

A NEW GENERALIZATION OF BOAS THEOREM FOR SOME LORENTZ SPACES $\Lambda_q(\omega)$

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Abstract. Let $\Lambda_q(\omega)$, $q > 0$, denote the Lorentz space equipped with the (quasi) norm

$$\|f\|_{\Lambda_q(\omega)} := \left(\int_0^1 (f^*(t)\omega(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

for a function f on $[0, 1]$ and with ω positive and equipped with some additional growth properties. A generalization of Boas theorem in the form of a two-sided inequality is obtained in the case of both general regular system $\Phi = \{\varphi_k\}_{k=1}^\infty$ and generalized Lorentz $\Lambda_q(\omega)$ spaces.

1. Introduction

The following Hardy-Littlewood theorem is well known (see [26] and also [10], [4]):

THEOREM A. *If $f \geq 0$ and f decreases, $1 < p < \infty$, and a_n are the Fourier sine or cosine coefficients of f , then*

$$\sum_{n=1}^{\infty} |a_n|^p < \infty$$

if and only if

$$x^{p-2} f(x)^p \in L_p.$$

This theorem can be extended as follows (see [4]):

THEOREM B. *If $f \geq 0$ and f decreases, $1 < p < \infty$, $-1/p' < \gamma < 1/p$, then*

$$\sum_{n=1}^{\infty} n^{-\gamma p} |a_n|^p < \infty$$

converges if and only if

$$x^{p-2} x^{p\gamma+p-2} f(x)^p \in L_p.$$

Here and in the sequel $p' = \frac{p}{p-1}$ for $p > 1$.

A characterization for the function f to belong to the Lorentz space L_{pq} was obtained by R. P. Boas in [4]. This result deals with trigonometric Fourier coefficients for the class of monotone functions and reads:

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THEOREM C. *If $f \geq 0$ and f decreases, $1 < p < \infty$, $1 < q < \infty$, then $f \in L_{pq}$ if and only if $\{a_n\} \in l_{p'q}$.*

Some other results which are related to the Hardy-Littlewood theorem for the class of monotone functions were obtained in [25], [3], [1], [24], [5], [15], [8], [9], [12] and [7].

Boas theorem was generalized and complemented in various ways also for more general Lorentz spaces $\Lambda_q(\omega)$ in 1974 by L.-E. Persson for the case when $\Phi = \{e^{2\pi i k x}\}_{k=-\infty}^{+\infty}$ is trigonometric system (see. [20]–[23]). For example the following theorem was proved:

THEOREM D. *Let $p > 0$ and $\Phi = \{e^{2\pi i k t}\}_{k=-\infty}^{+\infty}$ be a trigonometrical system. Let ω be a nonnegative function on $[0, \infty)$. If there exists a positive number $\delta > 0$ satisfying that $\omega(t)t^{-\delta}$ is an increasing function of t and $\omega(t)t^{-1+\delta}$ is a decreasing function of t and if f is a nonnegative and a decreasing function on $[0, \frac{1}{2}]$, then*

$$\left(\int_0^1 (f^*(t)\omega(t))^p \frac{dt}{t} \right)^{\frac{1}{p}} < \infty,$$

if and only if

$$\left(\sum_{k=1}^{\infty} (k\omega\left(\frac{1}{k}\right)a_k^*)^p \frac{1}{n} \right)^{\frac{1}{p}} < \infty,$$

where $\{a_k^*\}_{k=1}^{\infty}$ is the nonincreasing rearrangement of the sequence $\{a_n\}_{k=1}^{\infty}$ of Fourier coefficients of f with respect to the system Φ .

The main aim of this paper is to derive the Boas theorem for the space $\Lambda_q(\omega)$ with respect to the regular system. Moreover, a new Boas type theorem for space $\Lambda_q(\omega)$ and for generalized monotone functions is proved and discussed.

The main results are formulated in Section 3. Note that the results in Theorem 1 is obviously related to [11] but we have chosed to put also this result in this more general frame in English. The proofs can be found in Section 4 and in Section 2 we have presented some necessary preliminaries.

CONVENTIONS. The letter $c(c_1, c_2, \text{etc.})$ means a constant which does not depend on the involved functions and it can be different in different occurences. Moreover, for $C, D > 0$ the notation $C \sim D$ means that there exist positive constants a_1 and a_2 such that $a_1 D \leq C \leq a_2 D$.

2. Preliminaries

Let f be a measurable function on $[0, 1]$ and μ is Lebesgue measure. The nonincreasing rearrangement f^* of a function f is defined as follows:

$$m(\sigma, f) := \mu\{x \in [0, 1] : |f(x)| > \sigma\},$$

$$f^*(t) := \inf\{\sigma : m(\sigma, f) \leq t\}.$$

Let $0 < q \leq \infty$ and ω be a nonnegative function on $[0, 1]$. The generalized Lorentz spaces $\Lambda_q(\omega)$ consists of the functions f on $[0, 1]$ such that $\|f\|_{\Lambda_q(\omega)} < \infty$, where

$$\|f\|_{\Lambda_q(\omega)} := \begin{cases} \left(\int_0^1 (f^*(t)\omega(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} & \text{for } 0 < q < \infty, \\ \sup_{0 \leq t \leq 1} f^*(t)\omega(t) & \text{for } q = \infty. \end{cases}$$

These spaces $\Lambda_q(\omega)$ coincide to the classical spaces L_{pq} in the case $\omega(t) = t^{\frac{1}{p}}$, $1 < p < \infty$ (see [16] and also e.g. [2]).

Let $\mu = \{\mu(k)\}_{k \in \mathbb{N}}$ be a sequence of positive number and the space $\lambda_q(\mu)$ consists of all sequences $a = \{a_k\}_{k=1}^{\infty}$ such that $\|a\|_{\lambda_q(\mu)} < \infty$, where

$$\|a\|_{\lambda_q(\mu)} := \begin{cases} \left(\sum_{k=1}^{\infty} (a_k^* \mu(k))^q \frac{1}{k} \right)^{\frac{1}{q}} & \text{for } 0 < q < \infty, \\ \sup_k a_k^* \mu(k) & \text{for } q = \infty. \end{cases}$$

Here, as usual, $\{a_k^*\}_{k=1}^{\infty}$ is the nonincreasing rearrangement of the sequence $\{|a_k|\}_{k=1}^{\infty}$.

Let the function f be periodic with period 1 and integrable on $[0, 1]$ and let $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ be an orthonormal system on $[0, 1]$. The numbers

$$a_k = a_k(f) = \int_0^1 f(x) \overline{\varphi_k(x)} dx, \quad k \in \mathbb{N}$$

are called the Fourier coefficients of the functions f with respect to the system $\Phi = \{\varphi_k\}_{k=1}^{\infty}$.

We say that the orthonormal system $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ is regular if there exists a constant B , such that

1) for every segment e from $[0, 1]$ and $k \in \mathbb{N}$ it yields that

$$\left| \int_e \varphi_k(x) dx \right| \leq B \min(|e|, 1/k),$$

2) for every segment w from \mathbb{N} and $t \in (0, 1]$ we have that

$$\left(\sum_{k \in w} \varphi_k(\cdot) \right)^*(t) \leq B \min(|w|, 1/t),$$

where $(\sum_{k \in w} \varphi_k(\cdot))^*(t)$ as usual denotes the nonincreasing rearrangement of the function $\sum_{k \in w} \varphi_k(x)$.

Examples of regular systems are all trigonometrical systems, the Walsh system and Prise's system. In [17], [19], [18] some results were obtained with respect to the regular system using network space.

Let $\delta > 0$ be a fixed parameter. Consider a nonnegative function $\omega(t)$ on $[0, 1]$. We define the following classes:

$$A_{\delta} := \{ \omega(t) : \omega(t)t^{-\frac{1}{2}-\delta} \text{ is an increasing function and } \omega(t)t^{-1+\delta} \text{ is a decreasing function} \},$$

$$B_\delta := \{ \omega(t) : \omega(t)t^{-\delta} \text{ is an increasing function and } \omega(t)t^{-1+\delta} \text{ is a decreasing function} \}.$$

Then the classes A and B can be defined as follows:

$$A = \bigcup_{\delta > 0} A_\delta.$$

and

$$B = \bigcup_{\delta > 0} B_\delta.$$

For the proof of our main results we need the following Theorem:

THEOREM E. *Let $\Phi = \{\varphi_k\}_{k=1}^\infty$ be a regular system and $f \stackrel{a.e.}{=} \sum_{k=1}^\infty a_k \varphi_k$.*

Let $1 \leq q \leq \infty$. If ω belongs to the class B , then

$$\left(\int_0^1 (\overline{f}(t)\omega(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} \leq c \left(\sum_{k=1}^\infty (a_k^* \mu(k))^q \frac{1}{k} \right)^{\frac{1}{q}},$$

where $\overline{f}(t) = \sup_{\xi \geq t} \frac{1}{\xi} \left| \int_0^\xi f(s) ds \right|$, $\mu(k) = k\omega\left(\frac{1}{k}\right)$ and the constant c does not depend on f .

This is just a slight generalization of Theorem 2 in [14] (see also [11]). For the reader’s convenience we include a proof in Appendix 1.

We also need the following technical Lemma:

LEMMA 1. *Let $1 \leq q \leq \infty$ and $1 \leq h \leq \infty$. If $\omega(t)$ belongs to the class B , then for any nonincreasing function f it yields that*

$$\left(\sum_{k=1}^\infty \left(\int_{2^{-k}}^{2^{-k+1}} (f(t)\omega(t))^h \frac{dt}{t} \right)^{\frac{q}{h}} \right)^{\frac{1}{q}} \sim \left(\int_0^1 (f(t)\omega(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}. \tag{1}$$

Proof. First we prove the following equivalence:

$$\left(\sum_{k=1}^\infty \left(\int_{2^{-k}}^{2^{-k+1}} (f(t)\omega(t))^h \frac{dt}{t} \right)^{\frac{q}{h}} \right)^{\frac{1}{q}} \sim \left(\sum_{k=1}^\infty (f(2^{-k})\omega(2^{-k}))^q \right)^{\frac{1}{q}}. \tag{2}$$

Let

$$\begin{aligned} I_h &:= \left(\sum_{k=1}^\infty \left(\int_{2^{-k}}^{2^{-k+1}} (f(t)\omega(t))^h \frac{dt}{t} \right)^{\frac{q}{h}} \right)^{\frac{1}{q}} \\ &= \left(\sum_{k=1}^\infty \left(\int_{2^{-k}}^{2^{-k+1}} (f(t)\omega(t)t^{-1+\delta}t^{1-\delta})^h \frac{dt}{t} \right)^{\frac{q}{h}} \right)^{\frac{1}{q}}. \end{aligned}$$

We use the fact that $\omega = \omega(t)$ belong to the class B . This means that there exists δ , $0 < \delta < 1$, such that $\omega(t)t^{-\delta}$ is an increasing function and $\omega(t)t^{-1+\delta}$ is a decreasing function. Then we have:

$$\begin{aligned} I_h &\leq \left(\sum_{k=1}^{\infty} \left(f(2^{-k})\omega(2^{-k})2^{-k(-1+\delta)} \left(\int_{2^{-k}}^{2^{-k+1}} t^{(1-\delta)h} \frac{dt}{t} \right)^{\frac{1}{h}} \right)^q \right)^{\frac{1}{q}} \\ &= c_1 \left(\sum_{k=1}^{\infty} \left(f(2^{-k})\omega(2^{-k})2^{k-k\delta}2^{k\delta-k} \right)^q \right)^{\frac{1}{q}} = c_1 \left(\sum_{k=1}^{\infty} \left(f(2^{-k})\omega(2^{-k}) \right)^q \right)^{\frac{1}{q}}. \\ I_h &= \left(\sum_{k=1}^{\infty} \left(\int_{2^{-k}}^{2^{-k+1}} \left(f(t)\omega(t)t^{-\delta}t^{\delta} \right)^h \frac{dt}{t} \right)^{\frac{q}{h}} \right)^{\frac{1}{q}} \\ &\geq \left(\sum_{k=1}^{\infty} \left(f(2^{-k+1})\omega(2^{-k})2^{k\delta} \left(\int_{2^{-k}}^{2^{-k+1}} t^{\delta h-1} dt \right)^{\frac{1}{h}} \right)^q \right)^{\frac{1}{q}} \\ &= c_2 \left(\sum_{k=1}^{\infty} \left(f(2^{-k+1})\omega(2^{-k}) \right)^q \right)^{\frac{1}{q}} \geq c_3 \left(\sum_{k=1}^{\infty} \left(f(2^{-k})\omega(2^{-k}) \right)^q \right)^{\frac{1}{q}}. \end{aligned}$$

Thus, (2) is proved, which, in particular means that $I_{h_1} \sim I_{h_2}$ for all h_1 and h_2 . Moreover, since f is nonincreasing and $\omega \in B$, it follows that

$$\left(\sum_{k=1}^{\infty} \left(f(2^{-k})\omega(2^{-k}) \right)^q \right)^{\frac{1}{q}} \sim \left(\int_0^1 (f(t)\omega(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

In particular, (1) follows and the proof is complete. \square

3. Main results

The main results of this paper are the following generalizations of the Boas theorem:

THEOREM 1. *Let $1 \leq q \leq \infty$ and $\omega \in B$. Let $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ be a regular system and let $f \stackrel{a.e.}{=} \sum_{k=1}^{\infty} a_k \varphi_k$. If f is a nonnegative and a nonincreasing function, then*

$$\left(\int_0^1 (f(t)\omega(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} \sim \left(\sum_{k=1}^{\infty} (a_k^* \mu(k))^q \frac{1}{k} \right)^{\frac{1}{q}},$$

where $\mu(k) = k\omega(\frac{1}{k})$.

We say that a function f on $[0, 1]$ is generalized monotone if there exists some constant $M > 0$ such that

$$|f(x)| \leq M \frac{1}{x} \left| \int_0^x f(t) dt \right|, \quad x > 0.$$

Our next main result reads:

THEOREM 2. *Let $1 \leq q \leq \infty$ and $\omega \in A$. Let $\Phi = \{\varphi_k\}_{k=1}^\infty$ be a regular system and let $f \stackrel{a.e.}{=} \sum_{k=1}^\infty a_k \varphi_k$. If f is a nonnegative and a generalized monotone function, then*

$$\|f\|_{\Lambda_q(\omega, [0,1])} \sim \left(\sum_{k=1}^\infty (a_k^* \mu(k))^q \frac{1}{k} \right)^{\frac{1}{q}},$$

where $\mu(k) = k\omega(\frac{1}{k})$.

4. Proofs of the main results

Proof of Theorem 1. The necessary part is similar to that in Theorem E. Indeed, since f is a nonincreasing function, then $f(t) \leq \overline{f(t)}$, $0 < t < 1$, so that

$$\left(\int_0^1 (f(t)\omega(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} \leq \left(\int_0^1 (\overline{f(t)}\omega(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} \leq c \left(\sum_{k=1}^\infty (a_k^* \mu(k))^q \frac{1}{k} \right)^{\frac{1}{q}},$$

where $\overline{f}(t) = \sup_{\xi \geq t} \frac{1}{\xi} \int_0^\xi f(s) ds$. We prove the sufficient condition. The condition $\omega(t) \in B$

implies that there exists $\delta > 0$ such that $\omega(t)t^{-\delta}$ is an increasing and $\omega(t)t^{-1+\delta}$ is a decreasing function, i.e. $\mu(k)k^{-\delta}$ is increasing and $\mu(k)k^{-1+\delta}$ is decreasing. Then the following estimate holds:

$$\frac{1}{k} \sum_{n=1}^k \frac{\mu^q(n)}{n} \leq c \frac{\mu^q(k)}{k}, \quad k \in \mathbb{N}.$$

Indeed,

$$\frac{1}{k} \sum_{n=1}^k \frac{\mu^q(n)}{n} \leq \frac{1}{k} \mu^q(k) k^{-\delta} \sum_{n=1}^k \frac{1}{n^{1-\delta}} \sim \frac{\mu^q(k)}{k}.$$

Next, we use Theorem 2.4.12 (ii) from [6] to conclude that the following equality holds:

$$\lambda_q(\mu) = (\lambda_{q'}(\mu^{-1}k))', \quad \text{for } 1 < q < \infty,$$

where $(\lambda_{q'}(\mu^{-1}k))'$ is dual space for the space $\lambda_q(\mu)$. Hence, by applying the duality representation of the norm of a sequence a in the space $\lambda_q(\mu)$ (see [6]), we obtain that

$$\|a\|_{\lambda_q(\mu)} = \sup_{\|b\|_{\lambda_{q'}(\mu^{-1}k)}=1} \sum_{k=1}^\infty a_k b_k.$$

Now we use Parseval's formula and find that

$$\begin{aligned} \|a\|_{\lambda_q(\mu)} &= \sup_{\|b\|_{\lambda_{q'}(\mu^{-1k})}=1} \int_0^1 f(t)g(t)dt \\ &= \sup_{\|b\|_{\lambda_{q'}(\mu^{-1k})}=1} \sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} f(t)g(t)dt \\ &\leq \sup_{\|b\|_{\lambda_{q'}(\mu^{-1k})}=1} \sum_{k=0}^{\infty} \left| \int_{2^{-k-1}}^{2^{-k}} f(t)g(t)dt \right|. \end{aligned} \quad (3)$$

We apply the mean value theorem to the integral $\int_{2^{-k-1}}^{2^{-k}} f(t)g(t)dt$ to conclude that there exists ξ from $(2^{-k-1}, 2^{-k})$ such that

$$\begin{aligned} \left| \int_{2^{-k-1}}^{2^{-k}} f(t)g(t)dt \right| &= \left| f(2^{-k-1}) \int_{2^{-k-1}}^{\xi} g(t)dt \right| \\ &\leq f(2^{-k-1}) \left(\left| \int_0^{\xi} g(t)dt \right| + \left| \int_0^{2^{-k-1}} g(t)dt \right| \right) \\ &\leq f(2^{-k-1}) \left(2^{-k} \sup_{s \geq 2^{-k-2}} \frac{1}{s} \left| \int_0^s g(t)dt \right| + 2^{-k} \sup_{s \geq 2^{-k-2}} \frac{1}{s} \left| \int_0^s g(t)dt \right| \right) \\ &= 2 \cdot 2^{-k} \cdot f(2^{-k-1}) \cdot \overline{g(2^{-k-2})}, \end{aligned} \quad (4)$$

where $\overline{g(2^{-k-2})} = \sup_{s \geq 2^{-k-2}} \frac{1}{s} \left| \int_0^s g(t)dt \right|$.

Thus, by inserting (4) in (3), we conclude that

$$\begin{aligned} \|a\|_{\lambda_q(\mu)} &\leq 8 \sup_{\|b\|_{\lambda_{q'}(\mu^{-1k})}=1} \sum_{k=0}^{\infty} 2^{-k-2} f(2^{-k-1}) \overline{g(2^{-k-2})} \\ &= 8 \sup_{\|b\|_{\lambda_{q'}(\mu^{-1k})}=1} \sum_{k=0}^{\infty} \left(2^{-k-2} \left(\omega(2^{-k-2}) \right)^{-1} \overline{g(2^{-k-2})} \right) \cdot f(2^{-k-1}) \omega(2^{-k-2}) \\ &= 8 \sup_{\|b\|_{\lambda_{q'}(\mu^{-1k})}=1} \sum_{k=2}^{\infty} \left(2^{-k} \left(\omega(2^{-k}) \right)^{-1} \overline{g(2^{-k})} \right) \cdot f(2^{-k+1}) \omega(2^{-k}). \end{aligned}$$

Next, by using Hölder's inequality, we get that

$$\begin{aligned} \|a\|_{\lambda_q(\mu)} &\leq c_1 \sup_{\|b\|_{\lambda_{q'}(\mu^{-1k})}=1} \left(\sum_{k=2}^{\infty} \left(2^{-k} \left(\omega(2^{-k}) \right)^{-1} \overline{g(2^{-k})} \right)^{q'} \right)^{\frac{1}{q}} \\ &\quad \times \left(\sum_{k=2}^{\infty} \left(f(2^{-k+1}) \omega(2^{-k}) \right)^q \right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned} &\leq c_1 \sup_{\|b\|_{\lambda_{q'}(\mu^{-1k})}=1} \left(\sum_{k=1}^{\infty} \left(2^{-k} \left(\omega(2^{-k}) \right)^{-1} \overline{g(2^{-k+1})} \right)^{q'} \right)^{\frac{1}{q'}} \\ &\quad \times \left(\sum_{k=1}^{\infty} \left(f(2^{-k+1}) \omega(2^{-k}) \right)^q \right)^{\frac{1}{q}} \\ &= c_2 \sup_{\|b\|_{\lambda_{q'}(\mu^{-1k})}=1} \left(\sum_{k=1}^{\infty} \left(\frac{2^{-k(1-\delta)} \omega^{-1}(2^{-k})}{2^{-k(1-\delta)}} 2^{-k} \overline{g(2^{-k+1})} \right)^{q'} \right)^{\frac{1}{q'}} \\ &\quad \times \left(\sum_{k=1}^{\infty} \left(\frac{2^{k\delta} \omega(2^{-k})}{2^{k\delta}} f(2^{-k+1}) \int_{2^{-k}}^{2^{-k+1}} \frac{dt}{t} \right)^q \right)^{\frac{1}{q}}. \end{aligned}$$

Since $f(t)$ is a nonincreasing function for all $0 < t < 1$ and $\omega(t)$ belongs to B , then there exists $0 < \delta < 1$ such that $\omega(t)t^{-\delta}$ is an increasing and $\omega(t)t^{-1+\delta}$ is a decreasing function, we get that

$$\begin{aligned} \|a\|_{\lambda_q(\mu)} &\leq c_3 \sup_{\|b\|_{\lambda_{q'}(\mu^{-1k})}=1} \left(\sum_{k=1}^{\infty} \left(2^{k(1-\delta)} \int_{2^{-k}}^{2^{-k+1}} t^{1-\delta} \omega^{-1}(t) \overline{g(t)} dt \right)^{q'} \right)^{\frac{1}{q'}} \\ &\quad \times \left(\sum_{k=1}^{\infty} \left(2^{-k\delta} \int_{2^{-k}}^{2^{-k+1}} f(t) \omega(t) t^{-\delta} \frac{dt}{t} \right)^q \right)^{\frac{1}{q}} \\ &\leq c_4 \sup_{\|b\|_{\lambda_{q'}(\mu^{-1k})}=1} \left(\sum_{k=1}^{\infty} \left(\int_{2^{-k}}^{2^{-k+1}} t \omega^{-1}(t) \overline{g(t)} \frac{dt}{t} \right)^{q'} \right)^{\frac{1}{q'}} \left(\sum_{k=1}^{\infty} \left(\int_{2^{-k}}^{2^{-k+1}} f(t) \omega(t) \frac{dt}{t} \right)^q \right)^{\frac{1}{q}}. \end{aligned}$$

By now applying Lemma 1, we obtain that

$$\|a\|_{\lambda_q(\mu)} \leq c_5 \sup_{\|b\|_{\lambda_{q'}(\mu^{-1k})}=1} \left(\int_0^1 \left(t \omega^{-1}(t) \overline{g(t)} \right)^{q'} \frac{dt}{t} \right)^{\frac{1}{q'}} \cdot \left(\int_0^1 (f(t) \omega(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

Furthermore, by using Theorem E, we obtain the following estimate

$$\begin{aligned} \|a\|_{\lambda_q(\mu)} &\leq c_6 \sup_{\|b\|_{\lambda_{q'}(\mu^{-1k})}=1} \left(\sum_{k=1}^{\infty} (b_k^* k \mu^{-1}(k))^{q'} \frac{1}{k} \right)^{\frac{1}{q'}} \cdot \left(\int_0^1 (f(t) \omega(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &= c_6 \sup_{\|b\|_{\lambda_{q'}(\mu^{-1k})}=1} \|b\|_{\lambda_{q'}(\mu^{-1k})} \cdot \left(\int_0^1 (f(t) \omega(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &= c_6 \left(\int_0^1 (f(t) \omega(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}. \end{aligned}$$

The proof is complete. \square

Proof of Theorem 2. The condition $\omega(t) \in A$ implies that there exists $\delta > 0$ such that $\omega(t)t^{-\frac{1}{2}-\delta}$ is an increasing function and $\omega(t)t^{-1+\delta}$ is a decreasing function. The necessary condition follows in a similar way as in Theorem E. Indeed, let $x > 0$ and

$$f^{**}(x) := \sup_{|e|=x} \frac{1}{|e|} \int_e |f(t)| dt.$$

It is obvious that $f^*(x) \leq f^{**}(x)$. Since f is a generalized monotone function, it yields that

$$f^{**}(x) = \sup_{|e|=x} \frac{1}{|e|} \int_e |f(t)| dt \leq \sup_{|e|=x} \frac{1}{|e|} \int_e \overline{f(t)} dt = \frac{1}{x} \int_0^x \overline{f(t)} dt,$$

where $\overline{f}(t) = \sup_{\xi \geq t} \int_0^\xi f(s) ds$.

Thus, we obtain the following inequalities

$$\|f\|_{\Lambda_q(\omega)} \leq \|f^{**}\|_{\Lambda_q(\omega)} \leq M \left\| \frac{1}{x} \int_0^x \overline{f(t)} dt \right\|_{\Lambda_q(\omega)}. \quad (5)$$

We prove the following inequality

$$\left(\int_0^1 \left(\omega(x) \frac{1}{x} \int_0^x \overline{f(t)} dt \right)^q \frac{dx}{x} \right)^{\frac{1}{q}} \leq c \left(\int_0^1 \left(\overline{f(t)} \omega(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}. \quad (6)$$

Choose ε so that $-\frac{1}{q} + 1 - \delta < \varepsilon < -\frac{1}{q} + 1$. We consider for any $x > 0$

$$\int_0^x \overline{f(t)} dt = \int_0^x \overline{f(t)} t^\varepsilon t^{-\varepsilon} dt.$$

Next we use Hölder's inequality and the fact that $\varepsilon < -\frac{1}{q} + 1$ to find that

$$\begin{aligned} \int_0^x \overline{f(t)} dt &\leq c_1 \left(\int_0^x (\overline{f(t)} t^\varepsilon)^q dt \right)^{\frac{1}{q}} \left(\int_0^x (t^{-\varepsilon})^{q'} dt \right)^{\frac{1}{q'}} \\ &\sim \left(\int_0^x (\overline{f(t)} t^\varepsilon)^q dt \right)^{\frac{1}{q}} x^{-\varepsilon + \frac{1}{q}}. \end{aligned}$$

Moreover,

$$\begin{aligned} I &:= c_2 \left(\int_0^1 \left(\omega(x) x^{-\varepsilon + \frac{1}{q} - 1} \right)^q \left(\int_0^x (\overline{f(t)} t^\varepsilon)^q dt \right) \frac{dx}{x} \right)^{\frac{1}{q}} \\ &= c_2 \left(\int_0^1 (\overline{f(t)} t^\varepsilon)^q \left(\int_t^1 x^{-\varepsilon q - 1} \omega^q(x) \frac{dx}{x} \right) dt \right)^{\frac{1}{q}}. \end{aligned}$$

By now using the fact that $\omega(t)t^{-\frac{1}{2}-\delta}$ is an increasing function, we find that

$$I \leq c_2 \left(\int_0^1 (\overline{f}(t)t^\varepsilon)^q \left(\omega(t)t^{-\frac{1}{2}-\delta} \right)^q \left(\int_{\frac{1}{t}}^1 x^{\varepsilon q+1-\frac{1}{2}q-\delta q} \frac{dx}{x} \right) dt \right)^{\frac{1}{q}}.$$

Taking into account that $\varepsilon \geq -\frac{1}{q} + \frac{1}{2} + \delta$, we obtain that

$$I \leq c_3 \left(\int_0^1 (\overline{f}(t)\omega(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

Thus, we have proved the inequality (6). From (5) and (6) it follows that

$$\|f\|_{\Lambda_q(\omega)} \leq c_3 \|\overline{f}\|_{\Lambda_q(\omega)}.$$

By now applying Theorem E, we obtain that

$$\|f\|_{\Lambda_q(\omega)} \leq c_4 \|a\|_{\lambda_q(\mu)}.$$

Since each regular system is bounded orthonormal system, then the sufficient condition follows from Theorem 2 in [13].

The proof is complete. \square

5. Appendix 1

Proof of Theorem E. Assume that $\omega(t)$ belongs to the class B . This means that there exists $\delta > 0$ such that $\omega(t)t^{-\delta}$ is an increasing function and $\omega(t)t^{-1+\delta}$ is a decreasing function. Suppose that

$$\left(\sum_{k=1}^{\infty} (a_k^* \mu(k))^q \frac{1}{k} \right)^{\frac{1}{q}} < \infty$$

and $f \stackrel{\text{a.e.}}{=} \sum_{k=1}^{\infty} a_k \varphi_k$. It yields that

$$\begin{aligned} \left| \int_0^\xi f(s) ds \right| &= \left| \int_0^\xi \sum_{k \in \mathbb{N}} a_k \varphi_k(s) ds \right| \\ &\leq \sum_{k \in \mathbb{N}} |a_k| \left| \int_0^\xi \varphi_k(s) ds \right|, \text{ for all } \xi \in [0, 1]. \end{aligned}$$

According to the regularity assumption we have that

$$\left| \int_0^\xi \varphi_k(s) ds \right| \leq B \min \left(\xi, \frac{1}{k} \right), k \in \mathbb{N}.$$

Hence,

$$\begin{aligned} \sum_{k=1}^{\infty} |a_k| \left| \int_0^{\xi} \varphi_k(s) ds \right| &\leq c_1 \sum_{k=1}^{\infty} |a_k| \min \left(\xi, \frac{1}{k} \right) \\ &\leq c_1 \sum_{k=1}^{\infty} a_k^* \min \left(\xi, \frac{1}{k} \right) \\ &\leq c_1 \left(\sum_{k=1}^{\left[\frac{1}{\xi} \right]} a_k^* \xi + \sum_{k=\left[\frac{1}{\xi} \right]}^{\infty} a_k^* \frac{1}{k} \right). \end{aligned}$$

Consequently,

$$\left| \int_0^{\xi} f(s) ds \right| \leq c_1 \left(\sum_{k=1}^{\left[\frac{1}{\xi} \right]} a_k^* \xi + \sum_{k=\left[\frac{1}{\xi} \right]}^{\infty} a_k^* \frac{1}{k} \right)$$

and we have that

$$\begin{aligned} &\left(\int_0^1 (\overline{f(t)} \omega(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\leq c_1 \left(\int_0^1 \left(\omega(t) \sup_{\xi \geq t} \frac{1}{\xi} \left(\sum_{k=1}^{\left[\frac{1}{\xi} \right]} a_k^* \xi + \sum_{k=\left[\frac{1}{\xi} \right]}^{\infty} a_k^* \frac{1}{k} \right) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\leq c_1 \left(\int_0^1 \left(\omega(t) \sup_{\xi \geq t} \frac{1}{\xi} \left(\sum_{k=1}^{\left[\frac{1}{\xi} \right]} a_k^* \xi + \sum_{k=\left[\frac{1}{\xi} \right]}^{\left[\frac{1}{t} \right]} a_k^* \cdot \frac{1}{k} + \sum_{k=\left[\frac{1}{t} \right]}^{\infty} a_k^* \frac{1}{k} \right) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\leq c_1 \left(\int_0^1 \left(\omega(t) \sup_{\xi \geq t} \frac{1}{\xi} \left(\sum_{k=1}^{\left[\frac{1}{\xi} \right]} a_k^* \xi + \sum_{k=\left[\frac{1}{\xi} \right]}^{\left[\frac{1}{t} \right]} a_k^* \cdot \xi + \sum_{k=\left[\frac{1}{t} \right]}^{\infty} a_k^* \frac{1}{k} \right) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &= c_1 \left(\int_0^1 \left(\omega(t) \sup_{\xi \geq t} \frac{1}{\xi} \left(\sum_{k=1}^{\left[\frac{1}{t} \right]} a_k^* \xi + \sum_{k=\left[\frac{1}{t} \right]}^{\infty} a_k^* \frac{1}{k} \right) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &= c_1 \left(\int_0^1 \left(\omega(t) \left(\sum_{k=1}^{\left[\frac{1}{t} \right]} a_k^* + \sup_{\xi \geq t} \frac{1}{\xi} \cdot \sum_{k=\left[\frac{1}{t} \right]}^{\infty} a_k^* \frac{1}{k} \right) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\leq c_1 \left(\int_0^1 \left(\omega(t) \left(\sum_{k=1}^{\left[\frac{1}{t} \right]} a_k^* + \frac{1}{t} \cdot \sum_{k=\left[\frac{1}{t} \right]}^{\infty} a_k^* \frac{1}{k} \right) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned} &\leq c_1 \left(\int_0^1 \left(\omega(t) \sum_{k=1}^{\lfloor \frac{1}{t} \rfloor} a_k^* \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} + c_1 \left(\int_0^1 \left(\omega(t) \frac{1}{t} \sum_{k=\lfloor \frac{1}{t} \rfloor}^{\infty} a_k^* \frac{1}{k} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &:= c_1(I_1 + I_2). \end{aligned}$$

We consider first I_1 . Choose a small number ε such that $\frac{1}{q} - 1 - \delta < \varepsilon < \frac{1}{q} - 1$. Since $\omega(t)t^{-\delta}$ is an increasing function of t , it yields that

$$\begin{aligned} I_1 &= \left(\int_0^1 \left(\omega(t) \sum_{k=1}^{\lfloor \frac{1}{t} \rfloor} a_k^* \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &= \left(\int_0^1 \left(\frac{\omega(t)t^{-\delta}}{t^{-\delta}} \sum_{k=1}^{\lfloor \frac{1}{t} \rfloor} a_k^* \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\leq \left(\int_0^1 \left(t^\delta \sum_{k=1}^{\lfloor \frac{1}{t} \rfloor} \omega\left(\frac{1}{k}\right) \left(\frac{1}{k}\right)^{-\delta} a_k^* \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &= \left(\int_1^\infty \left(t^{-\delta} \sum_{k=1}^t \omega\left(\frac{1}{k}\right) \left(\frac{1}{k}\right)^{-\delta} a_k^* \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\sim \left(\sum_{n=1}^\infty \left(n^{-\delta} \sum_{k=1}^n \omega\left(\frac{1}{k}\right) \left(\frac{1}{k}\right)^{-\delta} a_k^* \right)^q \frac{1}{n} \right)^{\frac{1}{q}}. \end{aligned}$$

Next we use Hölder’s inequality and the fact that $\varepsilon > \frac{1}{q} - 1 - \delta$ to find that

$$\begin{aligned} I_1 &\leq c_2 \left(\sum_{n=1}^\infty \left(n^{-\delta} \left(\sum_{k=1}^n \left(\omega\left(\frac{1}{k}\right) k^{-\varepsilon} a_k^* \right)^q \right)^{\frac{1}{q}} \left(\sum_{k=1}^n k^{(\delta+\varepsilon)q'} \right)^{\frac{1}{q'}} \right)^q \frac{1}{n} \right)^{\frac{1}{q}} \\ &\sim \left(\sum_{n=1}^\infty n^{(-\delta)q} n^{(\delta+\varepsilon)q + \frac{q}{q'}} \frac{1}{n} \sum_{k=1}^n \left(\omega\left(\frac{1}{k}\right) k^{-\varepsilon} a_k^* \right)^q \right)^{\frac{1}{q}}. \end{aligned}$$

Here we interchange the order of summation and find that

$$I_1 \leq c_2 \left(\sum_{k=1}^\infty \left(\omega\left(\frac{1}{k}\right) k^{-\varepsilon} a_k^* \right)^q \sum_{n=k}^\infty n^{\varepsilon q + q - 2} \right)^{\frac{1}{q}}.$$

Furthermore, by also using that $\varepsilon < \frac{1}{q} - 1$, we have that

$$I_1 \leq c_3 \left(\sum_{k=1}^\infty \left(\omega\left(\frac{1}{k}\right) k a_k^* \right)^q \frac{1}{k} \right)^{\frac{1}{q}} = c_3 \left(\sum_{k=1}^\infty (\mu(k) a_k^*)^q \frac{1}{k} \right)^{\frac{1}{q}}. \tag{7}$$

Next, we estimate I_2 in a similar way. Choose ε such that $-1 + \frac{1}{q} < \varepsilon < -1 + \frac{1}{q} + \delta$. By now using the growth properties of $\omega(t)$ we find that

$$\begin{aligned} I_2 &= \left(\int_0^1 \left(\omega(t) \frac{1}{t} \sum_{k=\lceil \frac{1}{t} \rceil}^{\infty} a_k^* \frac{1}{k} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &= \left(\int_0^1 \left(\frac{\omega(t) t^{-1+\delta}}{t^{-1+\delta}} \frac{1}{t} \sum_{k=\lceil \frac{1}{t} \rceil}^{\infty} \frac{a_k^*}{k} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\leq \left(\int_0^1 \left(t^{-\delta} \sum_{k=\lceil \frac{1}{t} \rceil}^{\infty} \omega\left(\frac{1}{k}\right) k^{1-\delta} \frac{a_k^*}{k} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &= \left(\int_1^{\infty} \left(t^{\delta} \sum_{k=t}^{\infty} \omega\left(\frac{1}{k}\right) k^{1-\delta} \frac{a_k^*}{k} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\sim \left(\sum_{n=1}^{\infty} \left(n^{\delta} \sum_{k=n}^{\infty} \omega\left(\frac{1}{k}\right) k^{1-\delta} \frac{a_k^*}{k} \right)^q \frac{1}{n} \right)^{\frac{1}{q}}. \end{aligned}$$

Next we use Hölder's inequality and the fact that $\varepsilon < -1 + \frac{1}{q} + \delta$ to find that

$$\begin{aligned} I_2 &\leq c_4 \left(\sum_{n=1}^{\infty} \left(n^{\delta} \left(\sum_{k=n}^{\infty} \left(a_k^* \omega\left(\frac{1}{k}\right) k^{-\varepsilon} \right)^q \right)^{\frac{1}{q}} \left(\sum_{k=n}^{\infty} k^{(-\delta+\varepsilon)q'} \right)^{\frac{1}{q'}} \right)^q \frac{1}{n} \right)^{\frac{1}{q}} \\ &\sim \left(\sum_{n=1}^{\infty} n^{\varepsilon q + q - 2} \sum_{k=n}^{\infty} \left(a_k^* \omega\left(\frac{1}{k}\right) k^{-\varepsilon} \right)^q \right)^{\frac{1}{q}} \\ &= \left(\sum_{k=1}^{\infty} \left(a_k^* \omega\left(\frac{1}{k}\right) k^{-\varepsilon} \right)^q \sum_{n=1}^k n^{\varepsilon q + q - 2} \right)^{\frac{1}{q}}. \end{aligned}$$

By interchanging the order of summation and using the fact that $\varepsilon > -1 + \frac{1}{q}$, we obtain that

$$I_2 \leq c_5 \left(\sum_{k=1}^{\infty} (a_k^* \mu(k))^q \frac{1}{k} \right)^{\frac{1}{q}}. \quad (8)$$

To complete the proof we just combine (7) with (8). \square

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