Journal of
Mathematical
Inequalities
Volume 12, Number 3 (2018), 605–617 doi:10.7153/jmi-2018-12-46

A NEW CHARACTERIZATION OF CONVEXITY WITH RESPECT TO CHEBYSHEV SYSTEMS

ZSOLT PÁLES AND ÉVA SZÉKELYNÉ RADÁCSI

(*Communicated by J. Matkowski*)

Abstract. The notion of nth order convexity in the sense of Hopf and Popoviciu is defined via the nonnegativity of the $(n+1)$st order divided differences of a given real-valued function. In view of the well-known recursive formula for divided differences, the nonnegativity of $(n+1)$st order divided differences is equivalent to the $(n-k-1)$st order convexity of the kth order divided differences which provides a characterization of nth order convexity.

The aim of this paper is to apply the notion of higher-order divided differences in the context of convexity with respect to Chebyshev systems introduced by Karlin in 1968. Using a determinant identity of Sylvester, we then establish a formula for the generalized divided differences which enables us to obtain a new characterization of convexity with respect to Chebyshev systems. Our result generalizes that of Wąsowicz which was obtained in 2006. As an application, we derive a necessary condition for functions which can be written as the difference of two functions convex with respect to a given Chebyshev system.

1. Introduction

Denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ the sets of natural, integer, rational, and real numbers, respectively. Given a set $H \subseteq \mathbb{R}$, the set of positive elements of H is denoted by H_+. Thus, for instance, $\mathbb{N} = \mathbb{Z}_+$.

For a set $H \subseteq \mathbb{R}$, denote the simplex of strictly increasingly ordered n-tuples of H^n by $\sigma_n(H)$, i.e.,

$$\sigma_n(H) := \{(x_1,\dots,x_n) \in H^n \mid x_1 < \dots < x_n\}.$$

The set of those elements of H^n that have pairwise distinct coordinates will be denoted by $\tau_n(H)$, i.e.,

$$\tau_n(H) := \{(x_1,\dots,x_n) \in H^n \mid x_i \neq x_j \, (i \neq j)\}.$$

Obviously, $\sigma_n(H) \neq \emptyset$ and $\tau_n(H) \neq \emptyset$ if and only if the cardinality $|H|$ of H is at least n, furthermore, we have that $\sigma_n(H) \subseteq \tau_n(H) \subseteq H^n$.

Mathematics subject classification (2010): Primary 26B25, Secondary 39B62.
Keywords and phrases: Chebyshev system, generalized convexity, generalized divided difference.
The research was supported by the Hungarian Scientific Research Fund (OTKA) Grant K-111651 and by the EFOP-3.6.1-16-2016-00022 project. This project is co-financed by the European Union and the European Social Fund.

Provided that $|H| \geqslant n$, for a vector valued function $\omega = (\omega_1, \dots, \omega_n) : H \to \mathbb{R}^n$, the functional operator $\Phi_\omega : H^n \to \mathbb{R}$ is defined by

$$\Phi_\omega(x_1,\dots,x_n) := \begin{vmatrix} \omega_1(x_1) & \dots & \omega_1(x_n) \\ \vdots & \ddots & \vdots \\ \omega_n(x_1) & \dots & \omega_n(x_n) \end{vmatrix} \qquad \big((x_1,\dots,x_n) \in H^n\big).$$

We say that ω is an *n-dimensional positive (resp. negative) Chebyshev system over H* if Φ_ω is positive (resp. negative) over $\sigma_n(H)$, respectively.

The following systems are the most important particular cases for positive Chebyshev systems. For more important examples we refer to the books by Karlin [5] and Karlin–Studden [6].

(i) The function $\omega : \mathbb{R} \to \mathbb{R}^n$ given by $\omega(x) := (1, x, \dots, x^{n-1})$ is an n-dimensional positive Chebyshev system on $\mathbb{R}$. This system is called the *standard, or polynomial n-dimensional Chebyshev system.*

(ii) The function $\omega(x) := \big(1, \cos(x), \sin(x), \dots, \cos(nx), \sin(nx)\big)$ is a $(2n+1)$-dimensional positive Chebyshev system on any open interval I whose length is less than or equal to 2π.

(iii) The function $\omega(x) := \big(\cos(x), \sin(x), \dots, \cos(nx), \sin(nx)\big)$ is a $(2n)$-dimensional positive Chebyshev system on any open interval I whose length is less than or equal to π.

(iv) For the function $\omega(x) := (1, x^2)$, we get that $\Phi_\omega(x_1, x_2) = (x_1 + x_2)(x_2 - x_1)$. Therefore ω is a 2-dimensional positive Chebyshev system on $\mathbb{R}_+$, but it is not a Chebyshev system on $\mathbb{R}$ (observe that $\Phi_\omega(-1, 1) = 0$).

Given a positive Chebyshev system $\omega : H \to \mathbb{R}^n$, a function $f : H \to \mathbb{R}$ is called *ω-convex* (i.e., *convex with respect to the Chebyshev system ω*) if, $\Phi_{(\omega,f)}$ is nonnegative over $\sigma_{n+1}(H)$. A function $f : H \to \mathbb{R}$ is strictly ω-convex if a function $(\omega_1, \dots, \omega_n, f)$ is an $(n+1)$-dimensional positive Chebyshev system over H.

For $k \geqslant 0$, define the kth power function $p_k : \mathbb{R} \to \mathbb{R}$ by $p_k(x) := x^k$. As we have seen it before, $(p_0, \dots, p_{n-1})$ is an n-dimensional Chebyshev system. The notion of convexity with respect to this system, called polynomial convexity, was introduced by Hopf [4] and by Popoviciu [7]. The particular case, when $\omega = (p_0, p_1)$, simplifies to the notion of standard convexity, moreover, for $(x,y,z) \in \sigma_3(H)$, the inequality $\Phi_{(p_0,p_1,f)}(x,y,z) \geqslant 0$ is equivalent to

$$f(y) \leqslant \frac{z-y}{z-x} f(x) + \frac{y-x}{z-x} f(z).$$

It is easy to verify that this inequality holds if and only if, for all $p \in H$, the mapping

$$x \mapsto \frac{f(x) - f(p)}{x - p} \qquad (x \in H \setminus \{p\})$$

is nondecreasing. More generally, in view of the well-known recursive formula for divided differences, the nonnegativity of $(n+1)$ st order divided differences is equivalent to the monotonicity of the nth order divided differences which provides a characterization of nth order convexity.

The aim of this paper is to characterize higher-order convexity with respect to Chebyshev systems (cf. [2]) by applying the notion of related generalized higher-order divided differences introduced by Karlin in [5] and rediscovered in [8]. Using a determinant identity of Sylvester, we then establish a formula for the generalized divided differences of higher order and obtain new characterizations of convexity with respect to Chebyshev systems. Our result generalizes that of Wąsowicz which was obtained in [9]. As an application, we introduce the notion of ω-variation and we derive a necessary condition for functions that can be written as the difference of two ω-convex functions.

2. Characterizations of convexity with respect to Chebyshev systems

In the sequel, we will need the following classical formula which is termed Sylvester's Determinant Identity in the literature [1], [3].

THEOREM. *Let $n \in \mathbb{N}$ and $A : \{1,\dots,n\} \times \{1,\dots,n\} \to \mathbb{R}$ be an $n \times n$ matrix. For $1 \leqslant k \leqslant n-1$ define the $(n-k)\times(n-k)$ matrix $B_k : \{k+1,\dots,n\} \times \{k+1,\dots,n\} \to \mathbb{R}$ by*

$$B_k(i,j) := \det\left(A|_{\{1,\dots,k,i\}\times\{1,\dots,k,j\}}\right) \qquad (i,j \in \{k+1,\dots,n\}).$$

Then the following identity holds

$$\det(B_k) = \left(\det\left(A|_{\{1,\dots,k\}\times\{1,\dots,k\}}\right)\right)^{n-k-1} \det(A).$$

To formulate our main results below, we introduce the notions of divided differences with respect to Chebyshev systems. Let $n \in \mathbb{N}$, $H \subseteq \mathbb{R}$ with $|H| \geqslant n$ and let $\omega = (\omega_1,\dots,\omega_n) : H \to \mathbb{R}^n$ be an n-dimensional positive Chebyshev system over H. In the sequel, for the sake of convenience and brevity, given $k \in \{1,\dots,n\}$, we shall denote by $\omega_{\langle k\rangle}$ the k-tuple $(\omega_1,\dots,\omega_k)$. Thus, in particular, we have that

$$\omega_{\langle 1\rangle} = \omega_1, \qquad \omega_{\langle 2\rangle} = (\omega_1,\omega_2), \qquad \dots, \qquad \omega_{\langle n\rangle} = \omega.$$

For a function $f : H \to \mathbb{R}$ and $k \in \{1,\dots,n\}$, the generalized $(k-1)$*-st order ω-divided difference of f* (cf. [5]) is defined by

$$\left[x_1,\dots,x_k;f\right]_{\omega_{\langle k\rangle}} := \frac{\Phi_{(\omega_{\langle k-1\rangle},f)}(x_1,\dots,x_k)}{\Phi_{\omega_{\langle k\rangle}}(x_1,\dots,x_k)} \qquad \left((x_1,\dots,x_k) \in \tau_k(H)\right).$$

provided that $\omega_{\langle k\rangle}$ is a k-dimensional Chebyshev system. Clearly, if $\omega = (\omega_1,\dots,\omega_n) = (p_0,\dots,p_{n-1})$, then, $\left[x_1,\dots,x_k;f\right]_{\omega_{\langle k\rangle}}$ is equal to the standard $(k-1)$-st order divided difference $\left[x_1,\dots,x_k;f\right]$.

THEOREM 1. *Let $n,k \in \mathbb{N}$, $k < n$, $|H| \geqslant n$ and let $x_1 < \ldots < x_k$ be arbitrary elements of H. Let $\omega := (\omega_1, \ldots, \omega_n)$ be an n-dimensional positive Chebyshev system over H such that $\omega_{\langle k\rangle}$ and $\omega_{\langle k+1\rangle}$ are k and $(k+1)$-dimensional positive Chebyshev system over H, respectively. Then the following system of functions*

$$x \mapsto \big[x_1,\ldots,x_k,x;\omega_j\big]_{\omega_{\langle k+1\rangle}} \qquad (k+1 \leqslant j \leqslant n) \tag{1}$$

is an $(n-k)$-dimensional positive Chebyshev system over $H \setminus \{x_1,\ldots,x_k\}$.

Proof. Let $x_{k+1} < \ldots < x_n$ be arbitrary elements of $H \setminus \{x_1,\ldots,x_k\}$. Applying Sylvester's determinant identity for the matrix A defined by $A(i,j) := \omega_i(x_j)$, we get

$$\begin{aligned}\Phi_\omega(x_1,\ldots,x_n)&\cdot\big(\Phi_{\omega_{\langle k\rangle}}(x_1,\ldots,x_k)\big)^{n-k-1}\\ &= \begin{vmatrix} \Phi_{(\omega_{\langle k\rangle},\omega_{k+1})}(x_1,\ldots,x_k,x_{k+1}) & \ldots & \Phi_{(\omega_{\langle k\rangle},\omega_{k+1})}(x_1,\ldots,x_k,x_n) \\ \vdots & \ddots & \vdots \\ \Phi_{(\omega_{\langle k\rangle},\omega_n)}(x_1,\ldots,x_k,x_{k+1}) & \ldots & \Phi_{(\omega_{\langle k\rangle},\omega_n)}(x_1,\ldots,x_k,x_n) \end{vmatrix}.\end{aligned}$$

Then dividing the j-th column $(j \in \{1,\ldots,n-k\})$ of the determinant on the right hand side of the above identity by $\Phi_{\omega_{\langle k+1\rangle}}(x_1,\ldots,x_k,x_j)$, we arrive at the following equality:

$$\begin{aligned}&\frac{\Phi_\omega(x_1,\ldots,x_n)\cdot\big(\Phi_{\omega_{\langle k\rangle}}(x_1,\ldots,x_k)\big)^{n-k-1}}{\prod_{j=k+1}^n \Phi_{\omega_{\langle k+1\rangle}}(x_1,\ldots,x_k,x_j)}\\ &\qquad= \begin{vmatrix} \big[x_1,\ldots,x_k,x_{k+1};\omega_{k+1}\big]_{\omega_{\langle k+1\rangle}} & \cdots & \big[x_1,\ldots,x_k,x_n;\omega_{k+1}\big]_{\omega_{\langle k+1\rangle}} \\ \vdots & \ddots & \vdots \\ \big[x_1,\ldots,x_k,x_{k+1};\omega_n\big]_{\omega_{\langle k+1\rangle}} & \cdots & \big[x_1,\ldots,x_k,x_n;\omega_n\big]_{\omega_{\langle k+1\rangle}} \end{vmatrix}.\end{aligned} \tag{2}$$

In order to complete the proof of the theorem, it suffices to show that the left hand side of the above identity is positive for elements $x_{k+1} < \ldots < x_n$ of $H \setminus \{x_1,\ldots,x_k\}$.

Define the indices $\ell_{k+1},\ldots,\ell_n$ by

$$\ell_j := \begin{cases} \max\big\{i \in \{1,\ldots,k\} \mid x_i < x_j\big\} & \text{if } x_1 < x_j, \\ 0 & \text{if } x_j < x_1. \end{cases}$$

Now, using that $\omega_{\langle k+1\rangle}$ is a positive Chebyshev system, for $j \in \{k+1,\ldots,n\}$, we are going to show that

$$\operatorname{sign}\Phi_{\omega_{\langle k+1\rangle}}(x_1,\ldots,x_k,x_j) = (-1)^{k-\ell_j}. \tag{3}$$

If $x_j < x_1$, then $\ell_j = 0$ and

$$\operatorname{sign}\Phi_{\omega_{\langle k+1\rangle}}(x_1,\ldots,x_k,x_j) = (-1)^k \operatorname{sign}\Phi_{\omega_{\langle k+1\rangle}}(x_j,x_1,\ldots,x_k) = (-1)^k = (-1)^{k-\ell_j}.$$

If $x_1 < x_j < x_k$, then $x_{\ell_j} < x_j < x_{\ell_j+1}$, hence

$$\operatorname{sign}\Phi_{\omega_{\langle k+1\rangle}}(x_1,\ldots,x_k,x_j) = (-1)^{k-\ell_j}\operatorname{sign}\Phi_{\omega_{\langle k+1\rangle}}(x_1,\ldots,x_{\ell_j},x_j,x_{\ell_j+1},\ldots,x_k) = (-1)^{k-\ell_j}.$$

Finally, if $x_k < x_j$, then $\ell_j = k$ and

$$\operatorname{sign}\Phi_{\omega_{\langle k+1\rangle}}(x_1,\ldots,x_k,x_j) = 1 = (-1)^{k-\ell_j}.$$

Applying (3), we get

$$\operatorname{sign}\prod_{j=k+1}^{n}\Phi_{\omega_{\langle k+1\rangle}}(x_1,\ldots,x_k,x_j) = (-1)^{(n-k)k-(\ell_{k+1}+\cdots+\ell_n)}.$$

An analogous computation and the positive Chebyshev property of ω results that

$$\operatorname{sign}\Phi_{\omega}(x_1,\ldots,x_n) = (-1)^{(n-k)k-(\ell_{k+1}+\cdots+\ell_n)}.$$

Therefore, the left hand side of (2) is positive since $\omega_{\langle k\rangle}$ is also a positive Chebyshev system. □

THEOREM 2. *Let $n,k\in\mathbb{N}$, $k<n$, $|H|\geqslant n+1$. Let $\omega := (\omega_1,\ldots,\omega_n): H\to\mathbb{R}^n$ be an n-dimensional positive Chebyshev system over H such that $\omega_{\langle k\rangle}$ and $\omega_{\langle k+1\rangle}$ are k and $(k+1)$-dimensional positive Chebyshev system over H, respectively and let $f: H\to\mathbb{R}$ be a function. Then, for all $(x_1,\ldots,x_{n+1})\in\tau_{n+1}(H)$, the following identity is valid*

$$\frac{\Phi_{(\omega,f)}(x_1,\ldots,x_{n+1})\cdot\big(\Phi_{\omega_{\langle k\rangle}}(x_1,\ldots,x_k)\big)^{n-k}}{\prod_{j=k+1}^{n+1}\Phi_{\omega_{\langle k+1\rangle}}(x_1,\ldots,x_k,x_j)}
= \begin{vmatrix} \big[x_1,\ldots,x_k,x_{k+1};\omega_{k+1}\big]_{\omega_{\langle k+1\rangle}} & \cdots & \big[x_1,\ldots,x_k,x_{n+1};\omega_{k+1}\big]_{\omega_{\langle k+1\rangle}} \\ \vdots & \ddots & \vdots \\ \big[x_1,\ldots,x_k,x_{k+1};\omega_{n}\big]_{\omega_{\langle k+1\rangle}} & \cdots & \big[x_1,\ldots,x_k,x_{n+1};\omega_{n}\big]_{\omega_{\langle k+1\rangle}} \\ \big[x_1,\ldots,x_k,x_{k+1};f\big]_{\omega_{\langle k+1\rangle}} & \cdots & \big[x_1,\ldots,x_k,x_{n+1};f\big]_{\omega_{\langle k+1\rangle}} \end{vmatrix}. \tag{4}$$

Furthermore, the following statements are equivalent:

(i) *f is ω-convex on H;*

(ii) *For each ordered k-tuple $(x_1,\ldots,x_k)\in\sigma_k(H)$, the function $x\mapsto\big[x_1,\ldots,x_k,x;f\big]_{\omega_{\langle k+1\rangle}}$ is convex on $H\setminus\{x_1,\ldots,x_k\}$ with respect to the $(n-k)$-dimensional Chebyshev system defined by* (1)*;*

(iii) *There exists $\ell\in\{0,\ldots,k\}$ such that, for each ordered k-tuple $(x_1,\ldots,x_k)\in\sigma_k(H)$, the function $x\mapsto\big[x_1,\ldots,x_k,x;f\big]_{\omega_{\langle k+1\rangle}}$ is convex with respect to the $(n-k)$-dimensional Chebyshev system defined by* (1) *on $H\cap]-\infty,x_1[$ if $\ell=0$, on $H\cap]x_\ell,x_{\ell+1}[$ if $0<\ell<k$ and on $H\cap]x_k,+\infty[$ if $\ell=k$.*

Proof. The formula in (4) follows from Sylvester's determinant identity with the $(n+1)\times(n+1)$ matrix A defined by

$$A(i,j) := \begin{cases} \omega_i(x_j) & \text{if } (i,j)\in\{1,\ldots,n\}\times\{1,\ldots,n+1\}, \\ f(x_j) & \text{if } (i,j)\in\{n+1\}\times\{1,\ldots,n+1\} \end{cases}$$

in the same way as formula (2) in the proof of Theorem 1.

(i)$\Rightarrow$(ii). Assume that the function f is ω-convex, i.e. $\Phi_{(\omega,f)}$ is nonnegative over $\sigma_{n+1}(H)$. Let $k<n$, let $x_1,\dots,x_k\in\sigma_k(H)$ and $x_{k+1},\dots,x_{n+1}\in\sigma_{n+1-k}(H\setminus\{x_1,\dots,x_k\})$. With similar idea as in the previous proof we can prove that the left hand side of (4) is nonnegative, hence the right hand side of (4) is also nonnegative, then using the positive Chebyshev property of (1), we get that a function $x\mapsto\big[x_1,\dots,x_k,x;f\big]_{\omega_{\langle k+1\rangle}}$ is a convex function with respect to the Chebyshev system (1).

The implication (ii)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(i). Assume that (iii) holds for some $\ell\in\{0,\dots,k\}$. To prove that f is ω-convex, let $(x_1,\dots,x_{n+1})\in\sigma_{n+1}(H)$. Define

$$x'_i:=\begin{cases} x_i & \text{if } 1\leqslant i \text{ and } i\leqslant \ell,\\ x_{i+n-k+1} & \text{if } \ell+1\leqslant i \text{ and } i\leqslant k,\\ x_{i+\ell-k} & \text{if } k+1\leqslant i \text{ and } i\leqslant n+1.\end{cases}$$

Now observe that $(x'_1,\dots,x'_k)\in\sigma_k(H)$ and $(x'_{k+1},\dots,x'_{n+1})\in\sigma_{n-k+1}(H_\ell)$, where

$$H_\ell:=\begin{cases} H\cap\,]-\infty,x'_1[& \text{if } \ell=0,\\ H\cap\,]x'_\ell,x'_{\ell+1}[& \text{if } 0<\ell<k,\\ H\cap\,]x'_k,\infty[& \text{if } \ell=k.\end{cases}$$

To complete the proof, applying formula (4) for the $(n+1)$-tuple $(x'_1,\dots,x'_{n+1})$, we obtain:

$$\frac{\Phi_{(\omega,f)}(x'_1,\dots,x'_{n+1})\cdot\big(\Phi_{\omega_{\langle k\rangle}}(x'_1,\dots,x'_k)\big)^{n-k}}{\prod_{j=k+1}^{n+1}\Phi_{\omega_{\langle k+1\rangle}}(x'_1,\dots,x'_k,x'_j)} = \begin{vmatrix} \big[x'_1,\dots,x'_k,x'_{k+1};\omega_{k+1}\big]_{\omega_{\langle k+1\rangle}} & \cdots & \big[x'_1,\dots,x'_k,x'_{n+1};\omega_{k+1}\big]_{\omega_{\langle k+1\rangle}}\\ \vdots & \ddots & \vdots\\ \big[x'_1,\dots,x'_k,x'_{k+1};\omega_n\big]_{\omega_{\langle k+1\rangle}} & \cdots & \big[x'_1,\dots,x'_k,x'_{n+1};\omega_n\big]_{\omega_{\langle k+1\rangle}}\\ \big[x'_1,\dots,x'_k,x'_{k+1};f\big]_{\omega_{\langle k+1\rangle}} & \cdots & \big[x'_1,\dots,x'_k,x'_{n+1};f\big]_{\omega_{\langle k+1\rangle}} \end{vmatrix}. \tag{5}$$

We have that the right hand side of the above equality is nonnegative, since, by (iii), $\big[x'_1,\dots,x'_k,\cdot,f\big]_{\omega_{\langle k+1\rangle}}$ is convex with respect to the $(n-k)$-dimensional Chebyshev system defined by (1) (where the x_i s are replaced by x'_i) on H_ℓ. The subsystem $\omega_{\langle k\rangle}$ being a positive Chebyshev system, $\Phi_{\omega_{\langle k\rangle}}(x'_1,\dots,x'_k)>0$. Hence, (5) implies that

$$\frac{\Phi_{(\omega,f)}(x'_1,\dots,x'_{n+1})}{\prod_{j=k+1}^{n+1}\Phi_{\omega_{\langle k+1\rangle}}(x'_1,\dots,x'_k,x'_j)}\geqslant 0. \tag{6}$$

For $j\in\{k+1,\dots,n+1\}$, we have that $x'_j<x'_1$ if $\ell=0$, $x'_\ell<x'_j<x'_{\ell+1}$ if $0<\ell<k$, and $x'_k<x'_j$ if $\ell=k$, therefore

$$\operatorname{sign}\Phi_{\omega_{\langle k+1\rangle}}(x'_1,\dots,x'_k,x'_j)=(-1)^{k-\ell},$$

which yields that

$$\operatorname{sign} \prod_{j=k+1}^{n+1} \Phi_{\omega_{\langle k+1\rangle}}(x_1',\dots,x_k',x_j') = (-1)^{(n-k+1)(k-\ell)}. \tag{7}$$

Therefore, using (7) and inequality (6), after interchanging the appropriate columns of the determinant $\Phi_{(\omega,f)}(x_1,\dots,x_{n+1})$, we get that

$$\Phi_{(\omega,f)}(x_1,\dots,x_{n+1}) = (-1)^{(n-k+1)(k-\ell)}\Phi_{(\omega,f)}(x_1',\dots,x_{n+1}') \geqslant 0.$$

This completes the proof of the ω-convexity of f. □

The following result, which was established by Wąsowicz [9, Theorem 2], concerns the particular case $k=n-1$ of the previous theorem.

COROLLARY 3. *Let $n\in\mathbb{N}$, $n\geqslant 2$, $|H|\geqslant n+1$ and $\omega := (\omega_1,\dots,\omega_n): H\to\mathbb{R}^n$ be an n-dimensional positive Chebyshev system such that $\omega_{\langle n-1\rangle}$ is an $(n-1)$-dimensional positive Chebyshev system and let $f: H\to\mathbb{R}$ be a function. Then, for all $(x_1,\dots,x_{n+1})\in\tau_{n+1}(H)$, the following identity is valid*

$$\begin{aligned}\frac{\Phi_{(\omega,f)}(x_1,\dots,x_{n+1})\Phi_{\omega_{\langle n-1\rangle}}(x_1,\dots,x_{n-1})}{\Phi_\omega(x_1,\dots,x_{n-1},x_n)\Phi_\omega(x_1,\dots,x_{n-1},x_{n+1})}&\\ = \big[x_1,\dots,x_{n-1},x_{n+1};f\big]_\omega - \big[x_1,\dots,x_{n-1},x_n;f\big]_\omega.&\end{aligned} \tag{8}$$

Furthermore, the following statements are equivalent:

(i) f is ω-convex;

(ii) For each ordered $(n-1)$-tuple $(x_1,\dots,x_{n-1})\in\sigma_{n-1}(H)$, the function $x\mapsto \big[x_1,\dots,x_{n-1},x;f\big]_\omega$ is nondecreasing on $H\setminus\{x_1,\dots,x_{n-1}\}$;

(iii) There exists $\ell\in\{0,\dots,n-1\}$ such that, for each ordered $(n-1)$-tuple $(x_1,\dots,x_{n-1})\in\sigma_{n-1}(H)$, the function $x\mapsto\big[x_1,\dots,x_{n-1},x;f\big]_\omega$ is nondecreasing on $H\cap]-\infty,x_1[$ if $\ell=0$, on $H\cap]x_\ell,x_{\ell+1}[$ if $0<\ell<n-1$ and on $H\cap]x_{n-1},+\infty[$ if $\ell=n-1$.

Proof. If $k=n-1$ then the $n-k=1$ dimensional Chebyshev system defined by (1) is the constant function 1 and (8) is a particular case of (4). The convexity of the function $x\mapsto\big[x_1,\dots,x_{n-1},x;f\big]_\omega$ with respect to this Chebyshev system is equivalent to its nondecreasingness. Thus, Theorem 2 directly yields the equivalence of statements (i), (ii), and (iii). □

In what follows, we apply Theorem 2 to the n-dimensional polynomial system. For this, we shall need the following auxiliary statement. Recall that we have defined $p_n:\mathbb{R}\to\mathbb{R}$ by $p_n(x) := x^n$.

LEMMA 4. *Let $k \in \mathbb{N}$ and $x_1 < \ldots < x_k$ be arbitrary elements of H. Then the following equality is valid*

$$\big[x_1,\ldots,x_k;p_n\big] = \sum_{\substack{\alpha_1,\ldots,\alpha_k \geqslant 0,\\ \alpha_1+\ldots+\alpha_k=n-k+1}} x_1^{\alpha_1}\cdots x_k^{\alpha_k} \qquad (n\in\mathbb{N}\cup\{0\}). \tag{9}$$

Proof. The proof runs by induction on k. For $k=1$ the statement trivially holds. Assume that (9) is true for $k=m-1\in\mathbb{N}$, $m\geqslant 2$. By a well-known property of classical divided differences, we have

$$\big[x_1,\ldots,x_m;p_n\big] = \frac{\big[x_2,\ldots,x_m;p_n\big]-\big[x_1,\ldots,x_{m-1};p_n\big]}{x_m-x_1}.$$

By the induction hypothesis,

$$\begin{aligned}
\big[x_1,\ldots,x_m;p_n\big] &= \sum_{\substack{\alpha_2,\ldots,\alpha_m\geqslant 0,\\ \alpha_2+\ldots+\alpha_m=n-m+2}} \frac{x_2^{\alpha_2}\cdots x_m^{\alpha_m}}{x_m-x_1} - \sum_{\substack{\alpha_1,\ldots,\alpha_{m-1}\geqslant 0,\\ \alpha_1+\ldots+\alpha_{m-1}=n-m+2}} \frac{x_1^{\alpha_1}\cdots x_{m-1}^{\alpha_{m-1}}}{x_m-x_1}\\
&= \sum_{j=0}^{n-m+2}\left(\frac{x_m^j-x_1^j}{x_m-x_1}\sum_{\substack{\alpha_2,\ldots,\alpha_{m-1}\geqslant 0,\\ \alpha_2+\ldots+\alpha_{m-1}=n-m-j+2}} x_2^{\alpha_2}\cdots x_{m-1}^{\alpha_{m-1}}\right)\\
&= \sum_{\substack{\alpha_1,\ldots,\alpha_m\geqslant 0,\\ \alpha_1+\ldots+\alpha_m=n-m+1}} x_1^{\alpha_1}\cdots x_m^{\alpha_m}.
\end{aligned}$$

Thus we obtain (9) for $k=m$, which completes the proof of the induction. $\square$

COROLLARY 5. *Let $n,k\in\mathbb{N}$, $k<n$, $|H|\geqslant n+1$. Let $f:H\to\mathbb{R}$ be a function. Then the following statements are pairwise equivalent.*

- *(i) f is n-monotone, i.e., it is convex with respect to the Chebyshev system $(p_0,\ldots,p_{n-1})$;*
- *(ii) For each ordered k-tuple $(x_1,\ldots,x_k)\in\sigma_k(H)$, the function $x\mapsto\big[x_1,\ldots,x_k,x;f\big]$ is $(n-k)$-monotone (i.e., it is convex with respect to the Chebyshev system $(p_0,\ldots,p_{n-k-1})$) on the set $H\setminus\{x_1,\ldots,x_k\}$;*
- *(iii) There exists $\ell\in\{0,\ldots,k\}$ such that, for each ordered k-tuple $(x_1,\ldots,x_k)\in\sigma_k(H)$, the function $x\mapsto\big[x_1,\ldots,x_k,x;f\big]$ is $(n-k)$-monotone on $H\cap]-\infty,x_1[$ if $\ell=0$, on $H\cap]x_\ell,x_{\ell+1}[$ if $0<\ell<k$ and on $H\cap]x_k,+\infty[$ if $\ell=k$.*

Proof. Let $(x_1,\ldots,x_k)\in\sigma_k(H)$ be fixed. Define, for $\ell\geqslant 0$,

$$P_\ell := P_\ell(x_1,\ldots,x_k) := \sum_{\substack{\alpha_1,\ldots,\alpha_k\geqslant 0,\\ \alpha_1+\ldots+\alpha_k=\ell}} x_1^{\alpha_1}\cdots x_k^{\alpha_k}.$$

Observe that $P_0 = 1$, $P_1 = x_1 + \cdots + x_k$, etc. Using Lemma 4, for $j \in \{k, \ldots, n-1\}$, we obtain

$$\big[x_1,\ldots,x_k,x;p_j\big] = \sum_{\alpha=0}^{j-k}\Bigg(\sum_{\substack{\alpha_1,\ldots,\alpha_k\geqslant 0,\\ \alpha_1+\ldots+\alpha_k=j-k-\alpha}} x_1^{\alpha_1}\cdots x_k^{\alpha_k}\Bigg)x^{\alpha} = \sum_{\alpha=0}^{j-k} P_{j-k-\alpha}x^{\alpha}.$$

Therefore, performing elementary row operations on determinants (subtracting P_1 times the first row from the second, then subtracting P_2 times the first plus P_1 times the second row from the third, etc.), for the right hand side of (4), we obtain the following formula

$$\begin{vmatrix} \big[x_1,\ldots,x_k,x_{k+1};p_k\big] & \cdots & \big[x_1,\ldots,x_k,x_{n+1};p_k\big] \\ \vdots & \ddots & \vdots \\ \big[x_1,\ldots,x_k,x_{k+1};p_{n-1}\big] & \cdots & \big[x_1,\ldots,x_k,x_{n+1};p_{n-1}\big] \\ \big[x_1,\ldots,x_k,x_{k+1};f\big] & \cdots & \big[x_1,\ldots,x_k,x_{n+1};f\big] \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cdots & 1 \\ x_{k+1}+P_1 & \cdots & x_{n+1}+P_1 \\ \vdots & \ddots & \vdots \\ x_{k+1}^{n-k-1}+P_1x_{k+1}^{n-k-2}+\cdots+P_{n-k-1} & \cdots & x_{n+1}^{n-k-1}+P_1x_{n+1}^{n-k-2}+\cdots+P_{n-k-1} \\ \big[x_1,\ldots,x_k,x_{k+1};f\big] & \cdots & \big[x_1,\ldots,x_k,x_{n+1};f\big] \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cdots & 1 \\ x_{k+1} & \cdots & x_{n+1} \\ \vdots & \ddots & \vdots \\ x_{k+1}^{n-k-1} & \cdots & x_{n+1}^{n-k-1} \\ \big[x_1,\ldots,x_k,x_{k+1};f\big] & \cdots & \big[x_1,\ldots,x_k,x_{n+1};f\big] \end{vmatrix}.$$

Replacing the right hand side of (4) by the right hand side the above identity, it follows from Theorem 2 that the convexity of f with respect to the Chebyshev system $(\omega_1,\ldots,\omega_n) = (p_0,\ldots,p_{n-1})$ is equivalent to the monotonicity/convexity properties of the mapping $x \mapsto \big[x_1,\ldots,x_k,x;f\big]$ on $H\setminus\{x_1,\ldots,x_k\}$ or on the subintervals of $H\setminus\{x_1,\ldots,x_k\}$. $\square$

EXAMPLE. For a direct application of our results, let ω be the 3-dimensional positive Chebyshev system $\big(1,\cos,\sin\big)$ over the interval $H =]-\pi,0[$. Then $\omega_{\langle 2\rangle} = (1,\cos)$ is a 2-dimensional positive Chebyshev system and, for any function $f : H \to \mathbb{R}$, the following statements are equivalent

(i) f is ω-convex on H;

(ii) For each $x_1 \in H$, the function $x \mapsto \big[x_1,x;f\big]_{(1,\cos)} = \frac{f(x)-f(x_1)}{\cos(x)-\cos(x_1)}$ is convex on $H\setminus\{x_1\}$ with respect to Chebyshev system $\Big(1,-\operatorname{ctg}\big(\frac{x_1+(\cdot)}{2}\big)\Big)$.

Proof. By well-known trigonometrical identities

$$\sin(x)-\sin(y)=2\cos\Big(\frac{x+y}{2}\Big)\sin\Big(\frac{x-y}{2}\Big),$$
$$\cos(x)-\cos(y)=-2\sin\Big(\frac{x+y}{2}\Big)\sin\Big(\frac{x-y}{2}\Big),$$

for $x,y\in H$ with $x\neq y$, we have

$$\frac{\sin(x)-\sin(y)}{\cos(x)-\cos(y)}=-\operatorname{ctg}\Big(\frac{x+y}{2}\Big). \tag{10}$$

Using (10), the right hand side of (4) with $k=1$ can be written as

$$\begin{vmatrix} 1 & 1 & 1 \\ [x_1,x_2;\sin]_{(1,\cos)} & [x_1,x_3;\sin]_{(1,\cos)} & [x_1,x_4;\sin]_{(1,\cos)} \\ [x_1,x_2;f]_{(1,\cos)} & [x_1,x_3;f]_{(1,\cos)} & [x_1,x_4;f]_{(1,\cos)} \end{vmatrix}$$
$$=\begin{vmatrix} 1 & 1 & 1 \\ \frac{\sin(x_2)-\sin(x_1)}{\cos(x_2)-\cos(x_1)} & \frac{\sin(x_3)-\sin(x_1)}{\cos(x_3)-\cos(x_1)} & \frac{\sin(x_4)-\sin(x_1)}{\cos(x_4)-\cos(x_1)} \\ [x_1,x_2;f]_{(1,\cos)} & [x_1,x_3;f]_{(1,\cos)} & [x_1,x_4;f]_{(1,\cos)} \end{vmatrix}$$
$$=\begin{vmatrix} 1 & 1 & 1 \\ -\operatorname{ctg}\big(\frac{x_1+x_2}{2}\big) & -\operatorname{ctg}\big(\frac{x_1+x_3}{2}\big) & -\operatorname{ctg}\big(\frac{x_1+x_4}{2}\big) \\ [x_1,x_2;f]_{(1,\cos)} & [x_1,x_3;f]_{(1,\cos)} & [x_1,x_4;f]_{(1,\cos)} \end{vmatrix}.$$

By Theorem 2, the $(1,\cos,\sin)$-convexity of a function $f:H\to\mathbb{R}$ is equivalent to the nonnegativity of the above determinants, which exactly means that the function $x\mapsto[x_1,x;f]_{(1,\cos)}$ is convex with respect to Chebyshev system $\Big(1,\operatorname{ctg}\big(\frac{-x_1-(\cdot)}{2}\big)\Big)$. □

3. Differences of ω-convex functions

Let H be an open real interval throughout this section and let $\omega:H\to\mathbb{R}^n$ be a Chebyshev system. We introduce the notion of ω-variation which will turn out to be finite for differences of ω-convex functions.

Given a subinterval $[a,b]$, define the set of partitions $\mathscr{P}([a,b])$ of $[a,b]$ by

$$\mathscr{P}([a,b]):=\{(x_0,\dots,x_n)\mid n\in\mathbb{N},\, a=x_0<\dots<x_n=b\}.$$

The ω*-variation* of $f:H\to\mathbb{R}$ on $[a,b]$ is now defined by

$$V^{\omega}_{[a,b]}(f):=\sup\Big\{\sum_{i=0}^{m-n}\Big|[x_{i+1},\dots,x_{i+n};f]_\omega-[x_i,\dots,x_{i+n-1};f]_\omega\Big|:$$
$$m\geqslant n,\,(x_0,\dots,x_m)\in\mathscr{P}([a,b])\Big\}.$$

Applying the notion of ω-variation, the next theorem formulates a necessary condition in order that a function could be decomposed as the difference of two ω-convex functions. The sufficiency of this conditions remains an open problem.

THEOREM 6. *Let $\omega := (\omega_1,\dots,\omega_n) : H \to \mathbb{R}^n$ be an n-dimensional positive Chebyshev system such that $\omega_{\langle n-1\rangle}$ is an $(n-1)$-dimensional positive Chebyshev system and let $f : H \to \mathbb{R}$ be a function. If there exist ω-convex functions $g,h : H \to \mathbb{R}$ such that $f = g-h$, then, for all subinterval $[a,b] \subseteq H$, the ω-variation $V^{\omega}_{[a,b]}(f)$ is finite. Furthermore, for all elements $a_1,\dots,a_n,b_1,\dots,b_n \in H$ with $a_1 < \dots < a_n = a$ and $b = b_1 < \dots < b_n$, the inequality*

$$V^{\omega}_{[a,b]}(f) \leqslant \big[b_1,\dots,b_n;g+h\big]_{\omega} - \big[a_1,\dots,a_n;g+h\big]_{\omega} \tag{11}$$

holds.

Proof. Assume that f is of the form $f = g-h$, where $g,h : H \to \mathbb{R}$ are ω-convex functions. Let $a,b \in H$ with $a<b$ and fix $a_1 < \dots < a_n = a$ and $b = b_1 < \dots < b_n$ in H. Let $(x_0,\dots,x_m) \in \mathscr{P}([a,b])$ be an arbitrary partition of $[a,b]$ with $m \geqslant n$. Then, using the linearity of ω-divided differences and the triangle inequality, we get

$$\begin{aligned}
&\sum_{i=0}^{m-n} \Big| \big[x_{i+1},\dots,x_{i+n};f\big]_{\omega} - \big[x_i,\dots,x_{i+n-1};f\big]_{\omega} \Big| \\
&= \sum_{i=0}^{m-n} \Big| \big[x_{i+1},\dots,x_{i+n};g-h\big]_{\omega} - \big[x_i,\dots,x_{i+n-1};g-h\big]_{\omega} \Big| \\
&\leqslant \sum_{i=0}^{m-n} \Bigg(\Big| \big[x_{i+1},\dots,x_{i+n};g\big]_{\omega} - \big[x_i,\dots,x_{i+n-1};g\big]_{\omega} \Big| \\
&\qquad + \Big| \big[x_{i+1},\dots,x_{i+n};h\big]_{\omega} - \big[x_i,\dots,x_{i+n-1};h\big]_{\omega} \Big| \Bigg).
\end{aligned}$$

In view of the monotonicity property of ω-divided differences established in Corollary 3 for the ω-convex functions g and h, for $i \in \{0,\dots,m-n\}$, we have

$$\begin{aligned}
\Big| \big[x_{i+1},\dots,x_{i+n};g\big]_{\omega} - \big[x_i,\dots,x_{i+n-1};g\big]_{\omega} \Big| &= \big[x_{i+1},\dots,x_{i+n};g\big]_{\omega} - \big[x_i,\dots,x_{i+n-1};g\big]_{\omega}, \\
\Big| \big[x_{i+1},\dots,x_{i+n};h\big]_{\omega} - \big[x_i,\dots,x_{i+n-1};h\big]_{\omega} \Big| &= \big[x_{i+1},\dots,x_{i+n};h\big]_{\omega} - \big[x_i,\dots,x_{i+n-1};h\big]_{\omega}.
\end{aligned}$$

Thus, performing telescopic summation and using the linearity of ω-divided differences, we obtain

$$\begin{aligned}
&\sum_{i=0}^{m-n} \Bigg(\Big| \big[x_{i+1},\dots,x_{i+n};g\big]_{\omega} - \big[x_i,\dots,x_{i+n-1};g\big]_{\omega} \Big| + \Big| \big[x_{i+1},\dots,x_{i+n};h\big]_{\omega} \\
&\qquad - \big[x_i,\dots,x_{i+n-1};h\big]_{\omega} \Big| \Bigg) \\
&= \sum_{i=0}^{m-n} \Bigg(\big[x_{i+1},\dots,x_{i+n};g\big]_{\omega} - \big[x_i,\dots,x_{i+n-1};g\big]_{\omega} + \big[x_{i+1},\dots,x_{i+n};h\big]_{\omega} \\
&\qquad - \big[x_i,\dots,x_{i+n-1};h\big]_{\omega} \Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \Big(\big[x_{m-n+1},\dots,x_m;g\big]_\omega - \big[x_0,\dots,x_{n-1};g\big]_\omega\Big) + \Big(\big[x_{m-n+1},\dots,x_m;h\big]_\omega \\
&\qquad - \big[x_0,\dots,x_{n-1};h\big]_\omega\Big) \\
&= \big[x_{m-n+1},\dots,x_m;g+h\big]_\omega - \big[x_0,\dots,x_{n-1};g+h\big]_\omega.
\end{aligned}
$$

Now, using the inequalities $a_i < x_{i-1}$ and $x_{m-n+i} < b_i$ (which follow from the choice of $a_1,\dots,a_n$ and $b_1,\dots,b_n$), and applying again the monotonicity property of ω-divided differences established in Corollary 3 for the ω-convex function $g+h$, we get

$$
\begin{aligned}
-\big[x_0,\dots,x_{n-1};g+h\big]_\omega &\leqslant -\big[a_1,\dots,a_n;g+h\big]_\omega \\
\big[x_{m-n+1},\dots,x_m;g+h\big]_\omega &\leqslant \big[b_1,\dots,b_n;g+h\big]_\omega.
\end{aligned}
$$

Finally, combining the above three estimates, for every partition $(t_0,\dots,t_m)$ of $[a,b]$, we obtain

$$
\begin{aligned}
\sum_{i=0}^{m-n} \Big|\big[x_{i+1},\dots,x_{i+n};f\big]_\omega - \big[x_i,\dots,x_{i+n-1};f\big]_\omega\Big| \\
\leqslant \big[b_1,\dots,b_n;g+h\big]_\omega - \big[a_1,\dots,a_n;g+h\big]_\omega,
\end{aligned}
$$

which implies

$$
V_{[a,b]}^{\omega}(f) \leqslant \big[b_1,\dots,b_n;g+h\big]_\omega - \big[a_1,\dots,a_n;g+h\big]_\omega < +\infty.
$$

Thus, inequality (11) and the theorem is proved. □

For the case of higher-order convexity in the sense of Hopf and Popoviciu, the following characterization holds (cf. [7]), which, in one direction, is a consequence Theorem 6.

COROLLARY 7. *Let $\omega := (p_0,\dots,p_{n-1}) : H \to \mathbb{R}^n$ be the n-dimensional positive Chebyshev system and let $f : H \to \mathbb{R}$ be a function. Then, there exist ω-convex functions $g,h : H \to \mathbb{R}$ such that $f = g-h$ if and only if for all subinterval $[a,b] \subseteq H$, the ω-variation $V_{[a,b]}^{\omega}(f)$ is finite.*

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(Received July 5, 2016)

Zsolt Páles and Éva Székelyné Radácsi
Institute of Mathematics
University of Debrecen
H-4032 Debrecen, Egyetem tér 1, Hungary
e-mail: pales@science.unideb.hu
e-mail: radacsi.eva@science.unideb.hu