

A NEW FORM OF HILBERT INTEGRAL INEQUALITY

TSERENDORJ BATBOLD AND LAITH E. AZAR

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Abstract. In this paper by estimating the triple integral $\int_0^\infty \int_0^\infty \int_0^\infty \frac{f(x,y)g(z)}{(x+y+z)^k} dx dy dz$, we introduce a new form of the Hilbert inequality for three variables with a best constant factor. The reverse form and some equivalent forms are also considered.

1. Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $f, g \geq 0$ satisfy $0 < \int_0^\infty f^p(x) dx < \infty$, $0 < \int_0^\infty g^q(x) dx < \infty$, then the following equivalent inequalities hold

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\frac{\pi}{p})} \left(\int_0^\infty f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(x) dx \right)^{\frac{1}{q}}, \quad (1)$$

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx \right)^p dy < \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^p \int_0^\infty f^p(x) dx, \quad (2)$$

where the constant factors $\frac{\pi}{\sin(\frac{\pi}{p})}$ and $\left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^p$ are the best possible in (1) and (2) respectively. Inequality (1) was first studied by D. Hilbert at the end of the 19th century, hence, in his honor, it is referred to as the Hilbert inequality.

After its discovery, the Hilbert inequality was studied by numerous authors, who either reproved it using various techniques, or applied and generalized it in many different ways. Such generalizations included inequalities with more general kernels, weight functions and integration sets, extension to a multidimensional case, equivalent forms, and so forth. The resulting relations are usually referred to as the Hilbert-type inequalities. For a comprehensive inspection of the initial development of the Hilbert inequality, the reader is referred to classical monographs [4] and [11].

Although classical, the Hilbert inequality is still of interest to numerous mathematicians. Nowadays, more than a century after its discovery, this problem area offers

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diverse possibilities for generalizations and extensions (see [1]–[3], [5]–[7], [9], [10], [12]–[14]). The most important recent results regarding Hilbert-type inequalities are collected in a monograph [8], that provides a unified treatment of such inequalities.

In this paper by estimating the triple integral $\int_0^\infty \int_0^\infty \int_0^\infty \frac{f(x,y)g(z)}{(x+y+z)^\lambda} dx dy dz$, we introduce a new Hilbert-type inequality for three variables with a best constant factor. The reverse form and some equivalent forms are also considered.

The paper is organized in the following way: After this Introduction, in Section 2 we present the preliminaries and lemmas that are needed in our work. In Section 3, we introduce a new Hilbert-type inequality for triple integrals and give the reverse form of this inequality. In addition, we obtain inequalities with the best possible constant factors. Finally, in Section 4 we introduce some equivalent forms of the obtained inequalities.

2. Preliminaries and lemmas

Recall that the Gamma function $\Gamma(\theta)$ and the Beta function $B(\mu, \nu)$ are defined respectively by

$$\Gamma(\theta) = \int_0^\infty t^{\theta-1} e^{-t} dt, \quad \theta > 0,$$

$$B(\mu, \nu) = \int_0^\infty \frac{t^{\mu-1}}{(t+1)^{\mu+\nu}} dt, \quad \mu, \nu > 0,$$

and they satisfy the following relation

$$B(\mu, \nu) = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}.$$

By the definition of the Gamma function, the following equality holds

$$\frac{1}{(x+y+z)^\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-(x+y+z)t} dt. \quad (3)$$

To prove our main results we need the next two lemmas:

LEMMA 1. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $t > 0$, $\alpha < \frac{2}{q}$ and $f(x, y)$ is a positive function defined and integrable on $(0, \infty) \times (0, \infty)$, then the following inequality holds*

$$\begin{aligned} & \left(\int_0^\infty \int_0^\infty e^{-t(x+y)} f(x, y) dx dy \right)^p \\ & \leq (t^{\alpha q - 2} \Gamma(2 - \alpha q))^{\frac{p}{q}} \int_0^\infty \int_0^\infty (x+y)^{\alpha p} e^{-t(x+y)} f^p(x, y) dx dy. \end{aligned} \quad (4)$$

Proof. By the help of Hölder inequality, we obtain

$$\begin{aligned} & \left(\int_0^\infty \int_0^\infty e^{-t(x+y)} f(x,y) dx dy \right)^p \\ &= \left(\int_0^\infty \int_0^\infty \left\{ (x+y)^{-\alpha} e^{-\frac{t(x+y)}{q}} \right\} \left\{ (x+y)^\alpha e^{-\frac{t(x+y)}{p}} f(x,y) dx dy \right\} \right)^p \\ &\leq \left(\int_0^\infty \int_0^\infty (x+y)^{-\alpha q} e^{-t(x+y)} dx dy \right)^{\frac{p}{q}} \int_0^\infty \int_0^\infty (x+y)^{\alpha p} e^{-t(x+y)} f^p(x,y) dx dy. \end{aligned}$$

We compute the first integral on the right hand side of the above estimate by using the substitutions $y = ux$ and $x = \frac{v}{t(u+1)}$ respectively. Then we have

$$\begin{aligned} \int_0^\infty \int_0^\infty (x+y)^{-\alpha q} e^{-t(x+y)} dx dy &= \int_0^\infty (1+u)^{-\alpha q} \int_0^\infty x^{1-\alpha q} e^{-tx(1+u)} dx du \\ &= t^{\alpha q-2} \int_0^\infty (1+u)^{-2} du \int_0^\infty v^{1-\alpha q} e^{-v} dv \\ &= t^{\alpha q-2} \Gamma(2-\alpha q). \end{aligned}$$

Thus

$$\begin{aligned} & \left(\int_0^\infty \int_0^\infty e^{-t(x+y)} f(x,y) dx dy \right)^p \\ &\leq (t^{\alpha q-2} \Gamma(2-\alpha q))^{\frac{p}{q}} \int_0^\infty \int_0^\infty (x+y)^{\alpha p} e^{-t(x+y)} f^p(x,y) dx dy. \end{aligned}$$

The proof is now finished. \square

LEMMA 2. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $t > 0$, $\theta > -\frac{1}{p}$ and $g(z)$ is a positive function defined and integrable on $(0, \infty)$, then the following inequality holds

$$\int_0^\infty e^{-tz} g(z) dz \leq \left[t^{-\theta p-1} \Gamma(\theta p + 1) \right]^{\frac{1}{p}} \left(\int_0^\infty z^{-\theta q} e^{-tz} g^q(z) dz \right)^{\frac{1}{q}}. \tag{5}$$

Proof. Using Hölder inequality, we find

$$\begin{aligned} \int_0^\infty e^{-tz} g(z) dz &= \int_0^\infty \left\{ z^\theta e^{-\frac{tz}{p}} \right\} \left\{ z^{-\theta} e^{-\frac{tz}{q}} g(z) \right\} dz \\ &\leq \left(\int_0^\infty z^{\theta p} e^{-tz} dz \right)^{\frac{1}{p}} \left(\int_0^\infty z^{-\theta q} e^{-tz} g^q(z) dz \right)^{\frac{1}{q}} \\ &= \left[t^{-\theta p-1} \Gamma(\theta p + 1) \right]^{\frac{1}{p}} \left(\int_0^\infty z^{-\theta q} e^{-tz} g^q(z) dz \right)^{\frac{1}{q}}. \quad \square \end{aligned}$$

REMARK 1. Note that if $0 < p < 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then by the reverse form of Hölder inequality we may prove that the reverse form of both (4) and (5) holds.

3. Main results

In this section, we introduce the main two results in this paper. In Theorem 1 we introduce a new Hilbert-type inequality for triple integrals and in Theorem 2 we give the reverse form of this inequality.

THEOREM 1. Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$, $\gamma \in \left(\frac{-\lambda}{p}, \frac{\lambda}{q}\right)$, suppose that $f(x, y)$ is a positive function defined on $(0, \infty) \times (0, \infty)$, and $g(z) > 0$ on $(0, \infty)$.

If $\int_0^\infty \int_0^\infty (x+y)^{2p-\lambda-p\gamma-2} f^p(x, y) dx dy < \infty$ and $\int_0^\infty z^{q+q\gamma-\lambda-1} g^q(z) dz < \infty$, then

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{f(x, y)g(z)}{(x+y+z)^\lambda} dx dy dz \leq C \left(\int_0^\infty \int_0^\infty (x+y)^{2p-\lambda-p\gamma-2} f^p(x, y) dx dy \right)^{\frac{1}{p}} \left(\int_0^\infty z^{q+q\gamma-\lambda-1} g^q(z) dz \right)^{\frac{1}{q}}, \quad (6)$$

where the constant $C = B\left(\frac{\lambda}{p} + \gamma, \frac{\lambda}{q} - \gamma\right)$ is the best possible. In particular, we find:

1) if $\lambda = 1$, $\gamma = 0$ and $p = q = 2$, then (6) becomes

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{f(x, y)g(z)}{x+y+z} dx dy dz \leq \frac{\pi}{\sin \frac{\pi}{p}} \left(\int_0^\infty \int_0^\infty (x+y) f^2(x, y) dx dy \right)^{\frac{1}{2}} \left(\int_0^\infty g^2(z) dz \right)^{\frac{1}{2}},$$

2) if $\lambda = 2$, $\gamma = 0$ and $p = q = 2$, then (6) becomes

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{f(x, y)g(z)}{(x+y+z)^2} dx dy dz \leq \left(\int_0^\infty \int_0^\infty f^2(x, y) dx dy \right)^{\frac{1}{2}} \left(\int_0^\infty \frac{1}{z} g^2(z) dz \right)^{\frac{1}{2}}.$$

Proof. By using (3) and applying Hölder inequality, we have

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty \int_0^\infty \frac{f(x, y)g(z)}{(x+y+z)^\lambda} dx dy dz \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty \int_0^\infty \int_0^\infty f(x, y)g(z) \left(\int_0^\infty t^{\lambda-1} e^{-(x+y+z)t} dt \right) dx dy dz \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty \left(t^{\frac{\lambda-1}{p} + \gamma} \int_0^\infty \int_0^\infty e^{-(x+y)t} f(x, y) dx dy \right) \left(t^{\frac{\lambda-1}{q} - \gamma} \int_0^\infty e^{-zt} g(z) dz \right) dt \\ &\leq \frac{1}{\Gamma(\lambda)} \left(\int_0^\infty t^{\lambda-1+p\gamma} \left(\int_0^\infty \int_0^\infty e^{-(x+y)t} f(x, y) dx dy \right)^p dt \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_0^\infty t^{\lambda-1-q\gamma} \left(\int_0^\infty e^{-zt} g(z) dz \right)^q dt \right)^{\frac{1}{q}}. \end{aligned} \quad (7)$$

By Lemma 1 and Lemma 2, we obtain respectively

$$\begin{aligned} & \left(\int_0^\infty \int_0^\infty e^{-(x+y)t} f(x,y) dx dy \right)^p \\ & \leq [t^{\alpha q - 2} \Gamma(2 - \alpha q)]^{\frac{p}{q}} \int_0^\infty \int_0^\infty (x+y)^{\alpha p} e^{-(x+y)t} f^p(x,y) dx dy \end{aligned}$$

and

$$\left(\int_0^\infty e^{-tz} g(z) dz \right)^q \leq [t^{-\theta p - 1} \Gamma(\theta p + 1)]^{\frac{q}{p}} \int_0^\infty z^{-\theta q} e^{-tz} g^q(z) dz.$$

Substituting these two inequalities in (7) we have

$$\begin{aligned} I & \leq \frac{\Gamma(2 - \alpha q)^{\frac{1}{q}} \Gamma(\theta p + 1)^{\frac{1}{p}}}{\Gamma(\lambda)} \\ & \times \left(\int_0^\infty \int_0^\infty (x+y)^{\alpha p} f^p(x,y) \left(\int_0^\infty t^{\lambda + p\gamma + p\alpha - 2p + 1} e^{-(x+y)t} dt \right) dx dy \right)^{\frac{1}{p}} \\ & \times \left(\int_0^\infty z^{-\theta q} g^q(z) \left(\int_0^\infty t^{\lambda - q\gamma - q\theta - q} e^{-tz} dt \right) dz \right)^{\frac{1}{q}}. \end{aligned}$$

Since

$$\begin{aligned} & \int_0^\infty t^{\lambda + p\gamma + p\alpha - 2p + 1} e^{-(x+y)t} dt \\ & = (x+y)^{-\lambda - p\gamma - p\alpha + 2p - 2} \Gamma(\lambda + p\gamma + p\alpha - 2p + 2) \end{aligned}$$

and

$$\int_0^\infty t^{\lambda - q\gamma - q\theta - q} e^{-tz} dt = z^{-1 - \lambda + q\gamma + q\theta + q} \Gamma(\lambda - q\gamma - q\theta - q + 1),$$

it follows that

$$I \leq C_{\alpha, \theta} \left(\int_0^\infty \int_0^\infty (x+y)^{2p - \lambda - p\gamma - 2} f^p(x,y) dx dy \right)^{\frac{1}{p}} \left(\int_0^\infty z^{q + q\gamma - \lambda - 1} g^q(z) dz \right)^{\frac{1}{q}},$$

where

$$C_{\alpha, \theta} = \frac{\Gamma(2 - \alpha q)^{\frac{1}{q}} \Gamma(\theta p + 1)^{\frac{1}{p}} \Gamma(\lambda + p\gamma + p\alpha - 2p + 2)^{\frac{1}{p}} \Gamma(\lambda - q\gamma - q\theta - q + 1)^{\frac{1}{q}}}{\Gamma(\lambda)}.$$

Finally, set $\alpha = \frac{2p - p\gamma - \lambda}{pq}$ and $\theta = \frac{\lambda - q\gamma - q}{pq}$ we obtain inequality (6) with the constant $C_{\frac{2p - p\gamma - \lambda}{pq}, \frac{\lambda - q\gamma - q}{pq}} = C = B \left(\frac{\lambda}{p} + \gamma, \frac{\lambda}{q} - \gamma \right)$. Inequality (6) is proved. It remains to show that the constant factor C in (6) is the best possible. Define the functions $f_\varepsilon(x,y) = 0$, on $(0,1) \times (0,1)$ and $f_\varepsilon(x,y) = (x+y)^{\frac{\lambda - \varepsilon}{p} + \gamma - 2}$ on $[1,\infty) \times [1,\infty)$ and $g_\varepsilon(z) = 0$ for $z \in (0,1)$ and $g_\varepsilon(z) = z^{\frac{\lambda - \varepsilon}{q} - \gamma - 1}$ for $z \in [1,\infty)$ ($0 < \varepsilon < \lambda - q\gamma$). Suppose that the

constant $C = B\left(\frac{\lambda}{p} + \gamma, \frac{\lambda}{q} - \gamma\right)$ is not the best possible, then there exist $0 < K < C$ such that

$$\begin{aligned}
 I &\leq K \left(\int_0^\infty \int_0^\infty (x+y)^{2p-\lambda-p\gamma-2} f_\varepsilon^p(x,y) dx dy \right)^{\frac{1}{p}} \left(\int_0^\infty z^{q+q\gamma-\lambda-1} g_\varepsilon^q(z) dz \right)^{\frac{1}{q}} \\
 &= K \left(\int_1^\infty \int_1^\infty (x+y)^{-\varepsilon-2} dx dy \right)^{\frac{1}{p}} \left(\int_1^\infty z^{-\varepsilon-1} dz \right)^{\frac{1}{q}} \\
 &= K \left(\int_1^\infty x^{-1-\varepsilon} \int_{\frac{1}{x}}^\infty (1+u)^{-\varepsilon-2} du dx \right)^{\frac{1}{p}} \left(\frac{1}{\varepsilon} \right)^{\frac{1}{q}} \\
 &= \frac{K}{(\varepsilon+1)^{\frac{1}{p}}} \left(\int_1^\infty \frac{1}{(x+1)^{1+\varepsilon}} dx \right)^{\frac{1}{p}} \left(\frac{1}{\varepsilon} \right)^{\frac{1}{q}} = \frac{K}{\varepsilon} \frac{1}{2^{\frac{\varepsilon}{p}} (1+\varepsilon)^{\frac{1}{p}}}. \tag{8}
 \end{aligned}$$

On the other hand, if we estimate the left hand side we find (set $z = u(x+y)$)

$$\begin{aligned}
 I &=: \int_0^\infty \int_0^\infty \int_0^\infty \frac{f_\varepsilon(x,y)g_\varepsilon(z)}{(x+y+z)^\lambda} dx dy dz \\
 &= \int_1^\infty \int_1^\infty \int_1^\infty \frac{(x+y)^{\frac{\lambda-\varepsilon}{p}+\gamma-2} z^{\frac{\lambda-\varepsilon}{q}-\gamma-1}}{(x+y+z)^\lambda} dx dy dz \\
 &= \int_1^\infty \int_1^\infty (x+y)^{-\varepsilon-2} \int_{\frac{1}{x+y}}^\infty \frac{u^{\frac{\lambda-\varepsilon}{q}-\gamma-1}}{(u+1)^\lambda} du dx dy \\
 &= \int_1^\infty \int_1^\infty (x+y)^{-\varepsilon-2} \left[\int_0^\infty \frac{u^{\frac{\lambda-\varepsilon}{q}-\gamma-1}}{(u+1)^\lambda} du - \int_0^{\frac{1}{x+y}} \frac{u^{\frac{\lambda-\varepsilon}{q}-\gamma-1}}{(u+1)^\lambda} du \right] dx dy \\
 &= \frac{B\left(\frac{\lambda-\varepsilon}{q} - \gamma, \frac{\lambda}{p} + \gamma + \frac{\varepsilon}{q}\right)}{\varepsilon 2^\varepsilon (1+\varepsilon)} - \int_1^\infty \int_1^\infty (x+y)^{-\varepsilon-2} \int_0^{\frac{1}{x+y}} \frac{u^{\frac{\lambda-\varepsilon}{q}-\gamma-1}}{(u+1)^\lambda} du dx dy \\
 &> \frac{B\left(\frac{\lambda-\varepsilon}{q} - \gamma, \frac{\lambda}{p} + \gamma + \frac{\varepsilon}{q}\right)}{\varepsilon 2^\varepsilon (1+\varepsilon)} - \int_1^\infty \int_1^\infty (x+y)^{-\varepsilon-2} \int_0^{\frac{1}{x+y}} u^{\frac{\lambda-\varepsilon}{q}-\gamma-1} du dx dy \\
 &= \frac{B\left(\frac{\lambda-\varepsilon}{q} - \gamma, \frac{\lambda}{p} + \gamma + \frac{\varepsilon}{q}\right)}{\varepsilon 2^\varepsilon (1+\varepsilon)} - O(1). \tag{9}
 \end{aligned}$$

It is obvious that, when $\varepsilon \rightarrow 0^+$ from (8) and (9) we obtain a contradiction. Thus the proof of the theorem is completed. \square

THEOREM 2. Let $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$, $\gamma \in \left(\frac{-\lambda}{p}, \frac{\lambda}{q}\right)$, suppose that $f(x,y)$ is a positive function defined on $(0, \infty) \times (0, \infty)$, and $g(z) > 0$ on $(0, \infty)$.

If $\int_0^\infty \int_0^\infty (x+y)^{2p-\lambda-p\gamma-2} f^p(x,y) dx dy < \infty$ and $\int_0^\infty z^{q+q\gamma-\lambda-1} g^q(z) dz < \infty$, then

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{f(x,y)g(z)}{(x+y+z)^\lambda} dx dy dz \geq C \left(\int_0^\infty \int_0^\infty (x+y)^{2p-\lambda-p\gamma-2} f^p(x,y) dx dy \right)^{\frac{1}{p}} \left(\int_0^\infty z^{q+q\gamma-\lambda-1} g^q(z) dz \right)^{\frac{1}{q}}, \tag{10}$$

where the constant $C = B\left(\frac{\lambda}{p} + \gamma, \frac{\lambda}{q} - \gamma\right)$ is the best possible.

Proof. By using (3) and applying the reverse Hölder inequality, we get

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty \int_0^\infty \frac{f(x,y)g(z)}{(x+y+z)^\lambda} dx dy dz \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty \int_0^\infty \int_0^\infty f(x,y)g(z) \left(\int_0^\infty t^{\lambda-1} e^{-(x+y+z)t} dt \right) dx dy dz \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty \left(t^{\frac{\lambda-1}{p} + \gamma} \int_0^\infty \int_0^\infty e^{-(x+y)t} f(x,y) dx dy \right) \left(t^{\frac{\lambda-1}{q} - \gamma} \int_0^\infty e^{-zt} g(z) dz \right) dt \\ &\geq \frac{1}{\Gamma(\lambda)} \left(\int_0^\infty t^{\lambda-1+p\gamma} \left(\int_0^\infty \int_0^\infty e^{-(x+y)t} f(x,y) dx dy \right)^p dt \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_0^\infty t^{\lambda-1-q\gamma} \left(\int_0^\infty e^{-zt} g(z) dz \right)^q dt \right)^{\frac{1}{q}}. \end{aligned} \tag{11}$$

By Remark 1, we obtain respectively (remember that $q < 0$)

$$\begin{aligned} &\left(\int_0^\infty \int_0^\infty e^{-(x+y)t} f(x,y) dx dy \right)^p \\ &\geq t^{\alpha p - 2p + 2} \Gamma(2 - \alpha q)^{\frac{p}{q}} \int_0^\infty \int_0^\infty (x+y)^{\alpha p} e^{-(x+y)t} f^p(x,y) dx dy \end{aligned} \tag{12}$$

and

$$\left(\int_0^\infty e^{-z} g(z) dz \right)^q \leq t^{-\theta q - q + 1} \Gamma(\theta p + 1)^{\frac{q}{p}} \int_0^\infty z^{-\theta q} e^{-tz} g^q(z) dz.$$

From the last inequality we find

$$\begin{aligned} &\left(\int_0^\infty t^{\lambda-1-q\gamma} \left(\int_0^\infty e^{-z} g(z) dz \right)^q dt \right)^{\frac{1}{q}} \\ &\geq \Gamma(\theta p + 1)^{\frac{1}{p}} \left(\int_0^\infty t^{\lambda-q\gamma-\theta q} \int_0^\infty z^{-\theta q} e^{-tz} g^q(z) dz dt \right)^{\frac{1}{q}}. \end{aligned} \tag{13}$$

Substituting (12) and (13) in (11) we have

$$\begin{aligned}
 I &\geq \frac{\Gamma(2 - \alpha q)^{\frac{1}{q}} \Gamma(\theta p + 1)^{\frac{1}{p}}}{\Gamma(\lambda)} \\
 &\times \left(\int_0^\infty \int_0^\infty (x+y)^{\alpha p} f^p(x,y) \left(\int_0^\infty t^{\lambda + p\gamma + p\alpha - 2p + 1} e^{-(x+y)t} dt \right) dx dy \right)^{\frac{1}{p}} \\
 &\times \left(\int_0^\infty z^{-\theta q} g^q(z) \left(\int_0^\infty t^{\lambda - q\gamma - q\theta - q} e^{-zt} dt \right) dz \right)^{\frac{1}{q}}.
 \end{aligned}$$

After some calculations as we did in Theorem 1 and setting $\alpha = \frac{2p - p\gamma - \lambda}{pq}$ and $\theta = \frac{\lambda - q\gamma - q}{pq}$ we obtain inequality (10) with the constant $B\left(\frac{\lambda}{p} + \gamma, \frac{\lambda}{q} - \gamma\right)$.

To prove that the constant $B\left(\frac{\lambda}{p} + \gamma, \frac{\lambda}{q} - \gamma\right)$ is the best one, we define the functions $f_\varepsilon(x, y)$ and $g_\varepsilon(z)$ as in Theorem 1. If we assume that we can find a constant $L : L > B\left(\frac{\lambda}{p} + \gamma, \frac{\lambda}{q} - \gamma\right)$ such that (10) is valid when we replace $B\left(\frac{\lambda}{p} + \gamma, \frac{\lambda}{q} - \gamma\right)$ by L , then we obtain

$$\begin{aligned}
 \frac{L 2^{-\frac{\varepsilon}{p}} (1 + \varepsilon)^{-\frac{1}{p}}}{\varepsilon} &\leq \int_1^\infty \int_1^\infty \int_1^\infty \frac{(x+y)^{\frac{\lambda - \varepsilon}{p} + \gamma - 2} z^{\frac{\lambda - \varepsilon}{q} - \gamma - 1}}{(x+y+z)^\lambda} dx dy dz \\
 &= \int_1^\infty \int_1^\infty (x+y)^{-\varepsilon - 2} \int_{\frac{1}{x+y}}^\infty \frac{u^{\frac{\lambda - \varepsilon}{q} - \gamma - 1}}{(u+1)^\lambda} du dx dy \\
 &< \int_1^\infty \int_1^\infty (x+y)^{-\varepsilon - 2} \int_0^\infty \frac{u^{\frac{\lambda - \varepsilon}{q} - \gamma - 1}}{(u+1)^\lambda} du dx dy \\
 &= \frac{B\left(\frac{\lambda - \varepsilon}{q} - \gamma, \frac{\lambda}{p} + \gamma + \frac{\varepsilon}{q}\right)}{\varepsilon 2^\varepsilon (1 + \varepsilon)}.
 \end{aligned}$$

If we let $\varepsilon \rightarrow 0^+$ we obtain a contradiction. Thus the proof of the theorem is completed. \square

4. Equivalent forms

In this section we introduce some equivalent forms of the obtained inequalities in Theorem 1 and Theorem 2. All the inequalities are with a best constant factor.

THEOREM 3. *Under the assumptions of Theorem 1 we obtain the following two inequalities*

$$\begin{aligned}
 &\int_0^\infty z^{(p-1)\lambda - \gamma p - 1} \left(\int_0^\infty \int_0^\infty \frac{f(x,y)}{(x+y+z)^\lambda} dx dy \right)^p dz \\
 &\leq C^p \int_0^\infty \int_0^\infty (x+y)^{2p - \lambda - p\gamma - 2} f^p(x,y) dx dy
 \end{aligned} \tag{14}$$

and

$$\begin{aligned} & \int_0^\infty \int_0^\infty (x+y)^{(q-1)\lambda+\gamma q-2} \left(\int_0^\infty \frac{g(z)}{(x+y+z)^\lambda} dz \right)^q dx dy \\ & \leq C^q \int_0^\infty z^{\gamma q-\lambda+q-1} g(z) dz. \end{aligned} \tag{15}$$

Both of the inequalities (14) and (15) are equivalent to (6) and the constants $C^p = B^p \left(\frac{\lambda}{p} + \gamma, \frac{\lambda}{q} - \gamma \right)$ and $C^q = B^q \left(\frac{\lambda}{p} + \gamma, \frac{\lambda}{q} - \gamma \right)$ are the best possible.

Proof. To prove (14) we set $g(z) = z^{(p-1)\lambda-\gamma p-1} \left(\int_0^\infty \int_0^\infty \frac{f(x,y)}{(x+y+z)^\lambda} dx dy \right)^{p-1}$.

Applying inequality (6) we find

$$\begin{aligned} & \int_0^\infty z^{(p-1)\lambda-\gamma p-1} \left(\int_0^\infty \int_0^\infty \frac{f(x,y)}{(x+y+z)^\lambda} dx dy \right)^p dz \\ & = \int_0^\infty z^{(p-1)\lambda-\gamma p-1} \left(\int_0^\infty \int_0^\infty \frac{f(x,y)}{(x+y+z)^\lambda} dx dy \right)^{p-1} \left(\int_0^\infty \int_0^\infty \frac{f(x,y)}{(x+y+z)^\lambda} dx dy \right) dz \\ & = \int_0^\infty \int_0^\infty \int_0^\infty \frac{f(x,y)g(z)}{(x+y+z)^\lambda} dx dy dz \\ & \leq C \left(\int_0^\infty \int_0^\infty (x+y)^{2p-\lambda-p\gamma-2} f^p(x,y) dx dy \right)^{\frac{1}{p}} \left(\int_0^\infty z^{q+q\gamma-\lambda-1} g^q(z) dz \right)^{\frac{1}{q}} \\ & = C \left(\int_0^\infty \int_0^\infty (x+y)^{2p-\lambda-p\gamma-2} f^p(x,y) dx dy \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_0^\infty z^{(p-1)\lambda-\gamma p-1} \left(\int_0^\infty \int_0^\infty \frac{f(x,y)}{(x+y+z)^\lambda} dx dy \right)^p dz \right)^{\frac{1}{q}}. \end{aligned} \tag{16}$$

Dividing both sides of inequality (16) by $\left(\int_0^\infty z^{(p-1)\lambda-\gamma p-1} \left(\int_0^\infty \int_0^\infty \frac{f(x,y)}{(x+y+z)^\lambda} dx dy \right)^p dz \right)^{\frac{1}{q}}$ we obtain inequality (14). On the other hand, by virtue of Hölder inequality and (14), we find

$$\begin{aligned} & \int_0^\infty \int_0^\infty \int_0^\infty \frac{f(x,y)g(z)}{(x+y+z)^\lambda} dx dy dz \\ & = \int_0^\infty \left(z^{\frac{\lambda}{q}-\gamma-\frac{1}{p}} \int_0^\infty \int_0^\infty \frac{f(x,y)}{(x+y+z)^\lambda} dx dy \right) \left(z^{\frac{-\lambda}{q}+\gamma+\frac{1}{p}} g(z) \right) dz \\ & \leq \left(\int_0^\infty z^{(p-1)\lambda-\gamma p-1} \left(\int_0^\infty \int_0^\infty \frac{f(x,y)}{(x+y+z)^\lambda} dx dy \right)^p dz \right)^{\frac{1}{p}} \left(\int_0^\infty z^{\gamma q+q-\lambda-1} g^q(z) dz \right)^{\frac{1}{q}} \end{aligned}$$

$$\leq \left(C^p \int_0^\infty \int_0^\infty (x+y)^{2p-\lambda-p\gamma-2} f^p(x,y) dx dy \right)^{\frac{1}{p}} \left(\int_0^\infty z^{\gamma q+q-\lambda-1} g^q(z) dz \right)^{\frac{1}{q}}.$$

Therefore by using (14) we obtain (6). To prove the equivalence relation between (6) and (15) we set here $f(x,y) = (x+y)^{(q-1)\lambda+\gamma q-2} \left(\int_0^\infty \frac{g(z)}{(x+y+z)^\lambda} dz \right)^{q-1}$. By inequality (6) we get

$$\begin{aligned} & \int_0^\infty \int_0^\infty (x+y)^{(q-1)\lambda+\gamma q-2} \left(\int_0^\infty \frac{g(z)}{(x+y+z)^\lambda} dz \right)^q dx dy \\ &= \int_0^\infty \int_0^\infty (x+y)^{(q-1)\lambda+\gamma q-2} \left(\int_0^\infty \frac{g(z)}{(x+y+z)^\lambda} dz \right)^{q-1} \left(\int_0^\infty \frac{g(z)}{(x+y+z)^\lambda} dz \right) dx dy \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \frac{f(x,y)g(z)}{(x+y+z)^\lambda} dx dy dz \\ &\leq \left(\int_0^\infty \int_0^\infty (x+y)^{2p-\lambda-p\gamma-2} f^p(x,y) dx dy \right)^{\frac{1}{p}} \left(\int_0^\infty z^{\gamma q+q-\lambda-1} g^q(z) dz \right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty \int_0^\infty (x+y)^{(q-1)\lambda+\gamma q-2} \left(\int_0^\infty \frac{g(z)}{(x+y+z)^\lambda} dz \right)^q dx dy \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_0^\infty z^{\gamma q+q-\lambda-1} g^q(z) dz \right)^{\frac{1}{q}}. \end{aligned}$$

It's clear that from the last inequality we obtain (15). On the other hand using Hölder inequality and (15) respectively, we have

$$\begin{aligned} & \int_0^\infty \int_0^\infty \int_0^\infty \frac{f(x,y)g(z)}{(x+y+z)^\lambda} dx dy dz \\ &= \int_0^\infty \int_0^\infty \left[(x+y)^{-\frac{(q-1)\lambda+\gamma q-2}{q}} f(x,y) \right] \left[(x+y)^{\frac{(q-1)\lambda+\gamma q-2}{q}} \int_0^\infty \frac{g(z)}{(x+y+z)^\lambda} dz \right] dx dy \\ &\leq \left(\int_0^\infty \int_0^\infty (x+y)^{2p-\lambda-p\gamma-2} f^p(x,y) dx dy \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_0^\infty \int_0^\infty (x+y)^{(q-1)\lambda+\gamma q-2} \left(\int_0^\infty \frac{g(z)}{(x+y+z)^\lambda} dz \right)^q dx dy \right)^{\frac{1}{q}} \\ &\leq \left(\int_0^\infty \int_0^\infty (x+y)^{2p-\lambda-p\gamma-2} f^p(x,y) dx dy \right)^{\frac{1}{p}} \left(C^q \int_0^\infty z^{\gamma q+q-\lambda-1} g^q(z) dz \right)^{\frac{1}{q}}. \end{aligned}$$

Thus, the equivalence relation between (15) and (6) is proved. Moreover, since the constant in (6) is the best possible we may conclude that the constants in both inequalities (14) and (15) are also the best possible. The theorem is proved. \square

The next theorem gives the reverse forms of (14) and (15). Since the proofs of these forms are similar to the proofs of (14) and (15) here we omit it.

THEOREM 4. *Under the assumptions of Theorem 2 we obtain the following two inequalities*

$$\begin{aligned} & \int_0^\infty z^{(p-1)\lambda - \gamma p - 1} \left(\int_0^\infty \int_0^\infty \frac{f(x, y)}{(x + y + z)^\lambda} dx dy \right)^p dz \\ & \geq C^p \int_0^\infty \int_0^\infty (x + y)^{2p - \lambda - p\gamma - 2} f^p(x, y) dx dy \end{aligned} \quad (17)$$

and

$$\begin{aligned} & \int_0^\infty \int_0^\infty (x + y)^{(q-1)\lambda + \gamma q - 2} \left(\int_0^\infty \frac{g(z)}{(x + y + z)^\lambda} dz \right)^q dx dy \\ & \geq C^q \int_0^\infty z^{\gamma q - \lambda + q - 1} g^q(z) dz. \end{aligned} \quad (18)$$

Both of the inequalities (17) and (18) are equivalent to (10) and the constants $C^p = B^p \left(\frac{\lambda}{p} + \gamma, \frac{\lambda}{q} - \gamma \right)$ and $C^q = B^q \left(\frac{\lambda}{p} + \gamma, \frac{\lambda}{q} - \gamma \right)$ are the best possible.

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Tserendorj Batbold
Department of Mathematics & Institute of Mathematics
National University of Mongolia
P. O. Box 46A-104, Ulaanbaatar 14201, Mongolia
e-mail: tsbatbold@hotmail.com

Laith E. Azar
Department of Mathematics
Al al-Bayt University
P. O. Box 130095, Mafraq, Jordan
e-mail: azar1@aabu.edu.jo