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PARTIAL SUMS OF MITTAG–LEFFLER FUNCTION

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Abstract. In the present investigation, Mittag-Leffler function with their normalization are considered. In this paper, we will study the ratio of a function of the form (1.4) to its sequence of partial sums $(\mathbb{E}_{\lambda,\mu})_n(z) = z + \sum_{k=1}^{n} \frac{\Gamma(\mu)}{\Gamma(\lambda k+\mu)} z^{k+1}$. We will determine lower bounds for $\Re\left\{\frac{\mathbb{E}_{\lambda,\mu}(z)}{(\mathbb{E}_{\lambda,\mu})_n(z)}\right\}$, $\Re\left\{\frac{(\mathbb{E}_{\lambda,\mu})_n(z)}{\mathbb{E}_{\lambda,\mu}(z)}\right\}$, $\Re\left\{\frac{\mathbb{E}'_{\lambda,\mu}(z)}{(\mathbb{E}'_{\lambda,\mu})_n(z)}\right\}$ and $\Re\left\{\frac{(\mathbb{E}'_{\lambda,\mu})_n(z)}{\mathbb{E}'_{\lambda,\mu}(z)}\right\}$. Results obtained are new and their usefulness are depicted by deducing several interesting examples.

1. Introduction

Let $\mathscr{A}$ denote the family of all functions f that are analytic in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ and are of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbb{D}. \tag{1.1}$$

Recently, several researchers studied families of analytic functions involving special functions $\mathscr{F} \subset \mathscr{A}$, to find different conditions such that the members of $\mathscr{F}$ have certain geometric properties like univalency, starlikeness or convexity in $\mathbb{D}$. In this context many results are available in the literature regarding the hypergeometric functions [11, 19, 16, 15], Bessel functions [2, 3, 4, 5, 17], Wright function [18] and Mittag-Leffler functions [1].

The function $E_\lambda(z)$ defined by

$$E_\lambda(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n+1)} \qquad (z \in \mathbb{C},\ \Re(\lambda) > 0), \tag{1.2}$$

was introduced by Mittag-Leffler [12] and is, therefore, known as the Mittag-Leffler function. A more general function $E_{\lambda,\mu}(z)$, generalizing $E_\lambda(z)$, is defined by

$$E_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n+\mu)} \qquad (z,\lambda,\mu \in \mathbb{C};\ \Re(\lambda) > 0). \tag{1.3}$$

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This function, sometime called, a Mittang-Leffler type function, first appeared in a paper by Wiman [24, 25]. The Mittag-Leffler function arises naturally in the solution of fractional order differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. These functions interpolate between a purely exponential law and power-law like behavior of phenomena governed by ordinary kinetic equations and their fractional counterparts [8, 9, 7, 20]. The most essential properties of these entire functions, investigated by many mathematicians, can be found in [6].

Observe that, Mittag-Leffler function $E_{\lambda,\mu}$ does not belong to the family $\mathscr{A}$. Thus, it is natural to consider the following normalization of Mittag-Leffler function:

$$\begin{aligned} \mathbb{E}_{\lambda,\mu}(z) &= \Gamma(\mu) z E_{\lambda,\mu}(z) \\ &:= \sum_{n=0}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda n+\mu)} z^{n+1} \quad (z,\lambda,\mu \in \mathbb{C};\ \Re(\lambda)>0,\ \mu \neq 0,-1,\cdots). \end{aligned} \tag{1.4}$$

Whilst formula (1.4) holds for complex-valued λ, μ and $z \in \mathbb{C}$, however in this paper we shall restrict our attention to the case of real-valued λ, μ and $z \in \mathbb{D}$. Observe that, the function $\mathbb{E}_{\lambda,\mu}$ contains many well known functions as its special case, for example

$$\begin{cases} \mathbb{E}_{0,1}(z) = \dfrac{z}{1-z}, \quad \mathbb{E}_{1,1}(z) = ze^z, \quad \mathbb{E}_{2,1}(z) = z\cosh(\sqrt{z}), \\ \mathbb{E}_{1,2}(z) = e^z - 1, \quad \mathbb{E}_{1,3}(z) = \dfrac{2(e^z - z - 1)}{z}, \\ \mathbb{E}_{1,4}(z) = \dfrac{6(e^z - 1 - z) - 3z^2}{z^2}, \quad \mathbb{E}_{2,2}(z) = \sqrt{z}\sinh(\sqrt{z}), \\ \mathbb{E}_{3,1}(z) = \dfrac{z}{2}\left[e^{z^{1/3}} + 2e^{-\frac{1}{2}z^{1/3}}\cos\left(\frac{\sqrt{3}}{2}z^{1/3}\right)\right]. \end{cases} \tag{1.5}$$

If f, g are analytic functions in $\mathbb{D}$, then f is said to be subordinate to g, written as $f(z) \prec g(z)$ $(z \in \mathbb{D})$, if there exists an analytic function w with $w(0) = 0$ and $|w(z)| \leqslant 1$ $(z \in \mathbb{D})$ such that $f(z) = g(w(z))$. In particular, if g is univalent in $\mathbb{D}$, then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

For more details one can refer [10].

In this paper, we will study the ratio of a function of the form (1.4) to its sequence of partial sums

$$(\mathbb{E}_{\lambda,\mu})_n(z) = z + \sum_{k=1}^{n} \frac{\Gamma(\mu)}{\Gamma(\lambda k+\mu)} z^{k+1} \tag{1.6}$$

$$(\mathbb{E}_{\lambda,\mu})_0(z) = z. \tag{1.7}$$

We will determine lower bounds for $\Re\left\{\frac{\mathbb{E}_{\lambda,\mu}(z)}{(\mathbb{E}_{\lambda,\mu})_n(z)}\right\}$, $\Re\left\{\frac{(\mathbb{E}_{\lambda,\mu})_n(z)}{\mathbb{E}_{\lambda,\mu}(z)}\right\}$, $\Re\left\{\frac{\mathbb{E}'_{\lambda,\mu}(z)}{(\mathbb{E}'_{\lambda,\mu})_n(z)}\right\}$ and $\Re\left\{\frac{(\mathbb{E}'_{\lambda,\mu})_n(z)}{\mathbb{E}'_{\lambda,\mu}(z)}\right\}$.

For various known results concerning with partial sums of analytic univalent functions one can refer the works of Owa et. al [14], Sheil-Small [21], Silverman [22], Silvia [23] and Orhan et. al [13].

2. Main results

To prove main results we need following Lemma:

LEMMA 2.1. *If $\lambda \geqslant 1$ and $\mu \geqslant 1$ then the function $\mathbb{E}_{\lambda,\mu} : \mathbb{D} \to \mathbb{C}$ given by (1.4), satisfies the following inequalities:*

$$|\mathbb{E}_{\lambda,\mu}(z)| \leqslant \frac{\mu^2+\mu+1}{\mu^2} \tag{2.1}$$

and

$$|\mathbb{E}'_{\lambda,\mu}(z)| \leqslant \frac{\mu^3+2\mu^2+3\mu+1}{\mu^3}. \tag{2.2}$$

Proof. Under the hypothesis, it is easy to check that $\Gamma(n+\mu) \leqslant \Gamma(\lambda n+\mu)$, $n \in \mathbb{N}$, which is equivalent to

$$\frac{\Gamma(\mu)}{\Gamma(\lambda n+\mu)} \leqslant \frac{\Gamma(\mu)}{\Gamma(n+\mu)} = \frac{1}{\mu(\mu+1)(\mu+2)\cdots(\mu+n-1)}, \quad n \in \mathbb{N}. \tag{2.3}$$

Using this inequality, we get

$$\begin{aligned}
\left|\mathbb{E}_{\lambda,\mu}(z)\right| &\leqslant |z| + \sum_{n=1}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda n+\mu)} |z|^{n+1} \\
&\leqslant 1 + \sum_{n=1}^{\infty} \frac{\Gamma(\mu)}{\Gamma(n+\mu)} \\
&= 1 + \sum_{n=1}^{\infty} \frac{1}{\mu(\mu+1)\cdots(\mu+n-1)} \\
&\leqslant 1 + \frac{1}{\mu} \sum_{n=0}^{\infty} \frac{1}{(\mu+1)^n} \\
&= \frac{\mu^2+\mu+1}{\mu^2}.
\end{aligned}$$

Similarly

$$\begin{aligned}\left|\mathbb{E}'_{\lambda,\mu}(z)\right| &\leqslant 1+\sum_{n=1}^{\infty}\frac{(n+1)\Gamma(\mu)}{\Gamma(\lambda n+\mu)}|z|^n \leqslant 1+\sum_{n=1}^{\infty}\frac{n\Gamma(\mu)}{\Gamma(\lambda n+\mu)}+\sum_{n=1}^{\infty}\frac{\Gamma(\mu)}{\Gamma(\lambda n+\mu)}\\ &\leqslant 1+\sum_{n=1}^{\infty}\frac{n}{\mu(\mu+1)\cdots(\mu+n-1)}+\sum_{n=1}^{\infty}\frac{1}{\mu(\mu+1)\cdots(\mu+n-1)}\\ &\leqslant 1+\frac{1}{\mu}\left[1+\frac{2}{(\mu+1)}+\frac{3}{(\mu+1)^2}+\ldots\right]+\frac{1}{\mu}\sum_{n=0}^{\infty}\frac{1}{(\mu+1)^n}\\ &= 1+\frac{1}{\mu}\left[\frac{(\mu+1)}{\mu}\right]^2+\frac{\mu+1}{\mu^2}=\frac{\mu^3+2\mu^2+3\mu+1}{\mu^3}. \quad \square\end{aligned}$$

THEOREM 2.1. *If* $\lambda \geqslant 1,\ \mu \geqslant 1$ *and* $\mu^2-\mu-1 \geqslant 0$, *then*

$$\Re\left\{\frac{\mathbb{E}_{\lambda,\mu}(z)}{(\mathbb{E}_{\lambda,\mu})_n(z)}\right\} \geqslant \frac{\mu^2-\mu-1}{\mu^2} \quad (z\in\mathbb{D}), \tag{2.4}$$

and

$$\Re\left\{\frac{(\mathbb{E}_{\lambda,\mu})_n(z)}{\mathbb{E}_{\lambda,\mu}(z)}\right\} \geqslant \frac{\mu^2}{\mu^2+\mu+1} \quad (z\in\mathbb{D}). \tag{2.5}$$

Proof. It is easy to see from (2.1) of Lemma 2.1 that

$$1+\sum_{k=1}^{\infty} b_k \leqslant \frac{\mu^2+\mu+1}{\mu^2}$$

which is equivalent to

$$\frac{\mu^2}{\mu+1}\sum_{k=1}^{\infty} b_k \leqslant 1 \ \left(\text{where } b_k=\frac{\Gamma(\mu)}{\Gamma(\lambda k+\mu)}\right). \tag{2.6}$$

To prove 2.4, we have to show that

$$\frac{\mu^2}{\mu+1}\left[\frac{\mathbb{E}_{\lambda,\mu}(z)}{(\mathbb{E}_{\lambda,\mu}(z))_n}-\frac{\mu^2-\mu-1}{\mu^2}\right] \prec \frac{1+z}{1-z}. \tag{2.7}$$

Using definition of subordination, and putting the values of $\mathbb{E}_{\lambda,\mu}$ and $(\mathbb{E}_{\lambda,\mu}(z))_n$, we have

$$\frac{1+\sum_{k=1}^{n} b_k z^k+\frac{\mu^2}{\mu+1}\sum_{k=n+1}^{\infty} b_k z^k}{1+\sum_{k=1}^{n} b_k z^k}=\frac{1+w(z)}{1-w(z)}.$$

Our assertion 2.4 is true if we show that $w(0)=0$ and $|w(z)|<1$ provided $z\in\mathbb{D}$. Simplifying for $w(z)$, we get

$$w(z)=\frac{\frac{\mu^2}{\mu+1}\sum_{k=n+1}^{\infty} b_k z^k}{2+2\sum_{k=1}^{n} b_k z^k+\frac{\mu^2}{\mu+1}\sum_{k=n+1}^{\infty} b_k z^k}.$$

Obviously $w(0)=0$ and

$$|w(z)| \leqslant \frac{\frac{\mu^2}{\mu+1}\sum_{k=n+1}^{\infty} b_k}{2-2\sum_{k=1}^{n} b_k-\frac{\mu^2}{\mu+1}\sum_{k=n+1}^{\infty} b_k} \leqslant 1$$

provided

$$\sum_{k=1}^{n} b_k+\frac{\mu^2}{\mu+1}\sum_{k=n+1}^{\infty} b_k \leqslant 1. \tag{2.8}$$

It suffices to show that the left hand side of (2.8) is bounded above by left hand side of (2.6), which is equivalent to

$$\left(\frac{\mu^2}{\mu+1}-1\right)\sum_{k=1}^{n} b_k \geqslant 0.$$

This is true as $\mu^2-\mu-1 \geqslant 0$.

To prove the result (2.5), we write

$$\frac{\mu^2+\mu+1}{\mu+1}\left[\frac{(\mathbb{E}_{\lambda,\mu}(z))_n}{\mathbb{E}_{\lambda,\mu}(z)}-\frac{\mu^2}{\mu^2+\mu+1}\right]=\frac{1+w(z)}{1-w(z)}.$$

Substituting the values of $\mathbb{E}_{\lambda,\mu}$ and $(\mathbb{E}_{\lambda,\mu}(z))_n$ and simplifying for $w(z)$, we have

$$w(z)=\frac{-\frac{\mu^2+\mu+1}{\mu+1}\sum_{k=n+1}^{\infty} b_k z^k}{2+2\sum_{k=1}^{n} b_k z^k-\frac{\mu^2-\mu-1}{\mu+1}\sum_{k=n+1}^{\infty} b_k z^k}.$$

Obviously $w(0)=0$ and

$$|w(z)| \leqslant \frac{\frac{\mu^2+\mu+1}{\mu+1}\sum_{k=n+1}^{\infty} b_k}{2-2\sum_{k=1}^{n} b_k-\frac{\mu^2-\mu-1}{\mu+1}\sum_{k=n+1}^{\infty} b_k} \leqslant 1 \tag{2.9}$$

as (2.8) is true for $\mu^2-\mu-1 \geqslant 0$. □

THEOREM 2.2. *If $\lambda \geqslant 1$ and $\mu \geqslant 1$ with $\mu^3-2\mu^2-3\mu-1 \geqslant 0$, then*

$$\Re\left\{\frac{\mathbb{E}'_{\lambda,\mu}(z)}{(\mathbb{E}'_{\lambda,\mu})_n(z)}\right\} \geqslant \frac{\mu^3-2\mu^2-3\mu-1}{\mu^3} \quad (z \in \mathbb{D}), \tag{2.10}$$

and

$$\Re\left\{\frac{(\mathbb{E}'_{\lambda,\mu})_n(z)}{\mathbb{E}'_{\lambda,\mu}(z)}\right\} \geqslant \frac{\mu^3}{\mu^3+2\mu^2+3\mu+1} \quad (z \in \mathbb{D}). \tag{2.11}$$

Proof. It is easy to see from (2.2) of Lemma 2.1 that

$$1+\sum_{k=1}^{\infty} b_k(k+1) \leqslant \frac{\mu^3+2\mu^2+3\mu+1}{\mu^3}$$

which is equivalent to

$$\frac{\mu^3}{2\mu^2+3\mu+1}\sum_{k=1}^{\infty} b_k(k+1) \leqslant 1 \left(\text{where } b_k=\frac{\Gamma(\mu)}{\Gamma(\lambda k+\mu)}\right). \tag{2.12}$$

To prove (2.10), we have to show that

$$\frac{\mu^3}{2\mu^2+3\mu+1}\left[\frac{\mathbb{E}'_{\lambda,\mu}(z)}{(\mathbb{E}'_{\lambda,\mu}(z))_n}-\frac{\mu^3-2\mu^2-3\mu-1}{\mu^3}\right] \prec \frac{1+z}{1-z}. \tag{2.13}$$

Using definition of subordination, and putting the values of $\mathbb{E}_{\lambda,\mu}$ and $(\mathbb{E}_{\lambda,\mu}(z))_n$, we have

$$\frac{1+\sum_{k=1}^{n} b_k(k+1)z^k+\frac{\mu^3}{2\mu^2+3\mu+1}\sum_{k=n+1}^{\infty}(k+1)b_k z^k}{1+\sum_{k=1}^{n}(k+1)b_k z^k}=\frac{1+w(z)}{1-w(z)}.$$

Our assertion (2.4) is ture if we show that $w(0)=0$ and $|w(z)|<1$ provided $z\in\mathbb{D}$. Simplifying for $w(z)$, we get

$$w(z)=\frac{\frac{\mu^3}{2\mu^2+3\mu+1}\sum_{k=n+1}^{\infty}(k+1)b_k z^k}{2+2\sum_{k=1}^{n}(k+1)b_k z^k+\frac{\mu^3}{2\mu^2+3\mu+1}\sum_{k=n+1}^{\infty}(k+1)b_k z^k}$$

Obviously $w(0)=0$ and

$$|w(z)| \leqslant \frac{\frac{\mu^3}{2\mu^2+3\mu+1}\sum_{k=n+1}^{\infty}(k+1)b_k}{2-2\sum_{k=1}^{n}(k+1)b_k-\frac{\mu^3}{2\mu^2+3\mu+1}\sum_{k=n+1}^{\infty}(k+1)b_k} \leqslant 1$$

provided

$$\sum_{k=1}^{n}(k+1)b_k+\frac{\mu^3}{2\mu^2+3\mu+1}\sum_{k=n+1}^{\infty}(k+1)b_k \leqslant 1. \tag{2.14}$$

It suffices to show that the left hand side of (2.14) is bounded above by left hand side of (2.12), which is equivalent to

$$\left(\frac{\mu^3}{2\mu^2+3\mu+1}\right)\sum_{k=1}^{n} b_k \geqslant 0.$$

This is true in view of hypothesis.

To prove the result (2.11), we write

$$\frac{\mu^3+2\mu^2+3\mu+1}{2\mu^2+3\mu+1}\left[\frac{(\mathbb{E}'_{\lambda,\mu}(z))_n}{\mathbb{E}'_{\lambda,\mu}(z)}-\frac{\mu^3}{\mu^3+2\mu^2+3\mu+1}\right]=\frac{1+w(z)}{1-w(z)}.$$

Substituting the values of $\mathbb{E}'_{\lambda,\mu}$ and $(\mathbb{E}'_{\lambda,\mu}(z))_n$ and simplifying for $w(z)$, we have

$$w(z)=\frac{-\frac{\mu^3+2\mu^2+3\mu+1}{2\mu^2+3\mu+1}\sum\limits_{k=n+1}^{\infty}(k+1)b_kz^k}{2+2\sum\limits_{k=1}^{n}(k+1)b_kz^k+\left(1-\frac{\mu^3}{2\mu^2+3\mu+1}\right)\sum\limits_{k=n+1}^{\infty}(k+1)b_kz^k}.$$

Obviously $w(0)=0$ and

$$|w(z)|\leqslant\frac{\frac{\mu^3+2\mu^2+3\mu+1}{2\mu^2+3\mu+1}\sum\limits_{k=n+1}^{\infty}(k+1)b_k}{2-2\sum\limits_{k=1}^{n}(k+1)b_k-\frac{\mu^3-2\mu^2-3\mu-1}{2\mu^2+3\mu+1}\sum\limits_{k=n+1}^{\infty}(k+1)b_k}\leqslant 1 \tag{2.15}$$

as (2.14) is true under the hypothesis. □

3. Application

In view of (1.1) and Theorem 2.1 and Theorem 2.2, we get the following inequalities:

(1) Choosing $\lambda=1$, $\mu=2$ and $n=0$, the hypothesis of Theorem 2.1 is satified but hypothesis of Theorem 2.2 is not satisfied, thus we get the following results

$$\Re\left\{\frac{e^z-1}{z}\right\}\geqslant\frac{1}{4}\quad(z\in\mathbb{D}), \tag{3.1}$$

$$\Re\left\{\frac{z}{e^z-1}\right\}\geqslant\frac{4}{7}\quad(z\in\mathbb{D}). \tag{3.2}$$

(2) Choosing $\lambda=2$, $\mu=2$ and $n=0$, the hypothesis of Theorem 2.1 is satified but hypothesis of Theorem 2.2 is not satisfied, thus we get the following results

$$\Re\left\{\frac{\sinh\sqrt{z}}{\sqrt{z}}\right\}\geqslant\frac{1}{4}\quad(z\in\mathbb{D}), \tag{3.3}$$

$$\Re\left\{\frac{\sqrt{z}}{\sinh\sqrt{z}}\right\}\geqslant\frac{4}{7} \tag{3.4}$$

One can find out more examples in view of (1.1). The following are graphs of some of the functions discusses above. These figures depicts validity of our results.

i We have the image domains of $f_1(z) = \frac{e^z-1}{z}$, $f_2(z) = \frac{z}{e^z-1}$, in Figure 1.

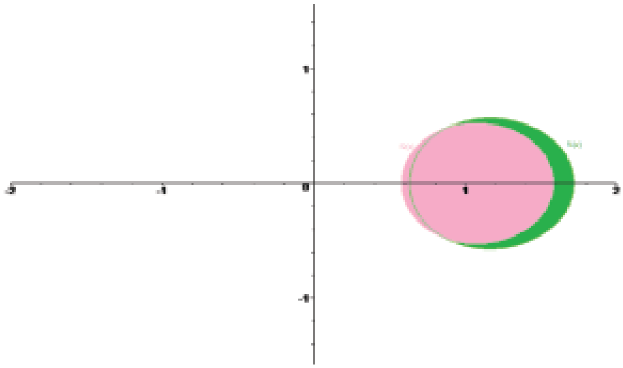

Figure 1.

ii We have the image domains of $f_3(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}}$, $f_4(z) = \frac{\sqrt{z}}{\sinh(\sqrt{z})}$, in Figure 2.

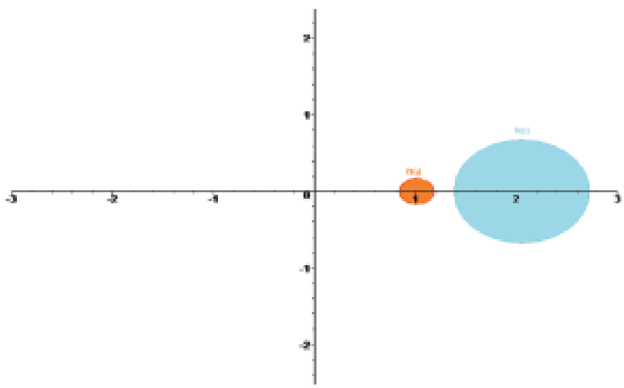

Figure 2.

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