for each } i \in \{1,2,\dots,s\},$$

then we are talking about the Cassels inequality. In that case, the constant on the right-hand side is equal to $\frac{(m+M)^2}{4mM}$.

In this paper we continue the research about the Cauchy and the Pólya-Szegö inequalities for two isotonic linear functionals. Also, we will give a new upper bound for the Chebyshev difference by the help of the Pólya-Szegö inequality. Let us mention some definitions and theorems related to that topic.

Let E be a non-empty set and L be a class of real-valued functions on E satisfying that a linear combination of functions from L is also in L and the function 1 belongs to L, $(1(t) = 1$ for $t \in E)$. A functional $A : L \to \mathbb{R}$ is called an isotonic linear functional if it is linear and if nonnegativity of $f \in L$ implies $A(f) \geqslant 0$.

A functional version of the Cauchy inequality is given in the following theorem, [13].

THEOREM 2. (The Cauchy inequality for one functional) *Let f, g, fg, f^2, $g^2 \in L$ and $A : L \to \mathbb{R}$ be a linear isotonic functional. Then*

$$A^2(fg) \leqslant A(f^2)A(g^2). \tag{4}$$

The Cassels type inequality for one functional is given in [6, p. 14]:

THEOREM 3. (The Cassels inequality for one functional) *Let f, g, $h \in L$ such that fgh, f^2h, $g^2h \in L$, $h \geqslant 0$. If $m, M > 0$ are such that*

$$mg \leqslant f \leqslant Mg,$$

then for any linear isotonic functional $A : L \to \mathbb{R}$ with $A(h) > 0$, we have

$$1 \leqslant \frac{A(f^2h)A(g^2h)}{A(fgh)} \leqslant \frac{(M+m)^2}{4mM}. \tag{5}$$

The paper is organized in the following way. After this introductory section we follow with Cauchy type inequalities involving two linear isotonic functionals. The third section is devoted to the reversed inequality, i.e. to the inequality of Pólya-Szegö type. Results, involving an upper bound for the Chebyshev difference, we give in the fourth section. In the last chapter we present several examples with applications.

2. The Cauchy inequality for two functionals

THEOREM 4. *Let f, g, fg, f^2, $g^2 \in L$ and A, B be linear isotonic functionals on L. Then*

$$A(f^2)B(g^2) + A(g^2)B(f^2) \geqslant 2A(fg)B(fg). \tag{6}$$

Proof. Acting on the inequality $(f(x)g(y)-f(y)g(x))^2 \geqslant 0$, i.e.

$$f^2(x)g^2(y)+f^2(y)g^2(x) \geqslant 2f(x)g(y)f(y)g(x)$$

by the functional A with respect to x and using isotonicity and linearity of A we get

$$g^2(y)A(f^2)+f^2(y)A(g^2) \geqslant 2g(y)f(y)A(fg).$$

Acting on the last inequality by B with respect to y we get the wanted inequality. □

REMARK 1. As pointed out by referee, inequality (6) was obtained in paper [4] for the first time.

Obviously, putting in (6) $A=B$ and dividing by 2 we get inequality (4). The Cauchy type inequality for two functionals is also considered in the following theorem.

THEOREM 5. *Let f, g, fg, f^2, $g^2 \in L$ and A, B be linear isotonic functionals on L. Then*

$$\Big[B(1)A(f^2)+A(1)B(f^2)\Big]\Big[B(1)A(g^2)+A(1)B(g^2)\Big] \geqslant \Big[B(1)A(fg)+A(1)B(fg)\Big]^2. \tag{7}$$

Proof. Let us define the pairs x and y as following:

$$x=\left(\sqrt{B(1)A(f^2)},\sqrt{A(1)B(f^2)}\right), \quad y=\left(\sqrt{B(1)A(g^2)},\sqrt{A(1)B(g^2)}\right).$$

Using the discrete Cauchy inequality (1) for the pairs x and y we get

$$\begin{aligned}&\Big[B(1)A(f^2)+A(1)B(f^2)\Big]\Big[B(1)A(g^2)+A(1)B(g^2)\Big]\\ &\qquad \geqslant \Big[B(1)\sqrt{A(f^2)A(g^2)}+A(1)\sqrt{B(f^2)B(g^2)}\Big]^2\\ &\qquad \geqslant \Big[B(1)|A(fg)|+A(1)|B(fg)|\Big]^2 \geqslant \Big[B(1)A(fg)+A(1)B(fg)\Big]^2,\end{aligned}$$

where in the second inequality we use result (4) twice. □

3. Pólya-Szegö type inequalities

In this section we prove several inequalities which can be considered as the Pólya-Szegö type inequalities. We consider a situation when the functions f and g are bounded by functions $\varphi_1,\varphi_2,\psi_1,\psi_2$, i.e. when f and g have variable bounds. This condition was motivated by certain investigation of fractional integrals, see [2, 10, 12].

THEOREM 6. *Let A and B be two isotonic linear functionals on L. Let f, g, φ_1, φ_2, ψ_1, ψ_2 be functions on E such that*

$$(H_1): \qquad 0<\varphi_1 \leqslant f \leqslant \varphi_2,\ 0<\psi_1 \leqslant g \leqslant \psi_2.$$

Then

$$\begin{aligned}\Big(B(1)A(\psi_1\psi_2 f^2)+A(1)B(\psi_1\psi_2 f^2)\Big)\Big(A(1)B(\varphi_1\varphi_2 g^2)+B(1)A(\varphi_1\varphi_2 g^2)\Big)\\ \leqslant \frac{1}{4}\Big(B(1)A((\varphi_1\psi_1+\varphi_2\psi_2)fg)+A(1)B((\varphi_1\psi_1+\varphi_2\psi_2)fg)\Big)^2\end{aligned} \tag{8}$$

and

$$\begin{aligned}A(1)B(1)\Big(A(\psi_1\psi_2 f^2)+A(\varphi_1\varphi_2 g^2)\Big)\Big(B(\psi_1\psi_2 f^2)+B(\varphi_1\varphi_2 g^2)\Big)\\ \leqslant \frac{1}{4}\Big(B(1)A((\varphi_1\psi_1+\varphi_2\psi_2)fg)+A(1)B((\varphi_1\psi_1+\varphi_2\psi_2)fg)\Big)^2\end{aligned} \tag{9}$$

provided all terms are well-defined.

Both inequalities involve

$$\begin{aligned}A(1)B(1)\sqrt{A(\psi_1\psi_2 f^2)A(\varphi_1\varphi_2 g^2)B(\psi_1\psi_2 f^2)B(\varphi_1\varphi_2 g^2)}\\ \leqslant \frac{1}{16}\Big(B(1)A((\varphi_1\psi_1+\varphi_2\psi_2)fg)+A(1)B((\varphi_1\psi_1+\varphi_2\psi_2)fg)\Big)^2\end{aligned} \tag{10}$$

and

$$A(\psi_1\psi_2 f^2)A(\varphi_1\varphi_2 g^2) \leqslant \frac{1}{4}A^2((\psi_1\varphi_1+\psi_2\varphi_2)fg). \tag{11}$$

Proof. From (H_1) we have that $\frac{\varphi_2(x)}{\psi_1(x)}-\frac{f(x)}{g(x)}$ and $\frac{f(x)}{g(x)}-\frac{\varphi_1(x)}{\psi_2(x)}$ are nonnegative. After multiplying them and after a simple calculation one can get that

$$(\varphi_1(x)\psi_1(x)+\varphi_2(x)\psi_2(x))f(x)g(x) \geqslant \psi_1(x)\psi_2(x)f^2(x)+\varphi_1(x)\varphi_2(x)g^2(x).$$

Acting with A we get

$$A((\varphi_1\psi_1+\varphi_2\psi_2)fg) \geqslant A(\psi_1\psi_2 f^2)+A(\varphi_1\varphi_2 g^2).$$

From here

$$B(1)\Big\{A((\varphi_1\psi_1+\varphi_2\psi_2)fg)-A(\psi_1\psi_2 f^2)-A(\varphi_1\varphi_2 g^2)\Big\} \geqslant 0.$$

In the same way we get

$$A(1)\Big\{B((\varphi_1\psi_1+\varphi_2\psi_2)fg)-B(\psi_1\psi_2 f^2)-B(\varphi_1\varphi_2 g^2)\Big\} \geqslant 0.$$

Adding these inequalities and using the AM-GM inequality we get:

$$\begin{aligned}
&B(1)A((\varphi_1\psi_1+\varphi_2\psi_2)fg)+A(1)B((\varphi_1\psi_1+\varphi_2\psi_2)fg)\\
&\quad\geqslant \Big\{B(1)A(\psi_1\psi_2 f^2]+A(1)B(\psi_1\psi_2 f^2)\Big\}+\Big\{A(1)B(\varphi_1\varphi_2 g^2)+B(1)A(\varphi_1\varphi_2 g^2)\Big\}\\
&\quad\geqslant 2\sqrt{B(1)A(\psi_1\psi_2 f^2)+A(1)B(\psi_1\psi_2 f^2)}\sqrt{A(1)B(\varphi_1\varphi_2 g^2)+B(1)A(\varphi_1\varphi_2 g^2)}
\end{aligned}$$

from where we easily obtain inequality (8).

Using once more the AM-GM inequality we get

$$B(1)A((\varphi_1\psi_1+\varphi_2\psi_2)fg)+A(1)B((\varphi_1\psi_1+\varphi_2\psi_2)fg)$$

$$\geqslant 2\sqrt{A(1)B(1)2\sqrt{A(\psi_1\psi_2 f^2)B(\psi_1\psi_2 f^2)}\cdot 2\sqrt{A(\varphi_1\varphi_2 g^2)B(\varphi_1\varphi_2 g^2)})}$$

from where we easily get (10). If we put $A=B$ we get

$$A(\psi_1\psi_2 f^2)A(\varphi_1\varphi_2 g^2)\leqslant \frac{1}{4}A^2((\psi_1\varphi_1+\psi_2\varphi_2)fg).$$

If the summands are ordered in another way and then the AM-GM inequality is used, we get

$$\begin{aligned}
&B(1)A((\varphi_1\psi_1+\varphi_2\psi_2)fg)+A(1)B((\varphi_1\psi_1+\varphi_2\psi_2)fg)\\
&\quad\geqslant B(1)\Big\{A(\psi_1\psi_2 f^2)+A(\varphi_1\varphi_2 g^2)\Big\}+A(1)\Big\{B(\psi_1\psi_2 f^2)+B(\varphi_1\varphi_2 g^2)\Big\}\\
&\quad\geqslant 2\sqrt{A(1)B(1)\Big\{A(\psi_1\psi_2 f^2)+A(\varphi_1\varphi_2 g^2)\Big\}\Big\{B(\psi_1\psi_2 f^2)+B(\varphi_1\varphi_2 g^2)\}}
\end{aligned}$$

from which inequality (9) follows. □

Inequality (11) is in fact, the functional version of the Polya-Szegö inequality when the functions f and g have variable bounds.

In the following corollary we discuss a case when bounds for the functions f and g are constants. Then it will be clear why the inequalities from Theorem 6 are called Polya-Szegö type inequalities.

COROLLARY 1. *Let A and B be two isotonic linear functionals on L. Let f, g be functions on E such that*

$$(H_1(c)):\qquad 0<m\leqslant f(t)\leqslant M,\ 0<n\leqslant g(t)\leqslant N \textit{ for } t\in E,$$

for some real numbers m, M, n and N. Then

$$\begin{aligned}
\Big(B(1)A(fg)+A(1)B(fg)\Big)^2 &\leqslant \Big(B(1)A(f^2)+A(1)B(f^2)\Big)\Big(A(1)B(g^2)+B(1)A(g^2)\Big)\\
&\leqslant \frac{(mn+MN)^2}{4mnMN}\Big(B(1)A(fg)+A(1)B(fg)\Big)^2, \qquad (12)
\end{aligned}$$

provided all terms are well-defined.

In particular, the functional variant of the Pólya-Szegö result is valid:

$$A(f^2)A(g^2) \leqslant \frac{(mn+MN)^2}{4mnMN} A^2(fg). \tag{13}$$

Moreover,

$$A(1)B(1)\sqrt{A(f^2)A(g^2)B(f^2)B(g^2)} \leqslant \frac{(mn+MN)^2}{16mnMN}\Big(B(1)A(fg)+A(1)B(fg)\Big)^2.$$

Proof. Putting in (8):

$$\varphi_1 = m, \quad \varphi_2 = M, \quad \psi_1 = n, \quad \psi_2 = N,$$

we get the most right inequality in (12). The most left inequality in (12) is the Cauchy inequality (7). Putting the same constants in (10) we obtain the last inequality in Corollary. Inequality (13) follows from (12) for $A = B$. □

Our attempts in a construction of the reversed Cauchy inequalities lead us to the following result.

LEMMA 1. *Let the assumptions of Theorem 6 hold. Then*

$$A(f^2)B(g^2) \leqslant A\left(\frac{\varphi_2 fg}{\psi_1}\right) B\left(\frac{\psi_2 fg}{\varphi_1}\right),$$

provided all terms are well-defined.

Proof. From (H_1) we have that $\frac{\varphi_2(x)}{\psi_1(x)} - \frac{f(x)}{g(x)} \geqslant 0$ and $f^2(x) \leqslant \frac{\varphi_2(x)f(x)g(x)}{\psi_1(x)}$. Acting with A and B we get

$$A(f^2) \leqslant A\left(\frac{\varphi_2 fg}{\psi_1}\right), \quad B(g^2) \leqslant B\left(\frac{\psi_2 fg}{\varphi_1}\right).$$

By multiplying both inequalities the proof is over. □

COROLLARY 2. *Let the assumptions of Corollary 1 hold. Then*

$$A(f^2)B(g^2) \leqslant \frac{MN}{mn} A(fg)B(fg),$$

provided all terms are well-defined. In particular,

$$A(f^2)A(g^2) \leqslant \frac{MN}{mn} A^2(fg). \tag{14}$$

REMARK 2. Note that inequality (13) is better than (14) , because

$$\frac{(mn+MN)^2}{4mMnN} = \frac{1}{4}\left(\frac{mn}{MN}+2+\frac{MN}{mn}\right) \leqslant \frac{1}{4}\left(3+\frac{MN}{mn}\right) \leqslant \frac{1}{4}\cdot 4\frac{MN}{mn} = \frac{MN}{mn}.$$

Note also that if $0 < m \leqslant f(x) \leqslant M$ by putting $g(x) = 1$ we get

$$A(1)A(f^2) \leqslant \frac{(m+M)^2}{4mM}A^2(f).$$

THEOREM 7. *Let the assumptions of Theorem 6 hold. Then*

$$A(f^2)B(\psi_1\psi_2)B(g^2)A(\varphi_1\varphi_2) \leqslant \frac{1}{4}\Big(A(\varphi_1 f)B(\psi_1 g)+A(\varphi_2 f)B(\psi_2 g)\Big)^2, \tag{15}$$

provided all terms are well-defined.

If the hypothesis $(H_1(c))$ is fulfilled, then

$$A(f^2)B(1)B(g^2)A(1) \leqslant \frac{(mn+MN)^2}{4mnMN}A^2(f)B^2(g). \tag{16}$$

Proof. From (H_1) we have that $\frac{\varphi_2(x)}{\psi_1(y)} - \frac{f(x)}{g(y)}$ and $\frac{f(x)}{g(y)} - \frac{\varphi_1(x)}{\psi_2(y)}$ are nonnegative. After multiplying them and making a simple calculation one can get that

$$\varphi_1(x)f(x)\psi_1(y)g(y) + \varphi_2(x)f(x)\psi_2(y)g(y) \geqslant \psi_1(y)\psi_2(y)f^2(x)+\varphi_1(x)\varphi_2(x)g^2(y).$$

Acting by A with respect of x and B with respect of y we get

$$A(\varphi_1 f)B(\psi_1 g)+A(\varphi_2 f)B(\psi_2 g) \geqslant A(f^2)B(\psi_1\psi_2)+B(g^2)A(\varphi_1\varphi_2).$$

Using the AM-GM inequality we obtain

$$A(\varphi_1 f)B(\psi_1 g)+A(\varphi_2 f)B(\psi_2 g) \geqslant 2\sqrt{A(f^2)B(\psi_1\psi_2)B(g^2)A(\varphi_1\varphi_2)},$$

from which the first inequality of Theorem is easily obtained. The last inequality follows by using the constant bounds m,n,M,N for f and g. $\square$

LEMMA 2. *Let the assumptions of Theorem 6 hold. Then*

$$A(fg)B(\varphi_1\psi_1+\varphi_2\psi_2)+B(fg)A(\varphi_1\psi_1+\varphi_2\psi_2)$$

$$\geqslant 4\Big\{A(\varphi_1 g)B(\varphi_2 g)A(\varphi_2 g)B(\varphi_1 g)A(\psi_1 f)B(\psi_2 f)A(\psi_2 f)B(\psi_1 f)\Big\}^{\frac{1}{4}},$$

provided all terms are well-defined. In particular,

$$A(fg)A(\varphi_1\psi_1+\varphi_2\psi_2) \geqslant 2\Big\{A(\varphi_1 g)A(\varphi_2 g)A(\psi_1 f)A(\psi_2 f)\Big\}^{\frac{1}{2}}.$$

Proof. Since $\dfrac{\varphi_2(x)}{\psi_1(y)} - \dfrac{f(x)}{g(y)}$ and $\dfrac{f(y)}{g(x)} - \dfrac{\varphi_1(y)}{\psi_2(x)}$ are nonnegative, acting by A and B we get

$$A(\varphi_2\psi_2)B(fg) - A(\varphi_2 g)B(\varphi_1 g) - A(\psi_2 f)B(\psi_1 f) + A(fg)B(\varphi_1\psi_1) \geqslant 0.$$

Changing the places of A and B we have

$$B(\varphi_2\psi_2)A(fg) - A(\varphi_1 g)B(\varphi_2 g) - A(\psi_1 f)B(\psi_2 f) + B(fg)A(\varphi_1\psi_1) \geqslant 0.$$

Summing the above two inequalities we get

$$A(fg)B(\varphi_1\psi_1 + \varphi_2\psi_2) + B(fg)A(\varphi_1\psi_1 + \varphi_2\psi_2) - A(\varphi_1 g)B(\varphi_2 g)$$
$$- A(\varphi_2 g)B(\varphi_1 g) - A(\psi_1 f)B(\psi_2 f) - A(\psi_2 f)B(\psi_1 f) \geqslant 0.$$

By applying the AM-GM inequality we obtain the first inequality in Lemma.

Putting $A = B$ we get

$$A(fg)A(\varphi_1\psi_1 + \varphi_2\psi_2) - A(\varphi_1 g)A(\varphi_2 g) - A(\psi_1 f)A(\psi_2 f) \geqslant 0$$

and after applying the AM-GM inequality we have

$$A(fg)A(\varphi_1\psi_1 + \varphi_2\psi_2) \geqslant 2\Big\{A(\varphi_1 g)A(\varphi_2 g)A(\psi_1 f)A(\psi_2 f)\Big\}^{\frac{1}{2}}. \quad \square$$

Replacing $\varphi_1 = m$, $\varphi_2 = M$, $\psi_1 = n$, $\psi_2 = N$ in the above inequalities we get the following corollary.

COROLLARY 3. *Let the assumptions of Corollary 1 hold. Then*

$$A(fg)(mn + MN) \geqslant mMA(g)^2 + nNA(f)^2$$

and

$$A^2(g)A^2(f) \leqslant \frac{(mn+MN)^2}{4mMnN}A^2(fg),$$

provided all terms are well-defined.

This result differs from Pólya-Szegö type result (13), because here on the left-hand side of the inequality the term $A^2(g)A^2(f)$ stays.

THEOREM 8. *Let A and B be two isotonic linear functionals on L and let $p,q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let f, g be positive functions on E and let m, M be positive real numbers such that $0 < m \leqslant \frac{f}{g} \leqslant M$. Then*

$$A(f)B(g) \leqslant \frac{2^{p-1}M^p}{p(M+1)^p}A(f^p + g^p)B(1) + \frac{2^{q-1}}{q(m+1)^q}B(f^q + g^q)A(1),$$

provided all terms are well-defined.

Proof. From the inequality $\frac{f(x)}{g(x)} \leqslant M$ we get

$$(M+1)f(x) \leqslant M(f+g)(x) \quad \text{and} \quad f^p(x) \leqslant \frac{M^p(f+g)(x)^p}{(M+1)^p}.$$

Acting with A we get

$$A(f^p) \leqslant \frac{M^p}{(M+1)^p} A((f+g)^p).$$

On the other hand $(m+1)g(x) \leqslant (f+g)(x)$ and

$$B(g^q) \leqslant \frac{1}{(m+1)^q} B((f+g)^q).$$

Acting by A with respect to x and by B with respect to y on the Young inequality

$$f(x)g(y) \leqslant \frac{f^p(x)}{p} + \frac{g^q(y)}{q}$$

we get

$$A(f)B(g) \leqslant \frac{A(f^p)}{p} B(1) + \frac{B(g^q)}{q} A(1)$$

and

$$A(f)B(g) \leqslant \frac{M^p}{p(M+1)^p} A((f+g)^p)B(1) + \frac{1}{q(m+1)^q} B((f+g)^q)A(1).$$

Using inequality $(a+b)^r \leqslant 2^{r-1}(a^r+b^r)$ we get the wanted inequality. □

4. Bounds for the Chebyshev difference

Let us mention some of the classical results, in which bounds for the Chebyshev difference are given. We write it in the functional variant, but as a historical remark, let us say that the first result of that type was an integral inequality proved in the thirties of the XXth century, nowdays known as the Grüss inequality.

The difference $T_A(f,g)$ given by

$$T_A(f,g) = A(fg)A(1) - A(f)A(g)$$

is called the Chebyshev difference or the Chebyshev functional. The most known estimation for $T_A(f,g)$ is the Grüss inequality which states that

$$T_A^2(f,g) \leqslant T_A(f,f)T_A(g,g)$$

and if the numbers m, M, n, N are such that $m \leqslant f(x) \leqslant M$, $n \leqslant g(x) \leqslant N$ for all $x \in [a,b]$, then

$$|T_A(f,g)| \leqslant \frac{1}{4}(M-m)(N-n)A^2(1).$$

Very recently, in [10, 11], the Grüss inequality and other bounds for the Chebyshev difference involving two linear functionals A and B were considered. Namely, the following theorem was proved in [10].

THEOREM 9. (The Grüss inequality for two functionals) *Let A and B be isotonic linear functionals on L. If f, g are functions such that f, g, $fg \in L$, then*

$$T(f,g)^2 \leqslant T(f,f)T(g,g), \tag{17}$$

where

$$T(f,g) = A(1)B(fg) + B(1)A(fg) - A(f)B(g) - A(g)B(f).$$

In this section we give a bound for $T(f,g)$ via the Pólya-Szegö inequality (8).

THEOREM 10. *Let A and B be two isotonic linear functionals on L. Let f, g, φ_1, φ_2, ψ_1, ψ_2 be functions on E such that (H_1) is satisfied with*

$$A(1)B(\varphi_1\varphi_2) + B(1)A(\varphi_1\varphi_2) \neq 0, \quad A(1)B(\psi_1\psi_2) + B(1)A(\psi_1\psi_2) \neq 0.$$

Then

$$T^2(f,g) \leqslant D(f,\varphi_1,\varphi_2) \cdot D(g,\psi_1,\psi_2), \tag{18}$$

where

$$D(u,v,w) = \frac{\Big\{B(1)A((v+w)u) + A(1)B((v+w)u)\Big\}^2}{4\Big[A(1)B(vw) + B(1)A(vw)\Big]} - 2A(u)B(u),$$

provided all terms are well-defined. In particular,

$$|A(1)A(fg) - A(f)A(g)| \leqslant \Big(G(f,\varphi_1,\varphi_2)G(g,\psi_1,\psi_2)\Big)^{\frac{1}{2}}, \tag{19}$$

where

$$G(u,v,w) = \frac{A(1)A^2((v+w)u)}{4A(vw)} - A^2(u).$$

Moreover, if the hypothesis $H_1(c)$ is fulfilled, then

$$|A(1)A(fg) - A(f)A(g)| \leqslant \frac{(M-m)(N-n)}{4\sqrt{mnMN}} A(f)A(g). \tag{20}$$

Proof. Putting in inequality (8) $g = \psi_1 = \psi_2 = 1$ we have

$$\Big(B(1)A(f^2) + A(1)B(f^2)\Big)\Big(A(1)B(\varphi_1\varphi_2) + B(1)A(\varphi_1\varphi_2)\Big) \leqslant \frac{1}{4}\Big(B(1)A((\varphi_1+\varphi_2)f) + A(1)B((\varphi_1+\varphi_2)f)\Big)^2,$$

i.e.

$$B(1)A(f^2) + A(1)B(f^2) \leqslant \frac{\Big\{A(1)B((\varphi_1+\varphi_2)f) + B(1)A((\varphi_1+\varphi_2)f)\Big\}^2}{4\Big(A(1)B(\varphi_1\varphi_2) + B(1)A(\varphi_1\varphi_2)\Big)}.$$

Using this estimation and the Grüss inequality for two functionals (17) we get

$$T(f,g)^2 \leqslant T(f,f)T(g,g) \leqslant D(f,\varphi_1,\varphi_2)\cdot D(g,\psi_1,\psi_2).$$

If $A = B$, then inequality (18) collapses in inequality (19), and (20) easily follows from (19). □

REMARK 3. If $A(f) = \int_a^b f(x)dx$, then inequality (20) can be found in the paper [7] as Theorem 1, i.e.

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)\right|$$
$$\leqslant \frac{(M-m)(N-n)}{4(b-a)^2\sqrt{mnMN}}\int_a^b f(x)dx\int_a^b g(x)dx,$$

where f and g are two positive integrable functions such that $0 < m \leqslant f \leqslant M$ and $0 < n \leqslant g \leqslant N$.

5. Examples and applications

We consider the case of piecewise affine, and in particular, step functions φ_1, ψ_1, φ_2, ψ_2. Let $E = \bigcup_{i=1}^s E_i$ and χ_i be the characteristic function of E_i. Let

$$\varphi_1(x) = \sum_{i=1}^s (k_i x + l_i)\chi_i(x), \quad \varphi_2(x) = \sum_{i=1}^s (K_i x + L_i)\chi_i(x), \tag{21}$$

$$\psi_1(x) = \sum_{i=1}^s (p_i x + r_i)\chi_i(x), \quad \psi_2(x) = \sum_{i=1}^s (P_i x + R_i)\chi_i(x), \tag{22}$$

with k_i, K_i, l_i, L_i, p_i, P_i, r_i, $R_i \in \mathbb{R}$ for $i = 1,\ldots,s$.

We say that the hypothesis $(H_1(l))$ is fulfilled if the hypothesis (H_1) with the piecewise affine functions φ_1, ψ_1, φ_2, ψ_2, defined as above is fulfilled. For functions which satisfy the hypothesis $(H_1(l))$ we obtain the following results.

THEOREM 11. *Let A and B be two isotonic linear functionals on L. If the hypothesis $(H_1(l))$ is fulfilled with $A(1)B(\varphi_1\varphi_2) + B(1)A(\varphi_1\varphi_2) \neq 0$, $A(1)B(\psi_1\psi_2) + B(1)A(\psi_1\psi_2) \neq 0$, then*

$$|A(1)B(fg) + A(fg)B(1) - A(f)B(g) - A(g)B(f)| \leqslant D(f,\varphi_1,\varphi_2)\cdot D(g,\psi_1,\psi_2), \tag{23}$$

where

$$D(f,\varphi_1,\varphi_2) = \frac{1}{4\left[A(1)B(\varphi_1\varphi_2) + B(1)A(\varphi_1\varphi_2)\right]}$$
$$\times\left\{B(1)\left(\sum_{i=1}^s (k_i+K_i)A(xf\chi_i) + \sum_{i=1}^s (l_i+L_i)A(f\chi_i)\right)\right.$$
$$\left. + A(1)\left(\sum_{i=1}^s (k_i+K_i)B(xf\chi_i) + \sum_{i=1}^s (l_i+L_i)B(f\chi_i)\right)\right\}^2 - 2A(f)B(f)$$

with

$$A(\varphi_1\varphi_2)=\sum_{i=1}^{s}k_iK_iA(x^2\chi_i)+\sum_{i=1}^{s}(k_iL_i+K_il_i)A(x\chi_i)+\sum_{i=1}^{s}l_iL_iA(\chi_i)$$

and similar for $B(\varphi_1\varphi_2)$ provided all terms are well-defined.

Proof. It is a consequence of Theorem 10 applied on the functions φ_1, ψ_1, φ_2, ψ_2 defined in (21) and (22). □

REMARK 4. It is interesting to see particular cases, i.e. when the bound-functions φ_1, ψ_1, φ_2, ψ_2 are step or piecewise linear functions. Namely, if

$$\varphi_1=\sum_{i=1}^{s}l_i\chi_i,\quad \varphi_2=\sum_{i=1}^{s}L_i\chi_i,\quad \psi_1=\sum_{i=1}^{s}r_i\chi_i,\quad \psi_2=\sum_{i=1}^{s}R_i\chi_i,$$

then, under the assumptions of Theorem 11, inequality (23) holds with

$$D(f,\varphi_1,\varphi_2)=\frac{\left\{B(1)\sum_{i=1}^{s}(l_i+L_i)A(f\chi_i)+A(1)\sum_{i=1}^{s}(l_i+L_i)B(f\chi_i)\right\}^2}{4\left[A(1)\sum_{i=1}^{s}l_iL_iB(\chi_i)+B(1)\sum_{i=1}^{s}l_iL_iA(\chi_i)\right]}-2A(f)B(f).$$

If φ_1, ψ_1, φ_2, ψ_2 are piecewise linear functions, i.e.

$$\varphi_1(x)=\sum_{i=1}^{s}k_ix\chi_i(x),\ \varphi_2(x)=\sum_{i=1}^{s}K_ix\chi_i(x),\ \psi_1(x)=\sum_{i=1}^{s}p_ix\chi_i(x),\ \psi_2(x)=\sum_{i=1}^{s}P_ix\chi_i(x),$$

then $D(f,\varphi_1,\varphi_2)$ becomes

$$D(f,\varphi_1,\varphi_2)=\frac{\left\{\sum_{i=1}^{s}(k_i+K_i)\Big(B(1)A(xf\chi_i)+A(1)B(xf\chi_i)\Big)\right\}^2}{4\sum_{i=1}^{s}k_iK_i\Big(A(1)B(x^2\chi_i)+B(1)A(x^2\chi_i)\Big)}-2A(f)B(f).$$

6. Applications for different fractional integral operators

In the paper [12] the authors gave some results involving the Riemann-Liouville fractional integral operators $A=R^{\alpha}_{0,t}$ and $B=R^{\beta}_{0,t}$, where

$$R^{\alpha}_{0,t}\{f\}(t):=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds.$$

For example, the results given in Lemma 3.1, Corollary 3.1, Lemma 3.3 and Corollary 3.2 from that paper are, in fact, our results (11), (13), (15) and (16). As Theorem 3.6 in [12] we can find the following Chebyshev type inequality which we repeat here in the language of functionals.

Let f and g be two positive integrable functions on $[0,\infty)$. Assume that there exist four positive integrable functions φ_1, ψ_1, φ_2, ψ_2 satisfying (H_1). Then for $t>0$ and α, $\beta>0$, the following inequality is true

$$\begin{aligned} &|A(1)B(fg)+B(1)A(fg)-A(f)B(g)-A(g)B(f)|^2 \\ &\quad \leqslant \Big(G_1(f,\varphi_1,\varphi_2)(t)+G_2(f,\varphi_1,\varphi_2)(t)\Big)\cdot\Big(G_1(g,\psi_1,\psi_2)(t)+G_2(g,\psi_1,\psi_2)(t)\Big), \end{aligned} \tag{24}$$

where

$$G_1(u,v,w)(t)=B(1)\frac{A^2((v+w)u)}{4A(vw)}-A(u)B(u),$$

$$G_2(u,v,w)(t)=A(1)\frac{B^2((v+w)u)}{4B(vw)}-A(u)B(u), \quad A=R^{\alpha}_{0,t}, \quad B=R^{\beta}_{0,t}.$$

Comparing the right-hand side of (24) with the right-hand side of (18) we get that our upper bound given in (18) is smaller (so, better) than the upper bound given in (24). Also, in [12] examples including the step functions φ_1, ψ_1, φ_2, ψ_2 are given, but again, we point out that the recent upper bound for the Chebyshev difference given in Remark 4 for $s=1$ are better than bound from [12].

Investigation which is very similar to investigation described in [12] can be found in the paper [2] but for the generalized Riemann-Liouville k-fractional integral operators. In that paper we can find the Pólya-Szegö inequality (15) and the Chebyshev inequality (24) for two generalized Riemann-Liouville k-fractional integral operators $R^{\alpha,r}_{a,k}$ and $R^{\beta,r}_{a,k}$. Also, the example with linear functions φ_1, ψ_1, φ_2, ψ_2 is considered.

Pólya-Szegö type inequalities for two q-analogue of the Saigo fractional integrals $I_q^{\alpha,\beta,\eta}$ are considered in the paper [1], but there we can find only results when functions φ_1, ψ_1, φ_2, ψ_2 are constants.

Finally, Pólya-Szegö type inequalities for two Hadamard fractional integrals $D^{-\alpha}_{1,t}$ and $D^{-\beta}_{1,t}$ are given in the paper [3]. In the same paper result related to Theorem 8 is given.

As we can see, this paper contains a unified treatment for several classes of fractional integral operators as well for other isotonic linear functionals. Also, we obtain several new inequalities which are variants of the Pólya-Szegö inequality and give an upper bound for the Chebyshev difference which is better than bounds recently occured in the observed particular cases.

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