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NEW HERMITE–HADAMARD INEQUALITIES VIA FRACTIONAL INTEGRALS, WHOSE ABSOLUTE VALUES OF SECOND DERIVATIVES IS P–CONVEX

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Abstract. In this research article, authors have established a general integral identity for Riemann-Liouville fractional integrals. Some new results related to the left-hand side of Hermite-Hadamard type integral inequalities utilizing this integral identity for the class of functions whose second derivatives at some power are P-convex are obtained.The presented results have some closely connection with [M. E. Özdemir, C. Yıldız, A. O. Akdemir, E. Set, On some inequalities for s-convex functions and applications, Jounal of Inequalities and Applications, 2013:333]

1. Introduction

Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1}$$

is known in the literature as the Hadamard inequality for convex mapping. Both inequalities hold in the reversed direction if f is concave [1].We take note that Hadamard's inequality might be viewed as a refinement of the idea of convexity and it takes after effortlessly from Jensen's inequality. Note that few of the classical inequalities for means can be gotten from (1) for suitable specific selections of the mapping f. It is notable that the Hermite-Hadamard inequality plays an essential part in nonlinear analysis. In the course of the most recent decade, this classical inequality has been enhanced and summed up in various routes; there have been a extensive number of research articles written on this subject, (see, [2–15]) and the references therein.

Some important definitions and mathematical preliminaries of fractional calculus theory which are utilized further as a bit of this paper.

Let $I \subset \mathbb{R}$ be an interval. The function $f: I \to \mathbb{R}$ is said to be P-convex (or belongs to the class $P(I)$) if it is nonnegative and, for all $a, b \in I$ and $t \in [0,1]$, satisfies the inequality

$$f(ta+(1-t)b) \leqslant f(a)+f(b) \tag{2}$$

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Note that $P(I)$ contain all nonnegative convex functions and quasi convex functions. Since then numerous articles have appeared in the literature reflecting further applications in this category see [3, 6, 12, 15]

DEFINITION 1. Let $f \in L^1[a,b]$. The left-side and right-side Riemann-Liouville fractional integrals of order $\alpha > 0$ with $a \geqslant 0$ are defined by

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad a < x$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively, where $\Gamma(.)$ is Gamma function and its definition is $\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1} du$. It is to be noted that $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Properties relating to this operator and for useful details on Hermite-Hadamard type inequalities connected with fractional integral inequalities, readers are directed to [2]–[7], [8], [10], [13].

In [13] Sarikaya et al. proved a variant of Hermite-Hadamard's inequalities for fractional integral which follows as:

THEOREM 1. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leqslant a < b$ and $f \in L^1[a,b]$. If f is convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \leqslant \frac{f(a)+f(b)}{2} \tag{3}$$

with $\alpha > 0$

REMARK 1. For $\alpha = 1$, inequality (3) reduces to inequality (1).

In [11] Özdemir et al. proved some inequalities related to Hermite-Hadamard's inequalities for functions whose second derivatives in absolute value at certain powers are s-convex functions as follows:

THEOREM 2. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on I^0 (where I^0 is the interior of I) such that $f'' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f''|$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds:*

$$\begin{aligned} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| &\leqslant \frac{(b-a)^2}{8(s+1)(s+2)(s+3)} \\ &\quad \times \left\{ |f''(a)| + (s+1)(s+2) \left|f''\left(\frac{a+b}{2}\right)\right| + |f''(b)| \right\} \\ &\leqslant \frac{\left[1+(s+2)2^{1-s}\right](b-a)^2}{8(s+1)(s+2)(s+3)} \left\{ |f''(a)| + |f''(b)| \right\} \end{aligned}$$

PROPOSITION 1. *Under the assumptions of Theorem 2 for* $s = 1$ *by Theorem 2, then we get,*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{(b-a)^2}{48}\left[|f''(a)| + |f''(b)|\right] \tag{4}$$

THEOREM 3. *Let* $f : I \subset [0,\infty) \to \mathbb{R}$ *be a twice differentiable mapping on* I^0 *(where* I^0 *is the interior of* I*) such that* $f'' \in L[a,b]$*, where* $a,b \in I$ *with* $a < b$*. If* $|f''|^q$ *,* $p,q \geqslant 1$ *is* s*-convex on* $[a,b]$*, for some fixed* $s \in (0,1]$*, then the following inequality holds:*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leqslant \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{1/p}\left\{\left(\frac{2}{(s+1)(s+2)(s+3)}|f''(a)|^q + \frac{1}{(s+3)}\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right. \\
&\left.+\left(\frac{1}{(s+3)}\left|f''\left(\frac{a+b}{2}\right)\right|^q + \frac{2}{(s+1)(s+2)(s+3)}|f''(b)|^q\right)^{1/q}\right\}
\end{aligned}$$

PROPOSITION 2. *Under the assumptions of Theorem 3 for* $s = 1$ *by Theorem 3, then we get,*

$$\begin{aligned}
\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{(b-a)^2}{48}\left(\frac{3}{4}\right)^{1/q}&\left\{\left(\frac{1}{3}|f''(a)|^q + \left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right. \\
&\left.+\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q + \frac{1}{3}|f''(b)|^q\right)^{1/q}\right\} \qquad (5)
\end{aligned}$$

The aim of this paper is to provide a unified approach to establish Hermite-Hadamard type inequalities for Riemann-Liouville fractional integral using the convexity as well as concavity, for functions whose absolute values of second derivatives are P-convex. we will derive a general integral identity for convex functions.

2. Main results

To obtain our principal results we need the following Lemma:

LEMMA 1. *Let* $I \subset \mathbb{R}$ *be an open interval,* $a,b \in I$ *with* $a < b$ *and* $f : [a,b] \to \mathbb{R}$ *be a twice differentiable function such that* f'' *is integrable and* $0 < \alpha \leqslant 1$ *on* (a,b) *with* $a < b$*. If* $|f''|$ *is a convex on* $[a,b]$*, then the following identity for Riemann-Liouville fractional integrals holds:*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] - f\left(\frac{a+b}{2}\right) = \frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}\sum_{k=1}^4 I_k,$$

where

$$
\begin{aligned}
I_1 &= \int_0^1 (1-t)^{\alpha+1} f''\left(ta+(1-t)\frac{a+b}{2}\right)dt,\\
I_2 &= \int_0^1 ((1+t)^{\alpha+1} - 2^{\alpha}(1+t) + \alpha 2^{\alpha}(1-t)) f''\left(tb+(1-t)\frac{a+b}{2}\right)dt,\\
I_3 &= \int_0^1 (1-t)^{\alpha+1} f''\left(tb+(1-t)\frac{a+b}{2}\right)dt,\\
I_4 &= \int_0^1 ((1+t)^{\alpha+1} - 2^{\alpha}(1+t) + \alpha 2^{\alpha}(1-t)) f''\left(ta+(1-t)\frac{a+b}{2}\right)dt.
\end{aligned}
$$

A simple proof of this inequality can be done by integrating by parts. The details are left to the interested readers. Using Lemma 1 the following results can be obtained.

THEOREM 4. *Let $I \subset \mathbb{R}$ be an open interval, $a,b \in I$ with $a<b$ and $f:[a,b] \to \mathbb{R}$ be a twice differentiable function such that f'' is integrable and $0<\alpha \leqslant 1$ on (a,b) with $a<b$. If $|f''|$ is a P-convex on $[a,b]$, then we have the following inequality for Riemann-Liouville fractional integrals:*

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] - f\left(\frac{a+b}{2}\right)\right|\\
&\qquad \leqslant \frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}(Q_1+Q_2)\left\{|f''(a)| + 2\left|f''\left(\frac{a+b}{2}\right)\right| + |f''(b)|\right\} \quad (6)
\end{aligned}
$$

where

$$
\begin{aligned}
Q_1 &= \int_0^1 (1-t)^{\alpha+1} dt = \frac{1}{(\alpha+2)}\\
Q_2 &= \int_0^1 ((1+t)^{\alpha+1} - 2^{\alpha}(1+t) + \alpha 2^{\alpha}(1-t))dt\\
&= \frac{2^{\alpha+2}}{(\alpha+2)} - 2^{\alpha+2} + 2^{\alpha-1} - \frac{1}{\alpha+2} + \alpha 2^{\alpha-1}
\end{aligned}
$$

Proof. Since $|f''|$, is a P-convex function, by using Lemma 1, we get

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] - f\left(\frac{a+b}{2}\right)\right|\\
&\leqslant \frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)} \int_0^1 (1-t)^{\alpha+1}\left\{f''\left(ta+(1-t)\frac{a+b}{2}\right)dt + f''\left(tb+(1-t)\frac{a+b}{2}\right)dt\right\}\\
&\quad + \frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)} \int_0^1 ((1+t)^{\alpha+1} - 2^{\alpha}(1+t) + \alpha 2^{\alpha}(1-t))\\
&\quad \times \left\{f''\left(tb+(1-t)\frac{a+b}{2}\right)dt + f''\left(ta+(1-t)\frac{a+b}{2}\right)dt\right\}
\end{aligned}
$$

$$\leqslant \frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}\int_0^1\left|(1-t)^{\alpha+1}\right|\left|\left(f''\left(ta+(1-t)\frac{a+b}{2}\right)|dt+f''\left(tb+(1-t)\frac{a+b}{2}\right)\right)dt\right|$$
$$+\frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}\int_0^1\left|((1+t)^{\alpha+1}-2^{\alpha}(1+t)+\alpha 2^{\alpha}(1-t))\right|$$
$$\times\left|\left(f''\left(tb+(1-t)\frac{a+b}{2}\right)dt+f''\left(ta+(1-t)\frac{a+b}{2}\right)\right)dt\right|$$
$$\leqslant \frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}Q_1\left\{|f''(a)|+2\left|f''\left(\frac{a+b}{2}\right)\right|+|f''(b)|\right\}$$
$$+\frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}Q_2\left\{|f''(a)|+2\left|f''\left(\frac{a+b}{2}\right)\right|+|f''(b)|\right\}$$
$$\leqslant \frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}(Q_1+Q_2)\left\{|f''(a)|+2\left|f''\left(\frac{a+b}{2}\right)\right|+|f''(b)|\right\}$$

which completes the proof. □

COROLLARY 1. *Under the assumption of Theorem 4 if we put* $(|f''(a)|=|f''(b)|)=0$ *then inequality takes the following form*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leqslant \frac{(b-a)^2}{2^{\alpha+2}(\alpha+1)}(Q_1+Q_2)\left(\left|f''\left(\frac{a+b}{2}\right)\right|\right)$$

The corresponding version for powers of the absolute value of the derivative is incorporated in the following theorem.

THEOREM 5. *Let* $I\subset\mathbb{R}$ *be an open interval,* $a,b\in I$ *with* $a<b$ *and* $f:[a,b]\to\mathbb{R}$ *be a twice differentiable function such that* f'' *is integrable and* $0<\alpha\leqslant 1$ *on* (a,b) *with* $a<b$. *If* $|f''|^q$ *is a P-convex on* $[a,b]$, $q\geqslant 1$ *then we have the following inequality for Riemann-Liouville fractional integrals:*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leqslant \frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}(Q_1)^{1-1/q}\left[\left(Q_1\left\{|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{1/q}\right]$$
$$+\frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}(Q_2)^{1-1/q}\left[\left(Q_2\left\{|f''(b)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{1/q}\right] \tag{7}$$

Proof. Using the well-known power-mean integral inequality for $q > 1$, we have

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b)+J_{b^-}^{\alpha} f(a)\right]-f\left(\frac{a+b}{2}\right)\right| \\
\leqslant &\left|\frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}\int_0^1 (1-t)^{\alpha+1}\left(f''\left(ta+(1-t)\frac{a+b}{2}\right)\right)dt\right| \\
\leqslant &\frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}\left(\int_0^1 (1-t)^{\alpha+1}\,dt\right)^{1-1/q} \\
&\times\left[\left(\int_0^1 (1-t)^{\alpha+1}\left|f''\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{1/q}\right. \\
&\left.+\left(\int_0^1 (1-t)^{\alpha+1}\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{1/q}\right] \\
&+\frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}\left(\int_0^1 ((1+t)^{\alpha+1}-2^{\alpha}(1+t)+\alpha 2^{\alpha}(1-t))dt\right)^{1-1/q} \\
&\times\left[\left(\int_0^1 ((1+t)^{\alpha+1}-2^{\alpha}(1+t)+\alpha 2^{\alpha}(1-t))\left|f''\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{1/q}\right. \\
&\left.+\left(\int_0^1 ((1+t)^{\alpha+1}-2^{\alpha}(1+t)+\alpha 2^{\alpha}(1-t))\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{1/q}\right] \\
\leqslant &\frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}(Q_1)^{1-1/q}\left[\left(Q_1\left\{\left|f''(a)\right|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{1/q}\right. \\
&\left.+\left(Q_1\left\{\left|f''(b)\right|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{1/q}\right] \\
&+\frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}(Q_2)^{1-1/q}\left[\left(Q_2\left\{\left|f''(a)\right|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{1/q}\right. \\
&\left.+\left(Q_2\left\{\left|f''(b)\right|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{1/q}\right] \\
\leqslant &\frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}(Q_1)^{1-1/q}\left[\left(Q_1\left\{\left|f''(a)\right|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{1/q}\right] \\
&+\frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}(Q_2)^{1-1/q}\left[\left(Q_2\left\{\left|f''(b)\right|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{1/q}\right].
\end{aligned}
$$

Which completes the proof. □

COROLLARY 2. *On letting $\alpha = 1$ in Theorem 5, then inequality (7) becomes as:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right|$$
$$\leqslant \frac{(b-a)^2}{48}\left\{ \left(|f''(a)|^q + \left| f''\left(\frac{a+b}{2}\right)\right|^q \right)^{1/q} + \left(\left| f''\left(\frac{a+b}{2}\right)\right|^q + |f''(b)|^q \right)^{1/q} \right\}$$

In the following, we obtain estimate of Hermite-Hadamard inequality (3) for concave functions.

THEOREM 6. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b) such that $f''\in L[a,b]$. If $|f''|^q$ is concave on $[a,b]$ for some fixed $p\geqslant 1$ with $q=\frac{p}{p-1}$, and $|f''|$ is a linear map, then the following inequality for fractional integrals holds for $\alpha > 0$:*

$$\left| \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)\right] - f\left(\frac{a+b}{2}\right)\right|$$
$$\leqslant \frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}\left[Q_1\left\{ \left| f''\left(\left\{\frac{Q_4 a + Q_3\left(\frac{a+b}{2}\right)}{Q_1}\right\}\right)\right| + \left| f''\left(\left\{\frac{Q_4 b + Q_3\left(\frac{a+b}{2}\right)}{Q_1}\right\}\right)\right|\right\}\right.$$
$$\left. +Q_2\left| f''\left(\left\{\frac{(Q_5 a + Q_6\left(\frac{a+b}{2}\right)}{Q_2}\right\}\right)\right| + f''\left\{\frac{(Q_5 b + Q_6\left(\frac{a+b}{2}\right)}{Q_2}\right\}\right]. \tag{8}$$

Where

$$Q_1 = \int_0^1 (1-t)^{\alpha+1}dt = \frac{1}{(\alpha+2)}$$
$$Q_2 = \int_0^1 ((1+t)^{\alpha+1} - 2^\alpha(1+t) + \alpha 2^\alpha(1-t))dt$$
$$= \frac{2^{\alpha+2}}{(\alpha+2)} - 2^{\alpha+2} + 2^{\alpha-1} - \frac{1}{\alpha+2} + \alpha 2^{\alpha-1}$$
$$Q_3 = \int_0^1 (1-t)^{\alpha+2}dt = \frac{1}{\alpha+3}$$
$$Q_4 = \int_0^1 t(1-t)^{\alpha+1}dt = \frac{1}{(\alpha+2)(\alpha+3)}$$
$$Q_5 = \int_0^1 \left\{(1+t)^{\alpha+1} - 2^\alpha(1+t) + \alpha 2^\alpha(1-t)\right\}tdt$$
$$= \frac{2^{\alpha+2}}{\alpha+2} - \frac{2^{\alpha+3}}{(\alpha+2)(\alpha+3)} + \frac{1}{(\alpha+2)(\alpha+3)} + \alpha\frac{2^\alpha}{6} - \frac{5.2^\alpha}{6}$$
$$Q_6 = \int_0^1 \left\{(1+t)^{\alpha+1} - 2^\alpha(1+t) + \alpha 2^\alpha(1-t)\right\}(1-t)dt$$
$$= -\frac{1}{\alpha+2} + \frac{2^{\alpha+3}}{(\alpha+2)(\alpha+3)} - \frac{1}{(\alpha+2)(\alpha+3)} - \frac{2^{\alpha+1}}{3} + \alpha\frac{2^\alpha}{3}$$

Proof. Using the concavity of $|f''|^q$ and the power-mean inequality, we obtain

$$|f''(tx+(1-t)y)|^q > t|f''(x)|^q+(1-t)|f''(y)|^q \geqslant \left(t|f''(x)|+(1-t)|f''(y)|\right)^q$$

Hence

$$|f''(tx+(1-t)y)| \geqslant t|f''(x)|+(1-t)|f''(y)|,$$

so, $|f''|$ is also concave. By the Jensen integral inequality, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]-f\left(\frac{a+b}{2}\right)\right| \\
&\leqslant \frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}\left(\int_0^1 (1-t)^{\alpha+1}\,dt\right)\left|f''\left(\frac{\int_0^1 (1-t)^{\alpha+1}\left(ta+(1-t)\frac{a+b}{2}\right)dt}{\int_0^1 (1-t)^{\alpha+1}\,dt}\right)\right|^q \\
&\quad+\frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}\left(\int_0^1 (1-t)^{\alpha+1}\,dt\right)\left|f''\left(\frac{\int_0^1 (1-t)^{\alpha+1}\left(tb+(1-t)\frac{a+b}{2}\right)dt}{\int_0^1 (1-t)^{\alpha+1}\,dt}\right)\right|^q \\
&\quad+\frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}\left(\int_0^1 ((1+t)^{\alpha+1}-2^\alpha(1+t)+\alpha 2^\alpha(1-t))dt\right) \\
&\quad\times\left|f''\left(\frac{\int_0^1 ((1+t)^{\alpha+1}-2^\alpha(1+t)+\alpha 2^\alpha(1-t))\left(ta+(1-t)\frac{a+b}{2}\right)dt}{\int_0^1 ((1+t)^{\alpha+1}-2^\alpha(1+t)+\alpha 2^\alpha(1-t))dt}\right)\right|^q \\
&\quad+\frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}\left(\int_0^1 ((1+t)^{\alpha+1}-2^\alpha(1+t)+\alpha 2^\alpha(1-t))dt\right) \\
&\quad\times\left|f''\left(\frac{\int_0^1 ((1+t)^{\alpha+1}-2^\alpha(1+t)+\alpha 2^\alpha(1-t))\left(tb+(1-t)\frac{a+b}{2}\right)dt}{\int_0^1 ((1+t)^{\alpha+1}-2^\alpha(1+t)+\alpha 2^\alpha(1-t))dt}\right)\right|^q \\
&\leqslant \frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}(Q_1)\left|f''\left(\frac{\left(Q_4a+Q_3\left(\frac{a+b}{2}\right)\right)}{Q_1}\right)\right|^q \\
&\quad+\frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}(Q_1)\left|f''\left(\frac{\left(Q_4b+Q_3\left(\frac{a+b}{2}\right)\right)}{Q_1}\right)\right|^q \\
&\quad+\frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}(Q_2)\left|f''\left(\frac{\left(Q_5a+Q_6\left(\frac{a+b}{2}\right)\right)}{Q_2}\right)\right|^q \\
&\quad+\frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}(Q_2)\left|f''\left(\frac{\left(Q_5b+Q_6\left(\frac{a+b}{2}\right)\right)}{Q_2}\right)\right|^q \\
&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]-f\left(\frac{a+b}{2}\right)\right| \\
&\leqslant \frac{(b-a)^2}{2^{\alpha+3}(\alpha+1)}\times\left[Q_1\left\{\left|f''\left(\left\{\frac{Q_4a+Q_3\left(\frac{a+b}{2}\right)}{Q_1}\right\}\right)\right|+\left|f''\left(\left\{\frac{Q_4b+Q_3\left(\frac{a+b}{2}\right)}{Q_1}\right\}\right)\right|\right\}\right. \\
&\quad\left.+Q_2\left|f''\left(\left\{\frac{\left(Q_5a+Q_6\left(\frac{a+b}{2}\right)\right)}{Q_2}\right\}\right)\right|+f''\left\{\frac{\left(Q_5b+Q_6\left(\frac{a+b}{2}\right)\right)}{Q_2}\right\}\right]
\end{aligned}$$

The proof is completed. □

COROLLARY 3. *On letting* $\alpha = 1$ *in Theorem* 6*, then inequality* (8) *becomes as:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leqslant \frac{(b-a)^2}{48}\left[\left|f''\left(\frac{5a+3b}{8}\right)\right| + \left|f''\left(\frac{3a+5b}{8}\right)\right|\right]. \tag{9}$$

REMARK 2. Inequality (9) is an improvement of obtained inequality as in [11, Corollary 4]

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