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ULAM–HYERS STABILITY FOR MATRIX–VALUED FRACTIONAL DIFFERENTIAL EQUATIONS

ZHANPENG YANG, WENJUAN REN AND TIANZHOU XU

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Abstract. In this paper, some Ulam-Hyers stability results for matrix-valued fractional differential equations are obtained. We also establish some sufficient conditions for the stability of matrix-valued fractional differential equations.

1. Introduction

Matrix-valued differential equations are very important in various fields which including physics, statistics, optimization, economic, linear system and linear differential system problems [1, 2]. Recently, stability of fractional order system has attracted increasing interest due to its importance in control theory. In [3], Matignon firstly studied the stability of linear fractional differential systems with applications to control processing. Since then, many researchers have studied further on the stability of linear fractional order linear systems [4, 5]. Li et al. considered the Mittag-Leffler stability for fractional order nonlinear dynamic systems and proposed Lyapunov direct method for the stability of fractional order nonlinear systems [6, 7]. By using Bihari's and Bellman-Gronwall's inequality, Delavari et al. introduced an extension of Lyapunov direct method for fractional order systems [8]. Senol et al. [9] presents numerical methods for robust stability analysis of nonlinear fractional order systems.

The notion of Ulam-Hyers stability was proposed by Ulam [10]. Hyers firstly obtained some results on the Ulam stablity in the case of Banach space [11]. Aoki generalized Hyers' theorem for approximately additive mappings [12]. Th.M. Rassias provided a generalized version of Hyers' result which allows the Cauchy difference to be unbounded [13]. J. M. Rassias and Xu generalized the Hyers stability result by introducing two weaker conditions controlled by a product of different powers of norms and a mixed product-sum of powers of norms, respectively [14, 15, 16, 17]. Furthermore, Jung proved the Ulam–Hyers stability of linear functional equations [18, 19, 20]. By applying a fixed point theorem in a generalized complete metric space, Wang et al. presented the Hyers–Ulam–Rassias stability, Hyers–Ulam stability and four types of Mittag–Leffler–Ulam stability for fractional differential equations [21, 22, 23].

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Recently, Wang proved the Hyers–Ulam stability of additive and quadratic functional equations in matrix Banach spaces [24]. By using fixed point method, Lee give some Hyer-Ulam stablity results for the quadratic functional equation and the Cauchy additive functional equation in matrix random normed spaces [25, 26]. In this paper, we consider the Ulam-Hyers stability of some matrix differential equations and give some Ulam-Hyers stability results for matrix-valued fractional differential equations. And we also establish some sufficient conditions for the stability of matrix-valued fractional differential equations.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper.

2.1. Fractional differential operators

There are several different definitions of the fractional derivative (see [2]). We will use the following definitions:

(i) Riemann-Liouville fractional integral and derivative

$$I^{\alpha}\mu(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}\mu(\tau)\mathrm{d}\tau, \tag{1}$$

$$D^{\alpha}\mu(x) = D^n I^{n-\alpha}\mu(x), \tag{2}$$

where $\alpha > 0$, $n-1 < \alpha \leqslant n$ $(n \in \mathbb{N})$, D is the differential operator, I is the integral operator and $\mu(x)$ is a suitable function for $x > 0$.

The Riemann-Liouville derivative has some disadvantage when trying to model the real-world phenomena. Therefore, we will introduce a modified fractional differential operator proposed by Caputo on the theory of viscoelasticity[27].

(ii) Caputo fractional differential operators

$$D^{\alpha}\mu(x) = I^{n-\alpha}D^n\mu(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{\mu^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}}\mathrm{d}\tau, \tag{3}$$

where $\alpha > 0$, $x > 0$ and $n-1 < \alpha \leqslant n$ $(n \in \mathbb{N})$.

The fractional derivative of $\mu(x)$ in the Caputo sense is defined for $0 < \alpha < 1$ as

$$D^{\alpha}\mu(x) = \frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{\mu'(\tau)}{(x-\tau)^{\alpha}}\mathrm{d}\tau. \tag{4}$$

2.2. Mittag–Leffler matrix

The exponential function e^t plays a very important role in the theory of integer-order differential equations. Its two-parameter generalization is defined as [28]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}, \quad \alpha,\beta > 0. \tag{5}$$

Moreover, when $\beta = 1$, we denote by $E_\alpha(z)$.

DEFINITION 1. [1] Let $\alpha > 0$ and $A \in M_m$. The Mittag-Leffler matrix $E_\alpha(A)$ is defined as

$$E_\alpha(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(k\alpha+1)} = I_m + \frac{A}{\Gamma(\alpha+1)} + \frac{A^2}{\Gamma(2\alpha+1)} + \dots. \tag{6}$$

DEFINITION 2. [1] For $A \in M_m$ the Mittag-Leffler function $E_\alpha(At^\alpha)$ is defined as

$$E_\alpha(At^\alpha) = \sum_{k=0}^{\infty} \frac{A^k t^{\alpha k}}{\Gamma(k\alpha+1)}, \ \alpha > 0. \tag{7}$$

Since we make use the spectral decomposition of $E_\alpha(A)$ and $E_\alpha(At^\alpha)$, then we get the following representations:

$$E_\alpha(A) = \sum_{k=0}^{\infty} x_k y_k^T E_\alpha(\lambda_k), \quad E_\alpha(At^\alpha) = \sum_{k=0}^{\infty} x_k y_k^T E_\alpha(\lambda_k t^\alpha), \tag{8}$$

where $\{x_1, x_2, \dots, x_m\}$ and $\{y_1, y_2, \dots, y_m\}$ are the eigenvectors corresponding to the eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$ of A and A^T, respectively.

The Mittag-Leffler function has the following asymptotic expression.

LEMMA 1. [2] *Let $0 < \alpha < 2$ and β be an arbitrary complex number. Assume that μ is an arbitrary real number such that $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$. Then, for an arbitrary integer $p \geqslant 1$, we have*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}} - \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta-\alpha k)} + O(|z|^{-1-p}), \tag{9}$$

when $|arg(z)| \leqslant \mu$ and $|z| \to \infty$;

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta-\alpha k)} + O(|z|^{-1-p}), \tag{10}$$

when $\mu \leqslant |arg(z)| \leqslant \pi$ and $|z| \to \infty$.

REMARK 1. In Lemma 1, if $\beta = \alpha$, then we have

$$E_{\alpha,\alpha}(z) = \frac{1}{\alpha} z^{(1-\alpha)/\alpha} e^{z^{1/\alpha}} - \sum_{k=2}^{p} \frac{z^{-k}}{\Gamma(\alpha-\alpha k)} + O(|z|^{-1-p}), \tag{11}$$

when $|\arg(z)| \leqslant \mu$ and $|z| \to \infty$;

$$E_{\alpha,\alpha}(z) = -\sum_{k=2}^{p} \frac{z^{-k}}{\Gamma(\alpha-\alpha k)} + O(|z|^{-1-p}), \tag{12}$$

when $\mu \leqslant |\arg(z)| \leqslant \pi$ and $|z| \to \infty$.

LEMMA 2. *Let $A \in M_n$, $0 < \alpha < 2$, $\beta > 0$. Assume that μ is such that $(\pi\alpha/2) < \mu < \min\{\pi, \pi\alpha\}$ and that $C_1 > 0$ is real constant. Then*

$$\|E_{\alpha,\beta}(A)\| \leqslant \frac{C_1}{1+\|A\|}, \tag{13}$$

where $\mu \leqslant |arg(\lambda(A))| \leqslant \pi$, $\lambda(A)$ represents the eigenvalues of matrix A and $\|\cdot\|$ is the l_2-norm.

2.3. Ulam–Hyers stability

Let $(\mathscr{X}, \|\cdot\|)$ be a Banach space, $\mathscr{Y}$ be a nonempty subset of $\mathscr{X}$ and $T : \mathscr{Y} \to \mathscr{X}$ be an operator. Let us give the definition of Ulam–Hyers stability of an operator equation due to Rus [29]. If $T : \mathscr{Y} \to \mathscr{X}$ is an operator, let us consider the operator equation

$$T(x) = 0, \quad x \in \mathscr{Y} \tag{14}$$

and the inequation

$$\|T(y)\| \leqslant \varepsilon. \tag{15}$$

DEFINITION 3. The equation (14) is called Ulam–Hyers stable if for each solution μ of (15) there exists a solution ν of the operator equation (14) such that

$$\|\mu - \nu\| \leqslant c\varepsilon,$$

where c is a constant depended on T.

DEFINITION 4. The equation (14) is called generalized Ulam–Hyers stable if there exists $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ increasing, continuous at 0 and $\psi(0) = 0$ such that for each $\varepsilon > 0$ and for each solution μ of (15) there exists a solution ν of the operator equation (14) such that

$$\|\mu - \nu\| \leqslant \psi(\varepsilon).$$

3. Ulam–Hyers Stability of Matrix-valued Fractional Equations

In this section, we present our main results for Ulam–Hyers stability of some linear matrix fractional differential equations. Firstly, we consider the stability of the non-homogenous vector-valued fractional differential equation:

$$\begin{cases} D^\alpha \mu(t) = A\mu(t) + f(t), \\ \mu(0) = t_0,\ t \in [0, b), \end{cases} \tag{16}$$

where $A \in M_n$, $t_0 \in M_{n,1}$.

THEOREM 1. *If all the eigenvalues of A satisfy*

$$|arg(\lambda(A))| > \frac{\alpha\pi}{2} \tag{17}$$

Then, the system (16) is Ulam-Hyers stable.

Proof. It is easy to verify that the unique solution of the system (16) is

$$\mu(t)=E_\alpha(At^\alpha)t_0+\int_0^t (t-s)^{\alpha-1}E_\alpha(A(t-s)^\alpha)f(s)\mathrm{d}s. \tag{18}$$

Let us consider the inequation

$$\|D^\alpha\mu(t)-A\mu(t)-f(t)\|\leqslant\varepsilon. \tag{19}$$

A function $\nu\in M_{n,1}$ is a solution of (19) if and only if there exists a function $g\in M_{n,1}$ (which depend on ν) such that

(i) $\|g(t)\|\leqslant\varepsilon,\ \forall t\in[0,b)$,

(ii)

$$D^\alpha\nu(t)=A\nu(t)+f(t)+g(t),\ \forall t\in[0,b). \tag{20}$$

Then, ν is a solution of the following integral inequation

$$\left\|\nu(t)-E_\alpha(At^\alpha)t_0-\int_0^t (t-s)^{\alpha-1}E_\alpha(A(t-s)^\alpha)f(s)\mathrm{d}s\right\|\leqslant C\varepsilon, \tag{21}$$

where C is a constant.

In fact, by (20) we have

$$\nu(t)=E_\alpha(At^\alpha)t_0+\int_0^t (t-s)^{\alpha-1}E_\alpha(A(t-s)^\alpha)(f(s)+g(s))\mathrm{d}s. \tag{22}$$

Then, we get

$$\begin{aligned}
&\left\|\nu(t)-E_\alpha(At^\alpha)x_0-\int_0^t (t-s)^{\alpha-1}E_\alpha(A(t-s)^\alpha)f(s)\mathrm{d}s\right\|\\
&=\left\|\int_0^t (t-s)^{\alpha-1}E_\alpha(A(t-s)^\alpha)g(s)\mathrm{d}s\right\|\\
&=\left\|\int_0^t s^{\alpha-1}E_\alpha(As^\alpha)g(t-s)\mathrm{d}s\right\|\\
&\leqslant\varepsilon\int_0^t \left\|s^{\alpha-1}E_\alpha(As^\alpha)\right\|\mathrm{d}s.
\end{aligned} \tag{23}$$

Assume that all the eigenvalues of A satisfy (17). First, suppose that the matrix A is diagonalizable, i.e. there exists an invertible matrix T such that

$$\Lambda=T^{-1}AT=\mathrm{diag}(\lambda_1,\lambda_2,\dots,\lambda_n).$$

Then,

$$E_\alpha(As^\alpha)=TE_\alpha(\Lambda s^\alpha)T^{-1}=T\mathrm{diag}[E_\alpha(\lambda_1s^\alpha),E_\alpha(\lambda_2s^\alpha),\dots,E_\alpha(\lambda_ns^\alpha)]T^{-1},$$

and

$$\begin{aligned}
&\int_0^t \|s^{\alpha-1}E_\alpha(As^\alpha)\|\mathrm{d}s\\
&=\int_0^t \|T\mathrm{diag}(s^{\alpha-1}E_\alpha(\lambda_1s^\alpha),s^{\alpha-1}E_\alpha(\lambda_2s^\alpha),\dots,s^{\alpha-1}E_\alpha(\lambda_ns^\alpha))T^{-1}\|\mathrm{d}s.
\end{aligned} \tag{24}$$

We will show that there exists a positive constant C_0 such that

$$\int_0^t |s^{\alpha-1}E_\alpha(\lambda_i s^\alpha)|\mathrm{d}s \leqslant C_0,\ 1 \leqslant i \leqslant n.$$

Indeed, using (12) we find for $t > t_0(>0)$,

$$\begin{aligned}
&\int_0^t |s^{\alpha-1}E_\alpha(\lambda_i s^\alpha)|\mathrm{d}s\\
&= \int_0^{t_0} |s^{\alpha-1}E_\alpha(\lambda_i s^\alpha)|\mathrm{d}s + \int_{t_0}^t |s^{\alpha-1}E_\alpha(\lambda_i s^\alpha)|\mathrm{d}s\\
&= \int_0^{t_0} |s^{\alpha-1}E_\alpha(\lambda_i s^\alpha)|\mathrm{d}s + \int_{t_0}^t \left| s^{\alpha-1}\left(-\sum_{k=2}^{p} \frac{(\lambda_i s^\alpha)^{-k}}{\Gamma(1-\alpha k)} + O(|\lambda_i s^\alpha|^{-1-p}) \right)\right| \mathrm{d}s\\
&= \int_0^{t_0} |s^{\alpha-1}E_\alpha(\lambda_i s^\alpha)|\mathrm{d}s + \int_{t_0}^t \left| -\sum_{k=2}^{p} \frac{(\lambda_i)^{-k}s^{-\alpha k+\alpha-1}}{\Gamma(1-\alpha k)} + O(|\lambda_i|^{-1-p}s^{-\alpha p-1}) \right| \mathrm{d}s\\
&\leqslant \int_0^{t_0} s^{\alpha-1}E_\alpha(|\lambda_i| s^\alpha)\mathrm{d}s + \int_{t_0}^t \left\{ \sum_{k=2}^{p} \frac{|\lambda_i|^{-k}s^{-\alpha k+\alpha-1}}{|\Gamma(1-\alpha k)|} + O(|\lambda_i|^{-1-p}s^{-\alpha p-1}) \right\} \mathrm{d}s\\
&= \sum_{k=0}^{\infty} \frac{|\lambda_i|^k}{\Gamma(\alpha k+1)} \int_0^{t_0} s^{\alpha k+\alpha-1}\mathrm{d}s + \sum_{k=2}^{p} \frac{|\lambda_i|^{-k}}{|\Gamma(1-\alpha k)|}\int_{t_0}^t s^{-\alpha k+\alpha-1}\mathrm{d}s + O(|\lambda_i|^{-1-p}t^{-\alpha p})\\
&= \sum_{k=0}^{\infty} \frac{|\lambda_i|^k t_0^{\alpha k+\alpha}}{\Gamma(\alpha k+2)} + \sum_{k=2}^{p} \frac{|\lambda_i|^{-k}t^{-\alpha k+\alpha}}{(-\alpha k+1)|\Gamma(1-\alpha k)|} - \sum_{k=2}^{p} \frac{|\lambda_i|^{-k}t_0^{-\alpha k+\alpha}}{(-\alpha k+1)|\Gamma(1-\alpha k)|}\\
&\quad + O(|\lambda_i|^{-1-p}t^{-\alpha p}) \to t_0^\alpha E_{\alpha,2}(|\lambda_i|t_0^\alpha) + \sum_{k=2}^{p} \frac{|\lambda_i|^{-k}t_0^{-\alpha k+\alpha}}{|\Gamma(2-\alpha k)|} \leqslant C_0 \text{ as } t\to\infty.
\end{aligned} \tag{25}$$

It immediately follows that $\int_0^t \|s^{\alpha-1}E_\alpha(As^\alpha)\|\mathrm{d}s \leqslant C$ for any $t \geqslant 0$.

Next, assume that the matrix A is similar to a Jordan canonical form, i.e., there exists an invertible matrix T such that

$$J = T^{-1}AT = \mathrm{diag}(J_1, J_2, \ldots, J_r),$$

where J_i, $1 \leqslant i \leqslant r$ has the following form

$$\begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}_{n_i \times n_i},$$

and $\sum_{i=1}^r n_i = n$. Obviously,

$$E_\alpha(As^\alpha) = T\mathrm{diag}[E_\alpha(J_1 s^\alpha), E_\alpha(J_2 s^\alpha), \ldots, E_\alpha(J_r s^\alpha)]T^{-1},\ \ 1 \leqslant i \leqslant r.$$

$$
\begin{aligned}
E_\alpha(J_i s^\alpha) &= \sum_{k=0}^{\infty} \frac{(J_i s^\alpha)^k}{\Gamma(\alpha k+1)} = \sum_{k=0}^{\infty} \frac{(s^\alpha)^k}{\Gamma(\alpha k+1)} J_i^k \\
&= \sum_{k=0}^{\infty} \frac{(s^\alpha)^k}{\Gamma(\alpha k+1)} \begin{bmatrix} \lambda_i^k & C_k^1 \lambda_i^{k-1} & \cdots & C_k^{n_i-1} \lambda_i^{k-n_i+1} \\ & \lambda_i^k & \ddots & \vdots \\ & & \ddots & C_k^1 \lambda_i^{k-1} \\ & & & \lambda_i^k \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{(\lambda_i s^\alpha)^k}{\Gamma(\alpha k+1)} & \sum_{k=0}^{\infty} \frac{(s^\alpha)^k}{\Gamma(\alpha k+1)} C_k^1 \lambda_i^{k-1} & \cdots & \sum_{k=0}^{\infty} \frac{(s^\alpha)^k}{\Gamma(\alpha k+1)} C_k^{n_i-1} \lambda_i^{k-n_i+1} \\ & \sum_{k=0}^{\infty} \frac{(\lambda_i s^\alpha)^k}{\Gamma(\alpha k+1)} & \ddots & \vdots \\ & & \ddots & \sum_{k=0}^{\infty} \frac{(s^\alpha)^k}{\Gamma(\alpha k+1)} C_k^1 \lambda_i^{k-1} \\ & & & \sum_{k=0}^{\infty} \frac{(\lambda_i s^\alpha)^k}{\Gamma(\alpha k+1)} \end{bmatrix} \\
&= \begin{bmatrix} E_\alpha(\lambda_i s^\alpha) & \frac{1}{1!} \frac{\partial}{\partial \lambda_i} E_\alpha(\lambda_i s^\alpha) & \cdots & \frac{1}{(n_i-1)!} \left(\frac{\partial}{\partial \lambda_i}\right)^{n_i-1} E_\alpha(\lambda_i s^\alpha) \\ & E_\alpha(\lambda_i s^\alpha) & \ddots & \vdots \\ & & \ddots & \frac{1}{1!} \frac{\partial}{\partial \lambda_i} E_\alpha(\lambda_i s^\alpha) \\ & & & E_\alpha(\lambda_i s^\alpha) \end{bmatrix},
\end{aligned}
$$

where $C_k^j,\ 1 \leqslant j \leqslant n_i - 1$ are the binomial coefficients.

For $t > t_0 (> 0)$, we get

$$
\begin{aligned}
&\int_0^t \left| s^{\alpha-1} \frac{1}{j!} \left(\frac{\partial}{\partial \lambda_i}\right)^j E_\alpha(\lambda_i s^\alpha) \right| \mathrm{d}s \\
&= \int_0^{t_0} \left| s^{\alpha-1} \frac{1}{j!} \left(\frac{\partial}{\partial \lambda_i}\right)^j E_\alpha(\lambda_i s^\alpha) \right| \mathrm{d}s + \int_{t_0}^t \left| s^{\alpha-1} \frac{1}{j!} \left(\frac{\partial}{\partial \lambda_i}\right)^j E_\alpha(\lambda_i s^\alpha) \right| \mathrm{d}s \\
&\leqslant \int_0^{t_0} \sum_{k=0}^{\infty} \frac{k(k-1)\cdots(k-j+1)|\lambda|^{k-j} s^{\alpha k+\alpha-1}}{j!\Gamma(\alpha k+1)} \mathrm{d}s \\
&\quad + \int_{t_0}^t \left| s^{\alpha-1} \frac{1}{j!} \left(\frac{\partial}{\partial \lambda_i}\right)^j \left\{ -\sum_{k=2}^{p} \frac{(\lambda_i s^\alpha)^{-k}}{\Gamma(1-\alpha k)} + O(|\lambda_i s^\alpha|^{-1-p}) \right\} \right| \mathrm{d}s \\
&= \sum_{k=0}^{\infty} \frac{k(k-1)\cdots(k-j+1)|\lambda|^{k-j}}{j!\Gamma(\alpha k+1)} \int_0^{t_0} s^{\alpha k+\alpha-1} \mathrm{d}s \\
&\quad + \int_{t_0}^t \left| s^{\alpha-1} \left\{ -\sum_{k=2}^{p} \frac{(-1)^j (k+j-1)! \lambda_i^{-k-j} s^{-\alpha k}}{j!(k-1)!\Gamma(1-\alpha k)} + O(|\lambda_i|^{-1-p-j} s^{\alpha(-1-p)}) \right\} \right| \mathrm{d}s \\
&\leqslant \sum_{k=0}^{\infty} \frac{k(k-1)\cdots(k-j+1)|\lambda|^{k-j} t_0^{\alpha k+\alpha}}{j!\Gamma(\alpha k+2)} \\
&\quad + \int_{t_0}^t \left\{ \sum_{k=2}^{p} \frac{(k+j-1)! \lambda_i^{-k-j} s^{-\alpha k+\alpha-1}}{j!(k-1)!|\Gamma(1-\alpha k)|} + O(|\lambda_i|^{-1-p-j} s^{-\alpha p-1} \right\} \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&= t_0^{\alpha} \frac{1}{j!} \left(\frac{\partial}{\partial |\lambda_i|} \right)^j E_{\alpha,2}(|\lambda_i| t_0^{\alpha}) + O(|\lambda_i|^{-1-p-j} t^{-\alpha p}) \\
&\quad + \sum_{k=2}^{p} \frac{(k+j-1)! \lambda_i^{-k-j}}{j!(k-1)!|\Gamma(1-\alpha k)|} \left(\frac{t^{-\alpha k+\alpha}}{-\alpha k+\alpha} - \frac{t_0^{-\alpha k+\alpha}}{-\alpha k+\alpha} \right) \\
&\quad \to t_0^{\alpha} \frac{1}{j!} \left(\frac{\partial}{\partial |\lambda_i|} \right)^j E_{\alpha,2}(|\lambda_i| t_0^{\alpha}) + \sum_{k=2}^{p} \frac{(k+j-1)! |\lambda_i|^{-k-j} t_0^{-\alpha k+\alpha}}{j!(k-1)!|\Gamma(2-\alpha k)|} \leqslant C_0 \text{ as } t \to +\infty,
\end{aligned}
$$

where $1 \leqslant j \leqslant n_i - 1$. Thus, $\int_0^t \left\| s^{\alpha-1} E_{\alpha}(A s^{\alpha}) \right\| \mathrm{d}s$ is bounded.

Equation (18) and Equation (23) imply that there exists a constant C such that

$$
\|\mu(t) - \nu(t)\| \leqslant C\varepsilon. \tag{26}
$$

So the system (16) is Ulam-Hyers stable. The proof is completed. $\square$

THEOREM 2. *Consider the initial valued problem*

$$
\begin{cases} D^{\alpha}\mu(t) = A\mu(t) + f(t), \\ \mu(0) = C, \end{cases} \tag{27}
$$

where $A \in M_n$, $C \in M_{n,m}$ and $f(t)$, $\mu(t) \in M_{n,m}$ are matrix-valued functions. If the eigenvalues of A satisfy

$$
|arg(\lambda(A))| > \frac{\alpha\pi}{2}, \tag{28}
$$

then the system (27) is Ulam-Hyers stable.

Proof. By using the $Vec(\cdot)$-notation in linear algebra, it is easy to obtain the solution of the system

$$
\mu(t) = E_{\alpha}(At^{\alpha})C + \int_0^t (t-s)^{\alpha-1} E_{\alpha}(A(t-s)^{\alpha}) f(s) \mathrm{d}s. \tag{29}
$$

Consider the inequation

$$
\|\mu^{\alpha}(t) - A\mu(t) - f(t)\| \leqslant \varepsilon. \tag{30}
$$

A function $\nu \in M_{n,m}$ is a solution of (30) if and only if there exists a function $g \in M_{n,1}$ (which depend on ν) such that

(i) $\|g(t)\| \leqslant \varepsilon, \ \forall t \in [0,b)$

(ii)

$$
D^{\alpha}\nu(t) = A\nu(t) + f(t) + g(t), \ \forall t \in [0,b). \tag{31}
$$

Then, ν is a solution of the following integral inequation

$$
\left\| \nu(t) - E_{\alpha}(At^{\alpha})C - \int_0^t (t-s)^{\alpha-1} E_{\alpha}(A(t-s)^{\alpha}) f(s) \mathrm{d}s \right\| \leqslant C_1 \varepsilon, \tag{32}
$$

where C_1 is a constant.

In fact, by (29) we have

$$\nu(t)=E_\alpha(At^\alpha)C+\int_0^t (t-s)^{\alpha-1}E_\alpha(A(t-s)^\alpha)(f(s)+g(s))\mathrm{d}s. \tag{33}$$

In view of the proof of Theorem 1, we get

$$\begin{aligned}
&\left\|\nu(t)-E_\alpha(At^\alpha)C-\int_0^t (t-s)^{\alpha-1}E_\alpha(A(t-s)^\alpha)f(s)\mathrm{d}s\right\| \\
&=\left\|\int_0^t (t-s)^{\alpha-1}E_\alpha(A(t-s)^\alpha)g(s)\mathrm{d}s\right\|=\left\|\int_0^t s^{\alpha-1}E_\alpha(As^\alpha)g(t-s)\mathrm{d}s\right\| \\
&\leqslant \varepsilon\int_0^t \left\|s^{\alpha-1}E_\alpha(As^\alpha)\right\|\mathrm{d}s\leqslant C_1\varepsilon.
\end{aligned} \tag{34}$$

Thus,

$$\|\mu(t)-\nu(t)\|\leqslant C_1\varepsilon. \tag{35}$$

Then, the system (27) is Ulam-Hyers stable. The proof is completed. □

THEOREM 3. *Consider the following matrix fractional differential equation*

$$D^\alpha\mu(t)=A\mu(t)+\mu(t)B+f(t),\ \ \mu(0)=C,\ \ t\in(0,b), \tag{36}$$

where $A\in M_n$, $B\in M_m$, $C\in M_{n,m}$, $\alpha\in(0,1)$, *and* $f(t),\mu(t)\in M_{n,m}$ *are both matrix-valued functions. Assume that* $b<+\infty$ *and that all the eigenvalues of* A *and* B *satisfy*

$$|arg(\lambda(A))|>\frac{\alpha\pi}{2},\ \pi\geqslant|arg(\lambda(B))|\geqslant k\ (\alpha\pi/2<k<\min\{\pi,\pi\alpha\}). \tag{37}$$

The system (36) *is Ulam-Hyers stable.*

Proof. It is easy to vertify that the general solution of the system (36) is given as

$$\mu(t)=E_\alpha(At^\alpha)CE_\alpha(Bt^\alpha)+\int_0^t (t-s)^{\alpha-1}E_\alpha(A(t-s)^\alpha)f(s)E_\alpha(B(t-s)^\alpha)\mathrm{d}s. \tag{38}$$

Consider the following inequation

$$\|\mu^\alpha(t)-A\mu(t)-\mu(t)B-f(t)\|\leqslant\varepsilon. \tag{39}$$

A function $\nu\in M_{n,m}$ is a solution of (39) if and only if there exists a function $g\in M_{n,1}$ (which depend on ν) such that

(i) $\|g(t)\|\leqslant\varepsilon,\ \forall t\in(0,b)$

(ii)

$$\nu^\alpha(t)=A\nu(t)+\nu(t)B+f(t)+g(t),\ \forall t\in(0,b). \tag{40}$$

Then, ν is a solution of the following integral inequation

$$\left\|\nu(t)-E_\alpha(At^\alpha)CE_\alpha(Bt^\alpha)+\int_0^t (t-s)^{\alpha-1}E_\alpha(A(t-s)^\alpha)f(s)E_\alpha(B(t-s)^\alpha)\mathrm{d}s\right\|\leqslant C_2\varepsilon, \tag{41}$$

where C_2 is a constant. In fact, by (38) we have

$$\nu(t) = E_\alpha(At^\alpha)CE_\alpha(Bt^\alpha) + \int_0^t (t-s)^{\alpha-1}E_\alpha(A(t-s)^\alpha)(f(s)+g(s))E_\alpha(B(t-s)^\alpha)\mathrm{d}s.$$

Then, by Lemma 2 we get

$$\begin{aligned}
&\left\|\nu(t) - E_\alpha(At^\alpha)CE_\alpha(Bt^\alpha) - \int_0^t (t-s)^{\alpha-1}E_\alpha(A(t-s)^\alpha)f(s)E_\alpha(B(t-s)^\alpha)\mathrm{d}s\right\| \\
&= \left\|\int_0^t (t-s)^{\alpha-1}E_\alpha(A(t-s)^\alpha)g(s)E_\alpha(B(t-s)^\alpha)\mathrm{d}s\right\| \\
&= \left\|\int_0^t s^{\alpha-1}E_\alpha(As^\alpha)g(t-s)E_\alpha(Bs^\alpha)\mathrm{d}s\right\| \\
&\leqslant \int_0^t \left\|s^{\alpha-1}E_\alpha(As^\alpha)g(t-s)E_\alpha(Bs^\alpha)\right\|\mathrm{d}s \leqslant \varepsilon\int_0^t \left\|s^{\alpha-1}E_\alpha(As^\alpha)E_\alpha(Bs^\alpha)\right\|\mathrm{d}s \\
&\leqslant \varepsilon\int_0^t \left\|s^{\alpha-1}E_\alpha(As^\alpha)\right\|\left\|E_\alpha(Bs^\alpha)\right\|\mathrm{d}s \leqslant C_1\varepsilon\int_0^t \|E_\alpha(Bs^\alpha)\|\mathrm{d}s \leqslant \frac{C_1C_2}{1+\|B\|}t.
\end{aligned} \tag{42}$$

Thus, if $b < +\infty$

$$\|\mu(t) - \nu(t)\| \leqslant C_3\varepsilon. \tag{43}$$

Then, the system (36) is Ulam-Hyers stable. And if $b = +\infty$, the system (36) is not Ulam-Hyers stable. The proof is completed. □

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Zhanpeng Yang
Key Laboratory of Technology in Geo-spatial Information Processing and Application System
Institute of Electronics, Chinese Academy of Sciences
Beijing 100190, P. R. China
e-mail: zhanpengyang@mail.ie.ac.cn

Wenjuan Ren
Key Laboratory of Technology in Geo-spatial Information Processing and Application System
Institute of Electronics, Chinese Academy of Sciences
Beijing 100190, P. R. China
e-mail: wjren2011@mail.ie.ac.cn

Tianzhou Xu
School of Mathematics and Statistics
Beijing Institute of Technology
Beijing 100081, P. R. China
e-mail: xutianzhou@bit.edu.cn