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A NEW GENERALIZATION OF BOAS THEOREM FOR SOME LORENTZ SPACES $\Lambda_q(\omega)$

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Abstract. Let $\Lambda_q(\omega)$, $q>0$, denote the Lorentz space equipped with the (quasi) norm

$$\|f\|_{\Lambda_q(\omega)} := \left(\int_0^1 (f^*(t)\omega(t))^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

for a function f on $[0,1]$ and with ω positive and equipped with some additional growth properties. A generalization of Boas theorem in the form of a two-sided inequality is obtained in the case of both general regular system $\Phi=\{\varphi_k\}_{k=1}^{\infty}$ and generalized Lorentz $\Lambda_q(\omega)$ spaces.

1. Introduction

The following Hardy-Littlewood theorem is well known (see [26] and also [10], [4]):

THEOREM A. *If $f \geqslant 0$ and f decreases, $1<p<\infty$, and a_n are the Fourier sine or cosine coefficients of f, then*

$$\sum_{n=1}^{\infty} |a_n|^p < \infty$$

if and only if

$$x^{p-2} f(x)^p \in L_p.$$

This theorem can be extended as follows (see [4]):

THEOREM B. *If $f \geqslant 0$ and f decreases, $1<p<\infty$, $-1/p'<\gamma<1/p$, then*

$$\sum_{n=1}^{\infty} n^{-\gamma p}|a_n|^p < \infty$$

converges if and only if

$$x^{p-2}x^{p\gamma+p-2} f(x)^p \in L_p.$$

Here and in the sequel $p' = \frac{p}{(p-1)}$ for $p>1$.

A characterization for the function f to belong to the Lorentz space L_{pq} was obtained by R. P. Boas in [4]. This result deals with trigonometric Fourier coefficients for the class of monotone functions and reads:

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THEOREM C. *If $f \geqslant 0$ and f decreases, $1 < p < \infty$, $1 < q < \infty$, then $f \in L_{pq}$ if and only if $\{a_n\} \in l_{p'q}$.*

Some other results which are related to the Hardy-Littlewood theorem for the class of monotone functions were obtained in [25], [3], [1], [24], [5], [15], [8], [9], [12] and [7].

Boas theorem was generalized and complemented in various ways also for more general Lorentz spaces $\Lambda_q(\omega)$ in 1974 by L.-E. Persson for the case when $\Phi = \{e^{2\pi ikx}\}_{k=-\infty}^{+\infty}$ is trigonometric system (see. [20]–[23]). For example the following theorem was proved:

THEOREM D. *Let $p > 0$ and $\Phi = \{e^{2\pi ikt}\}_{k=-\infty}^{+\infty}$ be a trigonometrical system. Let ω be a nonnegative function on $[0,\infty)$. If there exists a positive number $\delta > 0$ satisfying that $\omega(t)t^{-\delta}$ is an increasing function of t and $\omega(t)t^{-1+\delta}$ is a decreasing function of t and if f is a nonnegative and a decreasing function on $[0,\frac{1}{2}]$, then*

$$\left(\int_0^1 (f^*(t)\omega(t))^p \frac{dt}{t}\right)^{\frac{1}{p}} < \infty,$$

if and only if

$$\left(\sum_{k=1}^{\infty} (k\omega\left(\frac{1}{k}\right) a_k^*)^p \frac{1}{n}\right)^{\frac{1}{p}} < \infty,$$

where $\{a_k^*\}_{k=1}^{\infty}$ is the nonincreasing rerrangement of the sequence $\{a_n\}_{k=1}^{\infty}$ of Fourier coefficients of f with respect to the system Φ.

The main aim of this paper is to derive the Boas theorem for the space $\Lambda_q(\omega)$ with respect to the regular system. Moreover, a new Boas type theorem for space $\Lambda_q(\omega)$ and for generalized monotone functions is proved and discussed.

The main results are formulated in Section 3. Note that the results in Theorem 1 is obviously related to [11] but we have chosed to put also this result in this more general frame in English. The proofs can be found in Section 4 and in Section 2 we have presented some necessary preliminaries.

CONVENTIONS. The letter $c(c_1, c_2, \text{etc.})$ means a constant which does not dependent on the involved functions and it can be different in different occurences. Moreover, for $C, D > 0$ the notation $C \sim D$ means that there exist positive constants a_1 and a_2 such that $a_1 D \leqslant C \leqslant a_2 D$.

2. Preliminaries

Let f be a measurable function on $[0,1]$ and μ is Lebesgue measure. The nonincreasing rerrangement f^* of a function f is defined as follows:

$$m(\sigma, f) := \mu\{x \in [0,1] : \ |f(x)| > \sigma\},$$

$$f^*(t) := inf\{\sigma : \ m(\sigma, f) \leqslant t\}.$$

Let $0 < q \leqslant \infty$ and ω be a nonnegative function on $[0,1]$. The generalized Lorentz spaces $\Lambda_q(\omega)$ consists of the functions f on $[0,1]$ such that $\|f\|_{\Lambda_q(\omega)} < \infty$, where

$$\|f\|_{\Lambda_q(\omega)} := \begin{cases} \left(\int_0^1 (f^*(t)\omega(t))^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{for } 0 < q < \infty, \\ \sup\limits_{0 \leqslant t \leqslant 1} f^*(t)\omega(t) & \text{for } q = \infty. \end{cases}$$

These spaces $\Lambda_q(\omega)$ coincide to the classical spaces L_{pq} in the case $\omega(t) = t^{\frac{1}{p}}$, $1 < p < \infty$ (see [16] and also e.g. [2]).

Let $\mu = \{\mu(k)\}_{k \in \mathbb{N}}$ be a sequence of positive number and the space $\lambda_q(\mu)$ consists of all sequences $a = \{a_k\}_{k=1}^{\infty}$ such that $\|a\|_{\lambda_q(\mu)} < \infty$, where

$$\|a\|_{\lambda_q(\mu)} := \begin{cases} \left(\sum_{k=1}^{\infty} \left(a_k^* \mu(k)\right)^q \frac{1}{k}\right)^{\frac{1}{q}} & \text{for } 0 < q < \infty, \\ \sup\limits_{k} a_k^* \mu(k) & \text{for } q = \infty. \end{cases}$$

Here, as usual, $\{a_k^*\}_{k=1}^{\infty}$ is the nonincreasing rearrangement of the sequence $\{|a_k|\}_{k=1}^{\infty}$.

Let the function f be periodic with period 1 and integrable on $[0,1]$ and let $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ be an orthonormal system on $[0,1]$. The numbers

$$a_k = a_k(f) = \int_0^1 f(x)\overline{\varphi_k(x)}dx, \;\; k \in \mathbb{N}$$

are called the Fourier coefficients of the functions f with respect to the system $\Phi = \{\varphi_k\}_{k=1}^{\infty}$.

We say that the orthonormal system $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ is regular if there exists a constant B, such that

1) for every segment e from $[0,1]$ and $k \in \mathbb{N}$ it yields that

$$\left|\int_e \varphi_k(x)dx\right| \leqslant B\min(|e|, 1/k),$$

2) for every segment w from $\mathbb{N}$ and $t \in (0,1]$ we have that

$$\left(\sum_{k \in w} \varphi_k(\cdot)\right)^*(t) \leqslant B\min(|w|, 1/t),$$

where $\left(\sum_{k \in w} \varphi_k(\cdot)\right)^*(t)$ as usual denotes the nonincreasing rerrangement of the function $\sum_{k \in w} \varphi_k(x)$.

Examples of regular systems are all trigonometrical systems, the Walsh system and Prise's system. In [17], [19], [18] some results were obtained with respect to the regular system using network space.

Let $\delta > 0$ be a fixed parameter. Consider a nonnegative function $\omega(t)$ on $[0,1]$. We define the following classes:

$$\begin{aligned} A_\delta := \{\omega(t) : \; & \omega(t)t^{-\frac{1}{2}-\delta} \text{is an increasing function and} \\ & \omega(t)t^{-1+\delta} \text{is a decreasing function}\}, \end{aligned}$$

$$B_\delta := \{\omega(t): \ \omega(t)t^{-\delta} \text{ is an increasing function and} \\ \omega(t)t^{-1+\delta} \text{ is a decreasing function}\}.$$

Then the classes A and B can be defined as follows:

$$A = \bigcup_{\delta>0} A_\delta.$$

and

$$B = \bigcup_{\delta>0} B_\delta.$$

For the proof of our main results we need the following Theorem:

THEOREM E. *Let $\Phi = \{\varphi_k\}_{k=1}^\infty$ be a regular system and $f \overset{a.e.}{=} \sum_{k=1}^\infty a_k\varphi_k$. Let $1 \leqslant q \leqslant \infty$. If ω belongs to the class B, then*

$$\left(\int_0^1 \left(\overline{f(t)}\omega(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \leqslant c\left(\sum_{k=1}^\infty (a_k^*\mu(k))^q \frac{1}{k}\right)^{\frac{1}{q}},$$

where $\overline{f}(t) = \sup_{\xi \geqslant t} \frac{1}{\xi}\left|\int_0^\xi f(s)ds\right|$, $\mu(k) = k\omega\left(\frac{1}{k}\right)$ and the constant c does not depend on f.

This is just a slight generalization of Theorem 2 in [14] (see also [11]). For the reader's convenience we include a proof in Appendix 1.

We also need the following techniquel Lemma:

LEMMA 1. *Let $1 \leqslant q \leqslant \infty$ and $1 \leqslant h \leqslant \infty$. If $\omega(t)$ belongs to the class B, then for any nonincreasing function f it yields that*

$$\left(\sum_{k=1}^\infty \left(\int_{2^{-k}}^{2^{-k+1}} (f(t)\omega(t))^h \frac{dt}{t}\right)^{\frac{q}{h}}\right)^{\frac{1}{q}} \sim \left(\int_0^1 (f(t)\omega(t))^q \frac{dt}{t}\right)^{\frac{1}{q}}. \tag{1}$$

Proof. First we prove the following equivalence:

$$\left(\sum_{k=1}^\infty \left(\int_{2^{-k}}^{2^{-k+1}} (f(t)\omega(t))^h \frac{dt}{t}\right)^{\frac{q}{h}}\right)^{\frac{1}{q}} \sim \left(\sum_{k=1}^\infty \left(f(2^{-k})\omega(2^{-k})\right)^q\right)^{\frac{1}{q}}. \tag{2}$$

Let

$$\begin{aligned} I_h &:= \left(\sum_{k=1}^\infty \left(\int_{2^{-k}}^{2^{-k+1}} (f(t)\omega(t))^h \frac{dt}{t}\right)^{\frac{q}{h}}\right)^{\frac{1}{q}} \\ &= \left(\sum_{k=1}^\infty \left(\int_{2^{-k}}^{2^{-k+1}} \left(f(t)\omega(t)t^{-1+\delta}t^{1-\delta}\right)^h \frac{dt}{t}\right)^{\frac{q}{h}}\right)^{\frac{1}{q}}. \end{aligned}$$

We use the fact that $\omega = \omega(t)$ belong to the class B. This means that there exists δ, $0<\delta<1$, such that $\omega(t)t^{-\delta}$ is an increasing function and $\omega(t)t^{-1+\delta}$ is a decreasing function. Then we have:

$$I_h \leqslant \left(\sum_{k=1}^{\infty}\left(f(2^{-k})\omega(2^{-k})2^{-k(-1+\delta)}\left(\int_{2^{-k}}^{2^{-k+1}} t^{(1-\delta)h}\frac{dt}{t}\right)^{\frac{1}{h}}\right)^{q}\right)^{\frac{1}{q}}$$

$$= c_1\left(\sum_{k=1}^{\infty}\left(f(2^{-k})\omega(2^{-k})2^{k-k\delta}2^{k\delta-k}\right)^{q}\right)^{\frac{1}{q}} = c_1\left(\sum_{k=1}^{\infty}\left(f(2^{-k})\omega(2^{-k})\right)^{q}\right)^{\frac{1}{q}}.$$

$$I_h = \left(\sum_{k=1}^{\infty}\left(\int_{2^{-k}}^{2^{-k+1}}\left(f(t)\omega(t)t^{-\delta}t^{\delta}\right)^{h}\frac{dt}{t}\right)^{\frac{q}{h}}\right)^{\frac{1}{q}}$$

$$\geqslant \left(\sum_{k=1}^{\infty}\left(f(2^{-k+1})\omega(2^{-k})2^{k\delta}\left(\int_{2^{-k}}^{2^{-k+1}} t^{\delta h-1}dt\right)^{\frac{1}{h}}\right)^{q}\right)^{\frac{1}{q}}$$

$$= c_2\left(\sum_{k=1}^{\infty}\left(f(2^{-k+1})\omega(2^{-k})\right)^{q}\right)^{\frac{1}{q}} \geqslant c_3\left(\sum_{k=1}^{\infty}\left(f(2^{-k})\omega(2^{-k})\right)^{q}\right)^{\frac{1}{q}}.$$

Thus, (2) is proved, which, in particular means that $I_{h_1} \sim I_{h_2}$ for all h_1 and h_2. Moreover, since f is nonincreasing and $\omega \in B$, it follows that

$$\left(\sum_{k=1}^{\infty}\left(f(2^{-k})\omega(2^{-k})\right)^{q}\right)^{\frac{1}{q}} \sim \left(\int_0^1 (f(t)\omega(t))^q\frac{dt}{t}\right)^{\frac{1}{q}}.$$

In particular, (1) follows and the proof is complete. $\square$

3. Main results

The main results of this paper are the following generalizations of the Boas theorem:

THEOREM 1. *Let $1 \leqslant q \leqslant \infty$ and $\omega \in B$. Let $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ be a regular system and let $f \overset{a.e.}{=} \sum_{k=1}^{\infty} a_k\varphi_k$. If f is a nonnegative and a nonincreasing function, then*

$$\left(\int_0^1 (f(t)\omega(t))^q\frac{dt}{t}\right)^{\frac{1}{q}} \sim \left(\sum_{k=1}^{\infty}(a_k^*\mu(k))^q\frac{1}{k}\right)^{\frac{1}{q}},$$

where $\mu(k) = k\omega(\frac{1}{k})$.

We say that a function f on $[0,1]$ is generalized monotone if there exists some constant $M>0$ such that

$$|f(x)| \leqslant M\frac{1}{x}\left|\int_0^x f(t)dt\right|, \;\; x>0.$$

Our next main result reads:

THEOREM 2. *Let* $1 \leqslant q \leqslant \infty$ *and* $\omega \in A$. *Let* $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ *be a regular system and let* $f \overset{a.e.}{=} \sum_{k=1}^{\infty} a_k\varphi_k$. *If* f *is a nonnegative and a generalized monotone function, then*

$$\|f\|_{\Lambda_q(\omega,[0,1])} \sim \left(\sum_{k=1}^{\infty}(a_k^*\mu(k))^q\frac{1}{k}\right)^{\frac{1}{q}},$$

where $\mu(k) = k\omega(\frac{1}{k})$.

4. Proofs of the main results

Proof of Theorem 1. The necessary part is similar to that in Theorem E. Indeed, since f is a nonincreasing function, then $f(t) \leqslant \overline{f(t)}$, $0<t<1$, so that

$$\left(\int_0^1 (f(t)\omega(t))^q\frac{dt}{t}\right)^{\frac{1}{q}} \leqslant \left(\int_0^1 \left(\overline{f(t)}\omega(t)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}} \leqslant c\left(\sum_{k=1}^{\infty}(a_k^*\mu(k))^q\frac{1}{k}\right)^{\frac{1}{q}},$$

where $\overline{f}(t) = \sup\limits_{\xi\geqslant t}\frac{1}{\xi}\int\limits_0^{\xi} f(s)ds$. We prove the sufficient condition. The condition $\omega(t)\in B$ implies that there exists $\delta>0$ such that $\omega(t)t^{-\delta}$ is an increasing and $\omega(t)t^{-1+\delta}$ is a decreasing function, i.e. $\mu(k)k^{-\delta}$ is increasing and $\mu(k)k^{-1+\delta}$ is decreasing. Then the following estimate holds:

$$\frac{1}{k}\sum_{n=1}^{k}\frac{\mu^q(n)}{n} \leqslant c\frac{\mu^q(k)}{k}, \;\; k\in\mathbb{N}.$$

Indeed,

$$\frac{1}{k}\sum_{n=1}^{k}\frac{\mu^q(n)}{n} \leqslant \frac{1}{k}\mu^q(k)k^{-\delta}\sum_{n=1}^{k}\frac{1}{n^{1-\delta}} \sim \frac{\mu^q(k)}{k}.$$

Next, we use Theorem 2.4.12 (ii) from [6] to conclude that the following equality holds:

$$\lambda_q(\mu) = \left(\lambda_{q'}(\mu^{-1}k)\right)', \;\; \text{for } 1<q<\infty,$$

where $\left(\lambda_{q'}(\mu^{-1}k)\right)'$ is dual space for the space $\lambda_q(\mu)$. Hence, by appling the duality representation of the norm of a sequence a in the space $\lambda_q(\mu)$ (see [6]), we obtain that

$$\|a\|_{\lambda_q(\mu)} = \sup_{\|b\|_{\lambda_{q'}(\mu^{-1}k)}=1}\sum_{k=1}^{\infty}a_kb_k.$$

Now we use Parseval's formula and find that

$$\begin{aligned}\|a\|_{\lambda_q(\mu)} &= \sup_{\|b\|_{\lambda_{q'}(\mu^{-1}k)}=1} \int_0^1 f(t)g(t)dt \\ &= \sup_{\|b\|_{\lambda_{q'}(\mu^{-1}k)}=1} \sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} f(t)g(t)dt \\ &\leqslant \sup_{\|b\|_{\lambda_{q'}(\mu^{-1}k)}=1} \sum_{k=0}^{\infty} \left| \int_{2^{-k-1}}^{2^{-k}} f(t)g(t)dt \right|. \qquad (3)\end{aligned}$$

We apply the mean value theorem to the integral $\int_{2^{-k-1}}^{2^{-k}} f(t)g(t)dt$ to conclude that there exists ξ from $\left(2^{-k-1},\ 2^{-k}\right)$ such that

$$\begin{aligned}\left|\int_{2^{-k-1}}^{2^{-k}} f(t)g(t)dt\right| &= \left|f(2^{-k-1}) \int_{2^{-k-1}}^{\xi} g(t)dt\right| \\ &\leqslant f(2^{-k-1}) \left(\left|\int_0^{\xi} g(t)dt\right| + \left|\int_0^{2^{-k-1}} g(t)dt\right| \right) \\ &\leqslant f(2^{-k-1}) \left(2^{-k} \sup_{s\geqslant 2^{-k-2}} \frac{1}{s} \left|\int_0^s g(t)dt\right| + 2^{-k} \sup_{s\geqslant 2^{-k-2}} \frac{1}{s} \left|\int_0^s g(t)dt\right| \right) \\ &= 2\cdot 2^{-k}\cdot f(2^{-k-1})\cdot \overline{g(2^{-k-2})}, \qquad (4)\end{aligned}$$

where $\overline{g(2^{-k-2})} = \sup\limits_{s\geqslant 2^{-k-2}} \frac{1}{s}\left|\int_0^s g(t)dt\right|$.

Thus, by inserting (4) in (3), we conclude that

$$\begin{aligned}\|a\|_{\lambda_q(\mu)} &\leqslant 8 \sup_{\|b\|_{\lambda_{q'}(\mu^{-1}k)}=1} \sum_{k=0}^{\infty} 2^{-k-2} f(2^{-k-1})\overline{g(2^{-k-2})} \\ &= 8 \sup_{\|b\|_{\lambda_{q'}(\mu^{-1}k)}=1} \sum_{k=0}^{\infty} \left(2^{-k-2}\left(\omega(2^{-k-2})\right)^{-1} \overline{g(2^{-k-2})}\right)\cdot f(2^{-k-1})\omega(2^{-k-2}) \\ &= 8 \sup_{\|b\|_{\lambda_{q'}(\mu^{-1}k)}=1} \sum_{k=2}^{\infty} \left(2^{-k}\left(\omega(2^{-k})\right)^{-1} \overline{g(2^{-k})}\right)\cdot f(2^{-k+1})\omega(2^{-k}).\end{aligned}$$

Next, by using Hölder's inequality, we get that

$$\begin{aligned}\|a\|_{\lambda_q(\mu)} \leqslant c_1 &\sup_{\|b\|_{\lambda_{q'}(\mu^{-1}k)}=1} \left(\sum_{k=2}^{\infty} \left(2^{-k}\left(\omega(2^{-k})\right)^{-1} \overline{g(2^{-k})}\right)^{q'} \right)^{\frac{1}{q'}} \\ &\times \left(\sum_{k=2}^{\infty} \left(f(2^{-k+1})\omega(2^{-k})\right)^{q} \right)^{\frac{1}{q}}\end{aligned}$$

$$
\begin{aligned}
&\leqslant c_1 \sup_{\|b\|_{\lambda_{q'}(\mu^{-1}k)}=1} \left(\sum_{k=1}^{\infty} \left(2^{-k} \left(\omega(2^{-k}) \right)^{-1} \overline{g(2^{-k+1})} \right)^{q'} \right)^{\frac{1}{q'}} \\
&\quad \times \left(\sum_{k=1}^{\infty} \left(f(2^{-k+1}) \omega(2^{-k}) \right)^{q} \right)^{\frac{1}{q}} \\
&= c_2 \sup_{\|b\|_{\lambda_{q'}(\mu^{-1}k)}=1} \left(\sum_{k=1}^{\infty} \left(\frac{2^{-k(1-\delta)} \omega^{-1}(2^{-k})}{2^{-k(1-\delta)}} 2^{-k} \overline{g(2^{-k+1})} \right)^{q'} \right)^{\frac{1}{q'}} \\
&\quad \times \left(\sum_{k=1}^{\infty} \left(\frac{2^{k\delta} \omega(2^{-k})}{2^{k\delta}} f(2^{-k+1}) \int_{2^{-k}}^{2^{-k+1}} \frac{dt}{t} \right)^{q} \right)^{\frac{1}{q}}.
\end{aligned}
$$

Since $f(t)$ is a nonincreasing function for all $0 < t < 1$ and $\omega(t)$ belongs to B, then there exists $0 < \delta < 1$ such that $\omega(t)t^{-\delta}$ is an increasing and $\omega(t)t^{-1+\delta}$ is a decreasing function, we get that

$$
\begin{aligned}
\|a\|_{\lambda_q(\mu)} &\leqslant c_3 \sup_{\|b\|_{\lambda_{q'}(\mu^{-1}k)}=1} \left(\sum_{k=1}^{\infty} \left(2^{k(1-\delta)} \int_{2^{-k}}^{2^{-k+1}} t^{1-\delta} \omega^{-1}(t) \overline{g(t)} dt \right)^{q'} \right)^{\frac{1}{q'}} \\
&\quad \times \left(\sum_{k=1}^{\infty} \left(2^{-k\delta} \int_{2^{-k}}^{2^{-k+1}} f(t)\omega(t) t^{-\delta} \frac{dt}{t} \right)^{q} \right)^{\frac{1}{q}} \\
&\leqslant c_4 \sup_{\|b\|_{\lambda_{q'}(\mu^{-1}k)}=1} \left(\sum_{k=1}^{\infty} \left(\int_{2^{-k}}^{2^{-k+1}} t\omega^{-1}(t) \overline{g(t)} \frac{dt}{t} \right)^{q'} \right)^{\frac{1}{q'}} \left(\sum_{k=1}^{\infty} \left(\int_{2^{-k}}^{2^{-k+1}} f(t)\omega(t) \frac{dt}{t} \right)^{q} \right)^{\frac{1}{q}}.
\end{aligned}
$$

By now applying Lemma 1, we obtain that

$$
\|a\|_{\lambda_q(\mu)} \leqslant c_5 \sup_{\|b\|_{\lambda_{q'}(\mu^{-1}k)}=1} \left(\int_0^1 \left(t\omega^{-1}(t) \overline{g(t)} \right)^{q'} \frac{dt}{t} \right)^{\frac{1}{q'}} \cdot \left(\int_0^1 (f(t)\omega(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}.
$$

Furthermore, by using Theorem E, we obtain the following estimate

$$
\begin{aligned}
\|a\|_{\lambda_q(\mu)} &\leqslant c_6 \sup_{\|b\|_{\lambda_{q'}(\mu^{-1}k)}=1} \left(\sum_{k=1}^{\infty} \left(b_k^* k \mu^{-1}(k) \right)^{q'} \frac{1}{k} \right)^{\frac{1}{q'}} \cdot \left(\int_0^1 (f(t)\omega(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} \\
&= c_6 \sup_{\|b\|_{\lambda_{q'}(\mu^{-1}k)}=1} \|b\|_{\lambda_{q'}(\mu^{-1}k)} \cdot \left(\int_0^1 (f(t)\omega(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} \\
&= c_6 \left(\int_0^1 (f(t)\omega(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}.
\end{aligned}
$$

The proof is complete. $\square$

Proof of Theorem 2. The condition $\omega(t) \in A$ implies that there exists $\delta > 0$ such that $\omega(t)t^{-\frac{1}{2}-\delta}$ is an increasing function and $\omega(t)t^{-1+\delta}$ is a decreasing function. The necessary condition follows in a similar way as in Theorem E. Indeed, let $x > 0$ and

$$f^{**}(x) := \sup_{|e|=x} \frac{1}{|e|} \int_e |f(t)| dt.$$

It is obvious that $f^*(x) \leqslant f^{**}(x)$. Since f is a generalized monotone function, it yields that

$$f^{**}(x) = \sup_{|e|=x} \frac{1}{|e|} \int_e |f(t)| dt \leqslant \sup_{|e|=x} \frac{1}{|e|} \int_e \overline{f(t)} dt = \frac{1}{x} \int_0^x \overline{f(t)} dt,$$

where $\overline{f}(t) = \sup\limits_{\xi \geqslant t} \int_0^\xi f(s) ds.$

Thus, we obtain the following inequalities

$$\|f\|_{\Lambda_q(\omega)} \leqslant \|f^{**}\|_{\Lambda_q(\omega)} \leqslant M \|\frac{1}{x} \int_0^x \overline{f(t)} dt\|_{\Lambda_q(\omega)}. \tag{5}$$

We prove the following inequality

$$\left(\int_0^1 \left(\omega(x) \frac{1}{x} \int_0^x \overline{f(t)} dt \right)^q \frac{dx}{x} \right)^{\frac{1}{q}} \leqslant c \left(\int_0^1 \left(\overline{f(t)} \omega(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}. \tag{6}$$

Choose ε so that $-\frac{1}{q} + 1 - \delta < \varepsilon < -\frac{1}{q} + 1$. We consider for any $x > 0$

$$\int_0^x \overline{f(t)} dt = \int_0^x \overline{f(t)} t^\varepsilon t^{-\varepsilon} dt.$$

Next we use Hölder's inequality and the fact that $\varepsilon < -\frac{1}{q} + 1$ to find that

$$\begin{aligned}
\int_0^x \overline{f(t)} dt &\leqslant c_1 \left(\int_0^x \left(\overline{f}(t) t^\varepsilon \right)^q dt \right)^{\frac{1}{q}} \left(\int_0^x \left(t^{-\varepsilon} \right)^{q'} dt \right)^{\frac{1}{q'}} \\
&\sim \left(\int_0^x \left(\overline{f}(t) t^\varepsilon \right)^q dt \right)^{\frac{1}{q}} x^{-\varepsilon + \frac{1}{q'}}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
I &:= c_2 \left(\int_0^1 \left(\omega(x) x^{-\varepsilon + \frac{1}{q'} - 1} \right)^q \left(\int_0^x \left(\overline{f}(t) t^\varepsilon \right)^q dt \right) \frac{dx}{x} \right)^{\frac{1}{q}} \\
&= c_2 \left(\int_0^1 \left(\overline{f}(t) t^\varepsilon \right)^q \left(\int_t^1 x^{-\varepsilon q - 1} \omega^q(x) \frac{dx}{x} \right) dt \right)^{\frac{1}{q}}.
\end{aligned}$$

By now using the fact that $\omega(t)t^{-\frac{1}{2}-\delta}$ is an increasing function, we find that

$$I \leqslant c_2 \left(\int_0^1 \left(\overline{f}(t)t^{\varepsilon}\right)^q \left(\omega(t)t^{-\frac{1}{2}-\delta}\right)^q \left(\int_{\frac{1}{t}}^1 x^{\varepsilon q+1-\frac{1}{2}q-\delta q} \frac{dx}{x} \right) dt \right)^{\frac{1}{q}}.$$

Taking into account that $\varepsilon \geqslant -\frac{1}{q}+\frac{1}{2}+\delta$, we obtain that

$$I \leqslant c_3 \left(\int_0^1 \left(\overline{f}(t)\omega(t)\right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

Thus, we have proved the inequality (6). From (5) and (6) it follows that

$$\|f\|_{\Lambda_q(\omega)} \leqslant c_3 \|\overline{f}\|_{\Lambda_q(\omega)}.$$

By now applying Theorem E, we obtain that

$$\|f\|_{\Lambda_q(\omega)} \leqslant c_4 \|a\|_{\lambda_q(\mu)}.$$

Since each regular system is bounded orthonormal system, then the sufficient condition follows from Theorem 2 in [13].

The proof is complete. □

5. Appendix 1

Proof of Theorem E. Assume that $\omega(t)$ belongs to the class B. This means that there exists $\delta > 0$ such that $\omega(t)t^{-\delta}$ is an increasing function and $\omega(t)t^{-1+\delta}$ is a decreasing function. Suppose that

$$\left(\sum_{k=1}^{\infty} \left(a_k^* \mu(k)\right)^q \frac{1}{k} \right)^{\frac{1}{q}} < \infty$$

and $f \overset{\text{a.e.}}{=} \sum_{k=1}^{\infty} a_k \varphi_k$. It yields that

$$\begin{aligned} \left| \int_0^{\xi} f(s)ds \right| &= \left| \int_0^{\xi} \sum_{k\in\mathbb{N}} a_k \varphi_k(s)ds \right| \\ &\leqslant \sum_{k\in\mathbb{N}} |a_k| \left| \int_0^{\xi} \varphi_k(s)ds \right|, \text{ for all } \xi \in [0,1]. \end{aligned}$$

According to the regularity assumption we have that

$$\left| \int_0^{\xi} \varphi_k(s)ds \right| \leqslant B \min\left(\xi, \frac{1}{k}\right), \ k \in \mathbb{N}.$$

Hence,

$$\begin{aligned}\sum_{k=1}^{\infty}|a_k|\left|\int_0^{\xi}\varphi_k(s)ds\right| &\leqslant c_1\sum_{k=1}^{\infty}|a_k|\min\left(\xi,\frac{1}{k}\right)\\ &\leqslant c_1\sum_{k=1}^{\infty}a_k^*\min\left(\xi,\frac{1}{k}\right)\\ &\leqslant c_1\left(\sum_{k=1}^{\left[\frac{1}{\xi}\right]}a_k^*\xi+\sum_{k=\left[\frac{1}{\xi}\right]}^{\infty}a_k^*\frac{1}{k}\right).\end{aligned}$$

Consequently,

$$\left|\int_0^{\xi}f(s)ds\right|\leqslant c_1\left(\sum_{k=1}^{\left[\frac{1}{\xi}\right]}a_k^*\xi+\sum_{k=\left[\frac{1}{\xi}\right]}^{\infty}a_k^*\frac{1}{k}\right)$$

and we have that

$$\begin{aligned}&\left(\int_0^1\left(\overline{f(t)}\omega(t)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\\ &\leqslant\ c_1\left(\int_0^1\left(\omega(t)\sup_{\xi\geqslant t}\frac{1}{\xi}\left(\sum_{k=1}^{\left[\frac{1}{\xi}\right]}a_k^*\xi+\sum_{k=\left[\frac{1}{\xi}\right]}^{\infty}a_k^*\frac{1}{k}\right)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\\ &\leqslant\ c_1\left(\int_0^1\left(\omega(t)\sup_{\xi\geqslant t}\frac{1}{\xi}\left(\sum_{k=1}^{\left[\frac{1}{\xi}\right]}a_k^*\xi+\sum_{k=\left[\frac{1}{\xi}\right]}^{\left[\frac{1}{t}\right]}a_k^*\cdot\frac{1}{k}+\sum_{k=\left[\frac{1}{t}\right]}^{\infty}a_k^*\frac{1}{k}\right)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\\ &\leqslant\ c_1\left(\int_0^1\left(\omega(t)\sup_{\xi\geqslant t}\frac{1}{\xi}\left(\sum_{k=1}^{\left[\frac{1}{\xi}\right]}a_k^*\xi+\sum_{k=\left[\frac{1}{\xi}\right]}^{\left[\frac{1}{t}\right]}a_k^*\cdot\xi+\sum_{k=\left[\frac{1}{t}\right]}^{\infty}a_k^*\frac{1}{k}\right)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\\ &=\ c_1\left(\int_0^1\left(\omega(t)\sup_{\xi\geqslant t}\frac{1}{\xi}\left(\sum_{k=1}^{\left[\frac{1}{t}\right]}a_k^*\xi+\sum_{k=\left[\frac{1}{t}\right]}^{\infty}a_k^*\frac{1}{k}\right)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\\ &=\ c_1\left(\int_0^1\left(\omega(t)\left(\sum_{k=1}^{\left[\frac{1}{t}\right]}a_k^*+\sup_{\xi\geqslant t}\frac{1}{\xi}\cdot\sum_{k=\left[\frac{1}{t}\right]}^{\infty}a_k^*\frac{1}{k}\right)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\\ &\leqslant\ c_1\left(\int_0^1\left(\omega(t)\left(\sum_{k=1}^{\left[\frac{1}{t}\right]}a_k^*+\frac{1}{t}\cdot\sum_{k=\left[\frac{1}{t}\right]}^{\infty}a_k^*\frac{1}{k}\right)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\end{aligned}$$

$$\leqslant c_1\left(\int_0^1\left(\omega(t)\sum_{k=1}^{\left[\frac{1}{t}\right]}a_k^*\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}+c_1\left(\int_0^1\left(\omega(t)\frac{1}{t}\sum_{k=\left[\frac{1}{t}\right]}^{\infty}a_k^*\frac{1}{k}\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}$$
$$:=c_1\left(I_1+I_2\right).$$

We consider first I_1. Choose a small number ε such that $\frac{1}{q}-1-\delta<\varepsilon<\frac{1}{q}-1$. Since $\omega(t)t^{-\delta}$ is an increasing function of t, it yields that

$$\begin{aligned}
I_1&=\left(\int_0^1\left(\omega(t)\sum_{k=1}^{\left[\frac{1}{t}\right]}a_k^*\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\\
&=\left(\int_0^1\left(\frac{\omega(t)t^{-\delta}}{t^{-\delta}}\sum_{k=1}^{\left[\frac{1}{t}\right]}a_k^*\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\\
&\leqslant\left(\int_0^1\left(t^{\delta}\sum_{k=1}^{\left[\frac{1}{t}\right]}\omega\left(\frac{1}{k}\right)\left(\frac{1}{k}\right)^{-\delta}a_k^*\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\\
&=\left(\int_1^\infty\left(t^{-\delta}\sum_{k=1}^{t}\omega\left(\frac{1}{k}\right)\left(\frac{1}{k}\right)^{-\delta}a_k^*\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\\
&\sim\left(\sum_{n=1}^\infty\left(n^{-\delta}\sum_{k=1}^{n}\omega\left(\frac{1}{k}\right)\left(\frac{1}{k}\right)^{-\delta}a_k^*\right)^q\frac{1}{n}\right)^{\frac{1}{q}}.
\end{aligned}$$

Next we use Hölder's inequality and the fact that $\varepsilon>\frac{1}{q}-1-\delta$ to find that

$$\begin{aligned}
I_1&\leqslant c_2\left(\sum_{n=1}^\infty\left(n^{-\delta}\left(\sum_{k=1}^{n}\left(\omega\left(\frac{1}{k}\right)k^{-\varepsilon}a_k^*\right)^q\right)^{\frac{1}{q}}\left(\sum_{k=1}^{n}k^{(\delta+\varepsilon)q'}\right)^{\frac{1}{q'}}\right)^q\frac{1}{n}\right)^{\frac{1}{q}}\\
&\sim\left(\sum_{n=1}^\infty n^{(-\delta)q}n^{(\delta+\varepsilon)q+\frac{q}{q'}}\frac{1}{n}\sum_{k=1}^{n}\left(\omega\left(\frac{1}{k}\right)k^{-\varepsilon}a_k^*\right)^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Here we interchange the order of summation and find that

$$I_1\leqslant c_2\left(\sum_{k=1}^\infty\left(\omega\left(\frac{1}{k}\right)k^{-\varepsilon}a_k^*\right)^q\sum_{n=k}^\infty n^{\varepsilon q+q-2}\right)^{\frac{1}{q}}.$$

Furthemore, by also using that $\varepsilon<\frac{1}{q}-1$, we have that

$$I_1\leqslant c_3\left(\sum_{k=1}^\infty\left(\omega\left(\frac{1}{k}\right)ka_k^*\right)^q\frac{1}{k}\right)^{\frac{1}{q}}=c_3\left(\sum_{k=1}^\infty(\mu(k)a_k^*)^q\frac{1}{k}\right)^{\frac{1}{q}}. \tag{7}$$

Next, we estimate I_2 in a similar way. Choose ε such that $-1+\frac{1}{q}<\varepsilon<-1+\frac{1}{q}+\delta$. By now using the growth properties of $\omega(t)$ we find that

$$\begin{aligned}
I_2 &= \left(\int_0^1 \left(\omega(t)\frac{1}{t}\sum_{k=\left[\frac{1}{t}\right]}^{\infty} a_k^*\frac{1}{k}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&= \left(\int_0^1 \left(\frac{\omega(t)t^{-1+\delta}}{t^{-1+\delta}}\frac{1}{t}\sum_{k=\left[\frac{1}{t}\right]}^{\infty} \frac{a_k^*}{k}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\leqslant \left(\int_0^1 \left(t^{-\delta}\sum_{k=\left[\frac{1}{t}\right]}^{\infty} \omega\left(\frac{1}{k}\right)k^{1-\delta}\frac{a_k^*}{k}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&= \left(\int_1^\infty \left(t^{\delta}\sum_{k=t}^{\infty} \omega\left(\frac{1}{k}\right)k^{1-\delta}\frac{a_k^*}{k}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\sim \left(\sum_{n=1}^{\infty} \left(n^{\delta}\sum_{k=n}^{\infty} \omega\left(\frac{1}{k}\right)k^{1-\delta}\frac{a_k^*}{k}\right)^q \frac{1}{n}\right)^{\frac{1}{q}}.
\end{aligned}$$

Next we use Hölder's inequality and the fact that $\varepsilon<-1+\frac{1}{q}+\delta$ to find that

$$\begin{aligned}
I_2 &\leqslant c_4 \left(\sum_{n=1}^{\infty}\left(n^{\delta}\left(\sum_{k=n}^{\infty}\left(a_k^*\omega\left(\frac{1}{k}\right)k^{-\varepsilon}\right)^q\right)^{\frac{1}{q}}\left(\sum_{k=n}^{\infty}k^{(-\delta+\varepsilon)q'}\right)^{\frac{1}{q'}}\right)^q\frac{1}{n}\right)^{\frac{1}{q}} \\
&\sim \left(\sum_{n=1}^{\infty} n^{\varepsilon q+q-2}\sum_{k=n}^{\infty}\left(a_k^*\omega\left(\frac{1}{k}\right)k^{-\varepsilon}\right)^q\right)^{\frac{1}{q}} \\
&= \left(\sum_{k=1}^{\infty}\left(a_k^*\omega\left(\frac{1}{k}\right)k^{-\varepsilon}\right)^q\sum_{n=1}^{k} n^{\varepsilon q+q-2}\right)^{\frac{1}{q}}.
\end{aligned}$$

By interchanging the order of summation and using the fact that $\varepsilon>-1+\frac{1}{q}$, we obtain that

$$I_2 \leqslant c_5\left(\sum_{k=1}^{\infty}\left(a_k^*\mu(k)\right)^q\frac{1}{k}\right)^{\frac{1}{q}}. \tag{8}$$

To complete the proof we just combine (7) with (8). $\square$

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