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FABER POLYNOMIAL COEFFICIENTS FOR GENERALIZED BI–SUBORDINATE FUNCTIONS OF COMPLEX ORDER

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Abstract. In this paper, we obtain the upper bounds for the n-th ($n \geqslant 3$) coefficients for generalized bi-subordinate functions of complex order by using Faber polynomial expansions. The results, which are presented in this paper, would generalize those in related works of several earlier authors.

1. Introduction

Let $\mathscr{A}$ be the class of analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathscr{S}$ be the class of function f that are univalent in $\mathbb{D}$ and are of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

A function $f \in \mathscr{A}$ is said to be subordinate to a function $g \in \mathscr{A}$, denoted by $f \prec g$, if there exists a function $w \in \mathscr{A}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. We let $\mathscr{S}^*$ consist of starlike functions $f \in \mathscr{A}$, that is $\operatorname{Re} zf'/f > 0$ in $\mathbb{D}$ and $\mathscr{C}$ consist of convex functions $f \in \mathscr{A}$, that is $1 + \operatorname{Re} zf''/f' > 0$ in $\mathbb{D}$. In terms of subordination, these conditions are, respectively, equivalent to

$$\mathscr{S}^* \equiv \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \right\}$$

and

$$\mathscr{C} \equiv \left\{ f \in \mathscr{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z} \right\}.$$

A generalization of the above two classes, according to Ma and Minda [20], are

$$\mathscr{S}^*(\varphi) \equiv \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}$$

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and

$$\mathscr{C}(\varphi) \equiv \left\{ f \in \mathscr{A} : \ 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}$$

where φ is a positive real part function normalized by $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps D onto a region starlike with respect to 1 and symmetric with respect to the real axis. Obvious extensions of the above two classes (see [21]) are

$$\mathscr{S}^*(\gamma;\varphi) \equiv \left\{ f \in \mathscr{A} : 1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1 \right) \prec \varphi(z); \quad \gamma \in \mathbb{C} \setminus \{0\} \right\}$$

and

$$\mathscr{C}(\gamma;\varphi) \equiv \left\{ f \in \mathscr{A} : 1 + \frac{1}{\gamma}\left(\frac{zf''(z)}{f'(z)} \right) \prec \varphi(z); \quad \gamma \in \mathbb{C} \setminus \{0\} \right\}.$$

In literature, the functions belonging to these classes are called Ma-Minda s*tarlike and convex of complex order* γ $(\gamma \in \mathbb{C} \diagdown \{0\})$, respectively.

Some of the special cases of the above two classes $S^*(\gamma;\varphi)$ and $C(\gamma;\varphi)$ are

(1) $\mathscr{S}^*(1,(1+Az)/(1+Bz)) = \mathscr{S}[A,B]$ and $\mathscr{C}(1,(1+Az)/(1+Bz)) = \mathscr{C}[A,B]$, $(-1 \leqslant B < A \leqslant 1)$ are classes of Janowski starlike and convex functions, respectively,

(2) $\mathscr{S}^*((1-\beta)e^{-i\delta}\cos\delta, (1+z)/(1-z)) = \mathscr{S}^*[\delta,\beta]$ and $\mathscr{C}((1-\beta)e^{-i\delta}\cos\delta,$ $(1+z)/(1-z)) = \mathscr{C}[\delta,\beta]$, $(|\delta| < \pi/2,\ 0 \leqslant \beta < 1)$ are classes of δ-spirallike and δ-Robertson univalent functions of order β, respectively,

(3) $\mathscr{S}^*(1,(1+(1-2\beta)z)/(1-z)) = \mathscr{S}^*(\beta)$ and $\mathscr{C}(1,(1+(1-2\beta)z))/(1-z)) = \mathscr{C}(\beta)$ $(0 \leqslant \beta < 1)$ are classes of starlike and convex functions of order β, respectively,

(4) $\mathscr{S}^*\left(1, \left(\frac{1+z}{1-z}\right)^{\beta}\right) = \mathscr{S}^*_{\beta}$ and $\mathscr{C}\left(1, \left(\frac{1+z}{1-z}\right)^{\beta}\right) = \mathscr{C}_{\beta}$ are class of strongly starlike and convex functions of order β, respectively,

(5) $\mathscr{S}^*\left(1, \sqrt{1+z}\right) = \mathscr{S}^*_L = \left\{ f \in \mathscr{A} : \left| \left(\frac{zf'(z)}{f(z)}\right)^2 - 1 \right| < 1 \right\}$ is class of lemniscate starlike functions,

(6) $\mathscr{S}^*(\gamma,(1+z)/(1-z)) = \mathscr{S}^*[\gamma]$ and $\mathscr{C}(\gamma,(1+z)/(1-z)) = \mathscr{C}[\gamma]$ $(\gamma \in \mathbb{C} \diagdown \{0\})$ are classes of starlike and convex functions of complex order, respectively,

(7) $\mathscr{S}^*(1, q_k(z)) = k - \mathscr{S}^*_P = \left\{ f \in \mathscr{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > k \left| \frac{zf'(z)}{f(z)} - 1 \right| \right\}$ is class of k-parabolik starlike functions,

(8) $\mathscr{C}(1, q_k(z)) = k - \mathscr{UCV} = \left\{ f \in \mathscr{A} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > k \left| \frac{zf''(z)}{f'(z)} \right| \right\}$ is class of k-uniformly convex functions.

Here, for $0 \leqslant k < \infty$ the function $q_k : \mathbb{D} \to \{w = u + iv \in \mathbb{C} : \ u^2 > k^2\left((u-1)^2 + v^2\right),$ $u > 0\}$ has the form $q_k(z) = 1 + Q_1 z + Q_2 z^2 + \ldots,$ $(z \in \mathbb{D})$ where

$$Q_1 = \begin{cases} \frac{2\mathscr{B}^2}{1-k^2}; & 0 \leqslant k < 1, \\ \frac{8}{\pi^2}; & k = 1, \\ \frac{\pi^2}{4(k^2-1)\sqrt{t}(1+t)\mathscr{K}^2(t)}; & k > 1, \end{cases} \quad , \quad Q_2 = \begin{cases} \frac{(\mathscr{B}^2+2)}{3} Q_1; & 0 \leqslant k < 1, \\ \frac{2}{3} Q_1; & k = 1, \\ \frac{\left[4\mathscr{K}^2(t)(t^2+6t+1) - \pi^2\right]}{24\sqrt{t}(1+t)\mathscr{K}^2(t)} Q_1; & k > 1, \end{cases} \tag{1.1}$$

with $\mathscr{B} = \frac{2}{\pi}\arccos k$ and $\mathscr{K}(t)$ is the complete elliptic integral of first kind (see [18]).

A function $f \in \mathscr{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and its inverse map f^{-1} are univalent in $\mathbb{D}$. Let σ be the class of functions $f \in \mathscr{S}$ that are bi-univalent in $\mathbb{D}$. For a brief history and interesting examples of functions which are in (or are not in) the class σ, including various properties of such functions we refer the reader to the work of Srivastava et al. [22] and references therein. Bounds for the first few coefficients of various subclasses of bi-univalent functions were obtained by a variety of authors including ([4, 5, 6, 7], [10], [19], [23, 24, 25, 26, 27]). Not much was known about the bounds of the general coefficients $a_n; n \geqslant 4$ of subclasses of σ up until the publication of the article [14] by Jahangiri and Hamidi and followed by a number of related publications (see [11]–[17]). In this paper, we apply the Faber polynomial expansions to certain subclasses of bi-univalent functions and obtain bounds for their $n-th; (n \geqslant 3)$ coefficients subject to a given gap series condition.

2. Coefficient estimates

In the sequel, it is assumed that φ is an analytic function with positive real part in the unit disk $\mathbb{D}$, satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$, and $\varphi(\mathbb{D})$ is symmetric with respect to the real axis. Such a function is known to be typically real with the series expansion $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$ and $B_1 > 0$. Motivated by a class of functions defined by the first author [7], we define the following comprehensive class of analytic functions

$$\mathscr{S}(\lambda,\gamma;\varphi) \equiv \left\{ f \in \mathscr{A} : 1 + \frac{1}{\gamma}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} - 1 \right) \prec \varphi(z); \right.$$
$$\left. 0 \leqslant \lambda \leqslant 1,\ \gamma \in \mathbb{C}\setminus\{0\} \right\}.$$

A function $f \in \mathscr{A}$ is said the be generalized bi-subordinate of complex order γ and type λ if both f and its inverse map $g = f^{-1}$ are in $\mathscr{S}(\lambda,\gamma;\varphi)$. As special cases of the class $\mathscr{S}(\lambda,\gamma;\varphi)$ we have $\mathscr{S}(0,\gamma;\varphi) \equiv \mathscr{S}^*(\gamma;\varphi)$ and $\mathscr{S}(1,\gamma;\varphi) \equiv \mathscr{C}(\gamma;\varphi)$.

In the following theorem we use the Faber polynomials introduced by Faber [9] to obtain a bound for the general coefficients of the bi-univalent functions in $\mathscr{S}(\lambda,\gamma;\varphi)$ subject to a gap series condition.

THEOREM 2.1. *Let* $0 \leqslant \lambda \leqslant 1$ *and* $\gamma \in \mathbb{C}\setminus\{0\}$. *If both functions* $f(z) = z + \sum_{n=2}^{\infty} \rho_n z^n$ *and its inverse map* $g = f^{-1}$ *are in* $\mathscr{S}(\lambda,\gamma;\varphi)$ *and* $\rho_m = 0;\ 2 \leqslant m \leqslant n-1$ *then*

$$|\rho_n| \leqslant \frac{|\gamma| B_1}{(n-1)(1+\lambda(n-1))}.$$

Proof. If we write $\Lambda(f(z)) = \lambda z f'(z) + (1-\lambda) f(z)$ then

$$f \in \mathscr{S}(\lambda,\gamma;\varphi) \ \Leftrightarrow\ 1 + \frac{1}{\gamma}\left(\frac{z\Lambda'(f(z))}{\Lambda(f(z))} - 1 \right) \prec \varphi(z)$$

$$g=f^{-1}\in\mathscr{S}(\lambda,\gamma;\varphi)\ \Leftrightarrow\ 1+\frac{1}{\gamma}\left(\frac{w\Lambda'(g(w))}{\Lambda(g(w))}-1\right)\prec\varphi(w).$$

We observe that $a_n=(1+\lambda(n-1))\rho_n$ for $\Lambda(f(z))=z+\sum_{n=2}^{\infty}a_nz^n$. Now, an application of Faber polynomial expansion to the power series $\mathscr{S}(\lambda,\gamma;\varphi)$ (e.g. see [2] or [3, equation (1.6)]) yields

$$1+\frac{1}{\gamma}\left(\frac{z\Lambda'(f(z))}{\Lambda(f(z))}-1\right)=1-\frac{1}{\gamma}\sum_{n=2}^{\infty}F_{n-1}\left(a_2,a_3,a_4,\ldots,a_n\right)z^{n-1} \tag{2.1}$$

where

$$F_{n-1}\left(a_2,a_3,\ldots,a_n\right)=\sum_{i_1+2i_2+\ldots+(n-1)i_{n-1}=n-1}A\left(i_1,i_2,\ldots,i_{n-1}\right)\left(a_2^{i_1}a_3^{i_2}\ldots a_n^{i_{n-1}}\right)$$

and

$$A\left(i_1,i_2,\ldots,i_{n-1}\right):=(-1)^{(n-1)+2i_1+\ldots+ni_{n-1}}\frac{\left(i_1+i_2+\ldots+i_{n-1}-1\right)!\,(n-1)}{\left(i_1!\right)\left(i_2!\right)\ldots\left(i_{n-1}!\right)}.$$

The first few terms of $F_{n-1}\left(a_2,a_3,\ldots,a_n\right)$ are

$$\begin{aligned}F_1&=-a_2,\ F_2=a_2^2-2a_3,\ F_3=-a_2^3+3a_2a_3-3a_4,\\ F_4&=a_2^4-4a_2^2a_3+4a_2a_4+2a_3^2-4a_5,\\ F_5&=-a_2^5+5a_2^3a_3+5a_2^2a_4-5\left(a_3^2-a_5\right)a_2+5a_3a_4-5a_6.\end{aligned}$$

By the same token, the coefficients of the inverse map $g=f^{-1}$ may be expressed by

$$g(w)=f^{-1}(w)=w+\sum_{n=2}^{\infty}\frac{1}{n}K_{n-1}^{-n}\left(a_2,a_3,\ldots,a_n\right)w^n=w+\sum_{n=2}^{\infty}\tau_nw^n$$

where

$$\begin{aligned}K_{n-1}^{-n}=&\ \frac{(-n)!}{(-2n+1)!\,(n-1)!}a_2^{n-1}+\frac{(-n)!}{(2(-n+1))!\,(n-3)!}a_2^{n-3}a_3\\ &+\frac{(-n)!}{(-2n+3)!\,(n-4)!}a_2^{n-4}a_4+\frac{(-n)!}{(2(-n+2))!\,(n-5)!}a_2^{n-5}\left[a_5+(-n+2)a_3^2\right]\\ &+\frac{(-n)!}{(-2n+5)!\,(n-6)!}a_2^{n-6}\left[a_6+(-2n+5)a_3a_4\right]+\sum_{j\geqslant 7}a_2^{n-j}V_j\end{aligned}$$

and V_j for $7\leqslant j\leqslant n$ is a homogeneous polynomial in the variables $a_3,a_4,\ldots,a_n$. Obviously,

$$1+\frac{1}{\gamma}\left(\frac{w\Lambda'(g(w))}{\Lambda(g(w))}-1\right)=1-\frac{1}{\gamma}\sum_{n=2}^{\infty}F_{n-1}\left(b_2,b_3,b_4,\ldots,b_n\right)w^{n-1} \tag{2.2}$$

where $b_n=(1+\lambda(n-1))\tau_n$. Since, both functions f and its inverse map $g=f^{-1}$ are in $\mathscr{S}(\lambda,\gamma;\varphi)$, by the definition of subordination, there exist two Schwarz functions

$u(z) = c_1z + c_2z^2 + \ldots + c_nz^n + \ldots,\ |u(z)| < 1,\ z \in \mathbb{D}$ and $v(w) = d_1w + d_2w^2 + \ldots + d_nw^n + \ldots,\ |v(w)| < 1,\ w \in \mathbb{D}$, so that

$$1 + \frac{1}{\gamma}\left(\frac{z\Lambda'(f(z))}{\Lambda(f(z))} - 1\right) = \varphi(u(z)) = 1 - \sum_{n=1}^{\infty} B_1 K_n^{-1}\left(c_1, c_2, \ldots, c_n, B_1, B_2, \ldots, B_n\right) z^n \tag{2.3}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{w\Lambda'(g(w))}{\Lambda(g(w))} - 1\right) = \varphi(v(w)) = 1 - \sum_{n=1}^{\infty} B_1 K_n^{-1}\left(d_1, d_2, \ldots, d_n, B_1, B_2, \ldots, B_n\right) w^n. \tag{2.4}$$

In general (e.g., see [1] and [2, equation (1.6)]), the coefficients $K_n^p := K_n^p(k_1, k_2, \ldots, k_n, B_1, B_2, \ldots, B_n)$ are given by

$$\begin{aligned} K_n^p = {} & \frac{p!}{(p-n)!n!}k_1^n\frac{B_n}{B_1} + \frac{p!}{(p-n+1)!\,(n-2)!}k_1^{n-2}k_2\frac{B_{n-1}}{B_1} \\ & + \frac{p!}{(p-n+2)!\,(n-4)!}k_1^{n-3}k_3\frac{B_{n-2}}{B_1} \\ & + \frac{p!}{(p-n+3)!\,(n-4)!}k_1^{n-4}\left[k_4\frac{B_{n-3}}{B_1} + \frac{p-n+3}{2}k_2^2\frac{B_{n-2}}{B_1}\right] \\ & + \frac{p!}{(p-n+4)!\,(n-5)!}k_1^{n-5}\left[k_5\frac{B_{n-4}}{B_1} + (p-n+4)k_2k_3\frac{B_{n-3}}{B_1}\right] + \sum_{j\geqslant 6} k_1^{n-j}X_j \end{aligned}$$

where X_j is a homogeneous polynomial of degree j in the variables $k_2, k_3, \ldots, k_n$.

For the coefficients of the Schwarz functions $u(z)$ and $v(w)$ we have $|c_n| \leqslant 1$ and $|d_n| \leqslant 1$ (e.g., see [8]). Comparing the corresponding coefficients of (2.1) and (2.3) yields

$$\frac{1}{\gamma}F_{n-1}\left(a_2, a_3, \ldots, a_n\right) = B_1K_n^{-1}\left(c_1, c_2, \ldots, c_n, B_1, B_2, \ldots, B_n\right) \tag{2.5}$$

which under the assumption $a_m = 0;\ 2 \leqslant m \leqslant n-1$ we get

$$-\frac{1}{\gamma}(n-1)a_n = -\frac{1}{\gamma}(n-1)(1+\lambda(n-1))\rho_n = -B_1c_{n-1}. \tag{2.6}$$

Similarly, comparing the corresponding coefficients of (2.2) and (2.4) gives

$$\frac{1}{\gamma}F_{n-1}\left(b_2, b_3, \ldots, b_n\right) = B_1K_{n-1}^{-1}\left(d_1, d_2, \ldots, d_{n-1}, B_1, B_2, \ldots, B_n\right) \tag{2.7}$$

which by the hypothesis, we obtain

$$-\frac{1}{\gamma}(n-1)b_n = -B_1d_{n-1}.$$

Note that, for $a_m = 0;\ 2 \leqslant m \leqslant n-1$ we have $b_n = -a_n$ and therefore

$$\frac{1}{\gamma}(n-1)a_n = \frac{1}{\gamma}(n-1)(1+\lambda(n-1))\rho_n = -B_1d_{n-1}. \tag{2.8}$$

Taking the absolute values of either of the equations (2.6) or (2.8) we obtain the required bound. □

To prove our next theorem, we shall need the following well-known lemma (see [8]).

LEMMA 2.1. ([8]) *Let the function* $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ *be so that* $\operatorname{Re}(p(z)) > 0$ *for* $z \in \mathbb{D}$. *Then for* $-\infty < \alpha < \infty$,

$$\left|p_2 - \alpha p_1^2\right| \leqslant \begin{cases} 2 - \alpha \left|p_1\right|^2; & \alpha < \frac{1}{2}, \\ 2 - (1-\alpha)\left|p_1\right|^2; & \alpha \geqslant \frac{1}{2} \end{cases}. \tag{2.9}$$

Let $\varphi(z) = \sum_{n=1}^{\infty} \varphi_n z^n$ be a Schwarz function so that $|\varphi(z)| < 1$, $z \in \mathbb{D}$. Set $p(z) = [1 + \varphi(z)]/[1 - \varphi(z)]$ where $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ is so that $\operatorname{Re}(p(z)) > 0$ for $z \in \mathbb{D}$. Comparing the corresponding coefficients of powers of z yields $p_1 = 2\varphi_1$ and $p_2 = 2\left(\varphi_2 + \varphi_1^2\right)$. Now, substituting for p_1 and p_2 and letting $\eta = 1 - 2\alpha$ in (2.9) we obtain

$$\left|\varphi_2 + \eta \varphi_1^2\right| \leqslant \begin{cases} 1 - (1-\eta)|\varphi_1|^2; \; \eta > 0, \\ 1 - (1+\eta)|\varphi_1|^2; \; \eta \leqslant 0 \end{cases}. \tag{2.10}$$

The following theorem gives bounds for the coefficient body (ρ_2, ρ_3) of the generalized bi-subordinate functions of complex order γ and type λ.

THEOREM 2.2. *Let* $0 \leqslant \lambda \leqslant 1$ *and* $\gamma \in \mathbb{C} \setminus \{0\}$. *If both functions* $f(z) = z + \sum_{n=2}^{\infty} \rho_n z^n$ *and its inverse map* $g = f^{-1}$ *are in* $\mathscr{S}(\lambda, \gamma; \varphi)$ *then*

$$\left|\rho_3 - \rho_2^2\right| \leqslant \begin{cases} \frac{|\gamma| B_1}{2(1+2\lambda)}; \; B_1 \geqslant |B_2|, \\ \frac{|\gamma B_2|}{2(1+2\lambda)}; \; B_1 < |B_2| \end{cases}.$$

Proof. For $n = 2$, (2.5) and (2.7) imply

$$\rho_2 = \frac{\gamma B_1 c_1}{1+\lambda} \text{ and } \rho_2 = -\frac{\gamma B_1 d_1}{1+\lambda}. \tag{2.11}$$

For $n = 3$, the equations (2.5) and (2.7), respectively, imply

$$\frac{2(1+2\lambda)\rho_3 - (1+\lambda)^2 \rho_2^2}{\gamma} = B_1 c_2 + B_2 c_1^2 \tag{2.12}$$

and

$$\frac{-2(1+2\lambda)\rho_3 + (3+6\lambda-\lambda^2)\rho_2^2}{\gamma} = B_1 d_2 + B_2 d_1^2. \tag{2.13}$$

Considering (2.11) we get $c_1 = -d_1$. Also, from (2.12), (2.13) and $B_1 > 0$ we find that

$$\rho_3 - \rho_2^2 = \frac{\gamma B_1}{4(1+2\lambda)} \left[\left(c_2 + \frac{B_2}{B_1} c_1^2\right) - \left(d_2 + \frac{B_2}{B_1} d_1^2\right) \right]. \tag{2.14}$$

Taking the absolute values of both sides of (2.14) gives

$$|\rho_3-\rho_2^2|\leqslant\frac{|\gamma|B_1}{4(1+2\lambda)}\left[\left|c_2+\frac{B_2}{B_1}c_1^2\right|+\left|d_2+\frac{B_2}{B_1}d_1^2\right|\right]. \tag{2.15}$$

If $B_2\leqslant 0$, then for $\eta=B_2/B_1$ apply (2.10) to (2.15) to get

$$|\rho_3-\rho_2^2|\leqslant\frac{|\gamma|B_1}{4(1+2\lambda)}\left\{\left[1-\left(\frac{B_1+B_2}{B_1}\right)|c_1|^2\right]+\left[1-\left(\frac{B_1+B_2}{B_1}\right)|d_1|^2\right]\right\}. \tag{2.16}$$

If $B_1+B_2>0$ then (2.16) yields $|\rho_3-\rho_2^2|\leqslant\frac{|\gamma|B_1}{2(1+2\lambda)}$.

If $B_1+B_2<0$ then for the maximum values $|c_1|=|d_1|=1$ the inequality (2.16) yields

$$|\rho_3-\rho_2^2|\leqslant\frac{|\gamma|B_1}{4(1+2\lambda)}\left\{2\left[1-\left(\frac{B_1+B_2}{B_1}\right)\right]\right\}=-\frac{|\gamma|B_2}{2(1+2\lambda)}.$$

If $B_2>0$, then for $\eta=B_2/B_1$ apply (2.10) to (2.15) to get

$$|\rho_3-\rho_2^2|\leqslant\frac{|\gamma|B_1}{4(1+2\lambda)}\left\{\left[1-\left(\frac{B_1-B_2}{B_1}\right)|c_1|^2\right]+\left[1-\left(\frac{B_1-B_2}{B_1}\right)|d_1|^2\right]\right\}. \tag{2.17}$$

If $B_1-B_2>0$ then (2.17) yields $|\rho_3-\alpha\rho_2^2|\leqslant\frac{|\gamma|B_1}{2(1+2\lambda)}$.

If $B_1-B_2<0$ then for the maximum values $|c_1|=|d_1|=1$ the inequality (2.17) yields

$$|\rho_3-\rho_2^2|\leqslant\frac{|\gamma|B_1}{4(1+2\lambda)}\left\{2\left[1-\left(\frac{B_1-B_2}{B_1}\right)\right]\right\}=\frac{|\gamma|B_2}{2(1+2\lambda)}.$$

This concludes the proof of Theorem 2.2. □

For different values of λ and γ, Theorems 2.2 and 2.1 yield the following interesting corollaries.

COROLLARY 2.3. *If both functions f and its inverse map $g=f^{-1}$ are in $\mathscr{S}^*(\gamma;\varphi)$, then*

$$|\rho_n|\leqslant\frac{|\gamma|B_1}{(n-1)},\quad \rho_m=0;\quad 2\leqslant m\leqslant n-1.$$

Taking $\varphi(z)=(1+Az)/(1+Bz)=1+(A-B)z-B(A-B)z^2+\ldots$ in Corollary 2.3, we obtain the result of Hamidi and Jahangiri (see [13]).

COROLLARY 2.4. *If both functions f and its inverse map $g=f^{-1}$ are in $\mathscr{C}(\gamma;\varphi)$, then*

$$|\rho_n|\leqslant\frac{|\gamma|B_1}{(n-1)n},\quad \rho_m=0;\quad 2\leqslant m\leqslant n-1.$$

COROLLARY 2.5. *If both functions f and its inverse map $g=f^{-1}$ are in $\mathscr{S}^*[\delta,\beta]$ and $\mathscr{C}[\delta,\beta]$, respectively, then*

$$|\rho_n|\leqslant\frac{2(1-\beta)\,|\cos\delta|}{(n-1)}\quad \textit{and}\quad |\rho_n|\leqslant\frac{2(1-\beta)\,|\cos\delta|}{n(n-1)},\quad \rho_m=0;\quad 2\leqslant m\leqslant n-1.$$

COROLLARY 2.6. *If both functions f and its inverse map $g=f^{-1}$ are in $k-\mathcal{S}_P^*$ and $k-\mathcal{UCV}$, respectively, then for $\rho_m=0;\ 2\leqslant m\leqslant n-1$ we have*

$$|\rho_n|\leqslant\frac{Q_1}{(n-1)},\quad \left|\rho_3-\rho_2^2\right|\leqslant\begin{cases}\frac{\mathcal{B}^2}{1-k^2}; & 0\leqslant k<1,\\ \frac{4}{\pi^2}; & k=1,\\ \frac{\pi^2}{8(k^2-1)\sqrt{t}(1+t)\mathcal{K}^2(t)}; & k>1\end{cases}$$

and

$$|\rho_n|\leqslant\frac{Q_1}{(n-1)n},\quad \left|\rho_3-\rho_2^2\right|\leqslant\begin{cases}\frac{\mathcal{B}^2}{3\left(1-k^2\right)}; & 0\leqslant k<1,\\ \frac{4}{3\pi^2}; & k=1,\\ \frac{\pi^2}{24(k^2-1)\sqrt{t}(1+t)\mathcal{K}^2(t)}; & k>1\end{cases}$$

where Q_1 is given by (1.1).

Proof. Let f and its inverse map $g=f^{-1}$ be in $k-\mathcal{S}_P^*$. We will show that $Q_1\geqslant|Q_2|$ for $k\geqslant 0$. First suppose $0\leqslant k<1$. Since $0\leqslant\arccos k\leqslant\frac{\pi}{2}$ we have $\frac{|Q_2|}{Q_1}=\frac{\mathcal{B}^2+2}{3}\leqslant 1$. For $k=1$ it is clear that $\frac{|Q_2|}{Q_1}=\frac{2}{3}<1$. Finally, for $k>1$ we get $\frac{|Q_2|}{Q_1}=\frac{\left[4\mathcal{K}^2(t)(t^2+6t+1)-\pi^2\right]}{24\sqrt{t}(1+t)\mathcal{K}^2(t)}\leqslant 1$. A similar argument can be used to justify the case for $k-\mathcal{UCV}$. $\square$

Determination of extremal functions for bi-univalent functions (in general) and for bi-subordinate functions (in particular) remains a challenge.

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