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INEQUALITIES OF THE JENSEN AND EDMUNDSON–LAH–RIBARIČ TYPE FOR 3–CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper we derive some Jensen and Edmundson-Lah-Ribarič type inequalities for positive linear functionals and 3-convex functions. Obtained results are then applied to generalized means and power means, as well as to the generalized f-divergence functional. Examples with Zipf-Mandelbrot law are given.

1. Introduction

The Jensen inequality is perhaps the most important, and certainly the most famous inequality in modern mathematics, and it has many applications in various branches of mathematics.

In this paper we refer to a general form of the Jensen inequality for positive linear functionals. In order to present our result, we first need to introduce the appropriate setting.

Let E be a nonempty set and let L be a vector space of real-valued functions $f\colon E\to\mathbb{R}$ having the properties:

(L1) $f,g\in L\Rightarrow(af+bg)\in L$ for all $a,b\in\mathbb{R}$;

(L2) $\mathbf{1}\in L$, i.e., if $f(t)=1$ for every $t\in E$, then $f\in L$.

We also consider positive linear functionals $A\colon L\to\mathbb{R}$. That is, we assume that:

(A1) $A(af+bg)=aA(f)+bA(g)$ for $f,g\in L$ and $a,b\in\mathbb{R}$;

(A2) $f\in L,\ f(t)\geqslant 0$ for every $t\in E\Rightarrow A(f)\geqslant 0$ (A is positive).

Since it was proved, the famous Jensen inequality and its converses have been extensively studied by many authors and have been generalized in numerous directions. Jessen [17] gave the following generalization of Jensen's inequality for convex functions (see also [25, p. 47]):

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THEOREM 1. ([17]) *Let L satisfy properties (L1) and (L2) on a nonempty set E, and assume that ϕ is a continous convex function on an interval $I \subset \mathbb{R}$. If A is a positive linear functional with $A(1) = 1$, then for all $f \in L$ such that $\phi(f) \in L$ we have $A(f) \in I$ and*

$$\phi(A(f)) \leqslant A(\phi(f)). \tag{1}$$

The following result is one of the most famous converses of the Jensen inequality known as the Edmundson-Lah-Ribarič inequality, and it was proved in [2] by Beesack and Pečarić (see also [25, p. 98]):

THEOREM 2. ([2]) *Let ϕ be convex on the interval $I = [m,M]$ such that $-\infty < m < M < \infty$. Let L satisfy conditions (L1) and (L2) on E and let A be any positive linear functional on L with $A(1) = 1$. Then for every $f \in L$ such that $\phi(f) \in L$ (so that $m \leqslant f(t) \leqslant M$ for all $t \in E$), we have*

$$A(\phi(f)) \leqslant \frac{M - A(f)}{M - m}\phi(m) + \frac{A(f) - m}{M - m}\phi(M). \tag{2}$$

For some recent results on the converses of the Jensen inequality, the reader is referred to [6], [12], [13], [14], [15], [16], [19], [20] and [26].

Unlike the results from the above mentioned papers, which require convexity of the involved functions, the main objective of this paper is to derive a class of inequalities of the Jensen and Edmundson-Lah-Ribarič type that hold for 3-convex functions.

Definition of the n-convex function is characterized by nth-order divided difference. The nth-order divided difference of a function $f\colon [a,b] \to \mathbb{R}$ at mutually distinct points $t_0, t_1, \ldots, t_n \in [a,b]$ is defined recursively by

$$[t_i]f = f(t_i), \quad i = 0, \ldots, n,$$
$$[t_0, \ldots, t_n]f = \frac{[t_1, \ldots, t_n]f - [t_0, \ldots, t_{n-1}]f}{t_n - t_0}.$$

The value $[t_0, \ldots, t_n]f$ is independent of the order of the points $t_0, \ldots, t_n$. This definition may be extended to include the case in which some or all the points coincide (see [25, p. 14]).

A function $f\colon [a,b] \to \mathbb{R}$ is said to be n-convex ($n \geqslant 0$) if and only if for all choices of $(n+1)$ distinct points $t_0, t_1, \ldots, t_n \in [a,b]$, we have $[t_0, \ldots, t_n]f \geqslant 0$.

This paper is organized in the following manner: main results, that are inequalities of the Jensen and Edmundson-Lah-Ribarič type for 3-convex functions, are given in Section 2; application of the main results to the generalized means, with examples to the power means, are given in Section 3; application of the main results to the generalized f-divergence functional is given in Section 4, and finally in section 5 the results from the previous section are applied to the Zipf-Mandelbrot law.

2. Results

Throughout this paper, whenever mentioning the interval $[m,M]$, we assume that $-\infty < m < M < \infty$ holds.

THEOREM 3. *Let L satisfy conditions (L1) and (L2) on a non-empty set E and let A be any positive linear functional on L with $A(\mathbf{1}) = 1$. Let ϕ be a 3-convex function on an interval of real numbers I whose interior contains the interval $[m,M]$. Then*

$$\begin{aligned}
&\frac{A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right]}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m}-\phi'_+(m)\right)\\
\leqslant&\frac{M-A(f)}{M-m}\phi(m)+\frac{A(f)-m}{M-m}\phi(M)-A\left(\phi(f)\right)\\
\leqslant&\frac{A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right]}{M-m}\left(\phi'_-(M)-\frac{\phi(M)-\phi(m)}{M-m}\right)
\end{aligned}\tag{3}$$

holds for any $f \in L$ such that $\phi \circ f \in L$ and $m \leqslant f(t) \leqslant M$ for $t \in E$. If the function $-\phi$ is 3-convex, then the inequalities are reversed.

Proof. We start with a scalar identity for $t \in [m,M]$:

$$\begin{aligned}
&\frac{M-t}{M-m}\phi(m)+\frac{t-m}{M-m}\phi(M)-\phi(t)\\
=&\frac{M-t}{M-m}\left(\phi(m)-\phi(t)\right)+\frac{t-m}{M-m}\left(\phi(M)-\phi(t)\right)\\
=&\frac{(M-t)(t-m)}{M-m}\left(\frac{\phi(M)-\phi(t)}{M-t}-\frac{\phi(t)-\phi(m)}{t-m}\right)\\
=&(M-t)(t-m)[m,t,M]\phi.
\end{aligned}$$

It follows that

$$\frac{M-t}{M-m}\phi(m)+\frac{t-m}{M-m}\phi(M)-\phi(t)=(M-t)(t-m)[m,t,M]\phi \tag{4}$$

holds for every $t \in [m,M]$.

Since the function ϕ is 3-convex, we have $[t_0,t_1,t_2,t_3]\phi \geqslant 0$ for every choice of the points $t_0,t_1,t_2,t_3 \in [m,M]$. Let $t_0 = m$, $t_3 = M$ and $t_1 < t_2$. From the definition and main properties of the divided differences we get the following relation:

$$\begin{aligned}
0 \leqslant [m,t_1,t_2,M]\phi = [t_1,m,M,t_2]\phi &= \frac{[m,M,t_2]\phi-[t_1,m,M]\phi}{t_2-t_1}\\
&=\frac{[m,t_2,M]\phi-[m,t_1,M]\phi}{t_2-t_1},
\end{aligned}$$

so we have obtained that

$$[m,t_2,M]\phi-[m,t_1,M]\phi \geqslant 0$$

holds for any $t_1 < t_2$, that is, the function $[m,t,M]\phi$ is non-decreasing on $[m,M]$. It follows that the function $[m,t,M]\phi$ attains its minimal and maximal value in the points m and M respectively. We can calculate those bounds:

$$
\begin{aligned}
[m,m,M]\phi &= \frac{1}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m} - \phi'_+(m)\right)\\
[m,M,M]\phi &= \frac{1}{M-m}\left(\phi'_-(M) - \frac{\phi(M)-\phi(m)}{M-m}\right)
\end{aligned} \tag{5}
$$

Now, from (4) and (5) we have

$$
\begin{aligned}
&\frac{(M-t)(t-m)}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m} - \phi'_+(m)\right)\\
&\leqslant \frac{M-t}{M-m}\phi(m) + \frac{t-m}{M-m}\phi(M) - \phi(t)\\
&\leqslant \frac{(M-t)(t-m)}{M-m}\left(\phi'_-(M) - \frac{\phi(M)-\phi(m)}{M-m}\right)
\end{aligned} \tag{6}
$$

for any $t \in [m,M]$. The function f satisfies the bounds

$$
m \leqslant f(t) \leqslant M,
$$

so we can replace t with $f(t)$ in (6) and obtain:

$$
\begin{aligned}
&\frac{(M-f(t))(f(t)-m)}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m} - \phi'_+(m)\right)\\
\leqslant &\frac{M-f(t)}{M-m}\phi(m) + \frac{f(t)-m}{M-m}\phi(M) - \phi(f(t))\\
\leqslant &\frac{(M-f(t))(f(t)-m)}{M-m}\left(\phi'_-(M) - \frac{\phi(M)-\phi(m)}{M-m}\right).
\end{aligned}
$$

Functional A is linear and positive, and such that $A(\mathbf{1}) = 1$, so when we apply it to the previous inequalities we get the following:

$$
\begin{aligned}
&\frac{A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right]}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m} - \phi'_+(m)\right)\\
\leqslant &\frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M) - A(\phi(f))\\
\leqslant &\frac{A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right]}{M-m}\left(\phi'_-(M) - \frac{\phi(M)-\phi(m)}{M-m}\right),
\end{aligned}
$$

which concludes the proof. □

Theorem 3 can be utilized for obtaining Jensen-type inequalities for 3-convex functions.

THEOREM 4. *Let L satisfy conditions (L1) and (L2) on a non-empty set E and let A be any positive linear functional on L with $A(\mathbf{1})=1$. Let ϕ be a 3-convex function on an interval of real numbers I whose interior contains the interval $[m,M]$. Then*

$$\begin{aligned}
&\frac{(M-A(f))(A(f)-m)}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m}-\phi_+'(m)\right)\\
&\quad-\frac{A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right]}{M-m}\left(\phi_-'(M)-\frac{\phi(M)-\phi(m)}{M-m}\right)\\
&\leqslant A\left(\phi(f)\right)-\phi(A(f))\leqslant\frac{(M-A(f))(A(f)-m)}{M-m}\left(\phi_-'(M)-\frac{\phi(M)-\phi(m)}{M-m}\right)\\
&\quad-\frac{A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right]}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m}-\phi_+'(m)\right)
\end{aligned}\tag{7}$$

holds for any $f\in L$ such that $\phi\circ f\in L$ and $m\leqslant f(t)\leqslant M$ for $t\in E$. If the function $-\phi$ is 3-convex, then the inequalities are reversed.

Proof. Function $\phi\circ f$ belongs to L, which means that the function f satisfies the bounds $m\leqslant f(t)\leqslant M$. It follows that $m\leqslant A(f)\leqslant M$, so we can replace t with $A(f)$ in the relation (6) and obtain

$$\begin{aligned}
&\frac{(M-A(f))(A(f)-m)}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m}-\phi_+'(m)\right)\\
\leqslant&\frac{M-A(f)}{M-m}\phi(m)+\frac{A(f)-m}{M-m}\phi(M)-\phi(A(f))\\
\leqslant&\frac{(M-A(f))(A(f)-m)}{M-m}\left(\phi_-'(M)-\frac{\phi(M)-\phi(m)}{M-m}\right).
\end{aligned}\tag{8}$$

When we multiply the relation (3) from Theorem 3 by -1 we get

$$\begin{aligned}
&-\frac{A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right]}{M-m}\left(\phi_-'(M)-\frac{\phi(M)-\phi(m)}{M-m}\right)\\
\leqslant&-\frac{M-A(f)}{M-m}\phi(m)-\frac{A(f)-m}{M-m}\phi(M)+A\left(\phi(f)\right)\\
\leqslant&-\frac{A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right]}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m}-\phi_+'(m)\right).
\end{aligned}\tag{9}$$

Inequalities (7) follow by adding (8) to (9). □

3. Applications to generalized means

Let $I=\langle a,b\rangle$, $-\infty\leqslant a<b\leqslant\infty$, and let $\psi\colon I\to\mathbb{R}$ be continuous and strictly monotonic. Suppose that L and A satisfy the conditions $L1$, $L2$ and $A1$, $A2$ with $A(\mathbf{1})=1$ on a non-empty set E, and that $\psi(f)\in L$ for some $f\in L$. Generalized mean for $f\in L$ with respect to the operator A and the function ψ is defined by

$$M_\psi(f,A)=\psi^{-1}\left(A(\psi(f))\right).\tag{10}$$

Some inequalities regarding the generalized mean and its special cases can be found in [1], [4], [16], [19] and [21].

The following results give us inequalities of the Edmundson-Lah-Ribarič and Jensen type respectively for the generalized means.

THEOREM 5. *Let $I \subset \mathbb{R}$ be such that its interior contains the interval $[m,M]$, and let $\psi,\chi : I \to \mathbb{R}$ be continuous and strictly monotonic. Suppose that L and A satisfy the conditions $L1$, $L2$ and $A1$, $A2$ with $A(\mathbf{1}) = 1$ on a non-empty set E, and let $f \in L$ be such that $\psi(f), \chi(f) \in L$. Let us assume that the function $\phi = \chi \circ \psi^{-1}$ is 3-convex. Then*

$$\begin{aligned}
&\frac{A\left([M_\psi \mathbf{1} - \psi(f)][\psi(f) - m_\psi \mathbf{1}]\right)}{M_\psi - m_\psi}\left(\frac{\chi(M)-\chi(m)}{\psi(M)-\psi(m)} - [\chi\circ\psi^{-1}]'_+(m_\psi)\right)\\
\leqslant&\frac{\psi(M) - A(\psi(f))}{\psi(M)-\psi(m)}\chi(m) + \frac{A(\psi(f)) - \psi(m)}{\psi(M)-\psi(m)}\chi(M) - \chi(M_\chi(f,A)) \qquad (11)\\
\leqslant&\frac{A\left([M_\psi \mathbf{1} - \psi(f)][\psi(f) - m_\psi \mathbf{1}]\right)}{M_\psi - m_\psi}\left([\chi\circ\psi^{-1}]'_-(M_\psi) - \frac{\chi(M)-\chi(m)}{\psi(M)-\psi(m)}\right)
\end{aligned}$$

for every $f \in L$ such that $m \leqslant f(t) \leqslant M$ for $t \in [m,M]$, where $[m_\psi, M_\psi] = \psi([m,M])$. If $-\phi$ is 3-convex, then the inequalities in (11) are reversed.

Proof. Function ψ is strictly monotonic. If ψ is increasing, then $m_\psi = \psi(m)$ and $M_\psi = \psi(M)$, and if ψ is decreasing, then $m_\psi = \psi(M)$ and $M_\psi = \psi(m)$. Since $m \leqslant f(t) \leqslant M$ for $t \in [m,M]$, we have $m_\psi \leqslant \psi(f(t)) \leqslant M_\psi$ for every $t \in [m,M]$. We see that the conditions of Theorem 3 are satisfied, so we can obtain (11) by making substitutions

$$m = m_\psi,\ M = M_\psi,\ \phi = \chi\circ\psi^{-1} \text{ and } f = \psi \circ f$$

in (3). $\square$

THEOREM 6. *Let $I \subset \mathbb{R}$ be such that its interior contains the interval $[m,M]$, and let $\psi,\chi : I \to \mathbb{R}$ be continuous and strictly monotonic. Suppose that L and A satisfy the conditions $L1$, $L2$ and $A1$, $A2$ with $A(\mathbf{1}) = 1$ on a non-empty set E, and let $f \in L$ be such that $\psi(f), \chi(f) \in L$. Let us assume that the function $\phi = \chi \circ \psi^{-1}$ is 3-convex. Then*

$$\begin{aligned}
&\frac{(M_\psi - A(\psi(f)))(A(\psi(f)) - m_\psi)}{M_\psi - m_\psi}\left(\frac{\chi(M)-\chi(m)}{\psi(M)-\psi(m)} - [\chi\circ\psi^{-1}]'_+(m_\psi)\right)\\
&\quad - \frac{A\left[(M_\psi \mathbf{1} - \psi(f))(\psi(f) - m_\psi \mathbf{1})\right]}{M_\psi - m_\psi}\left([\chi\circ\psi^{-1}]'_-(M_\psi) - \frac{\chi(M)-\chi(m)}{\psi(M)-\psi(m)}\right)\\
\leqslant&\chi(M_\chi(f,A)) - \chi(M_\psi(f,A)) \qquad (12)\\
\leqslant&\frac{(M_\psi - A(\psi(f)))(A(\psi(f)) - m_\psi)}{M_\psi - m_\psi}\left([\chi\circ\psi^{-1}]'_-(M_\psi) - \frac{\chi(M)-\chi(m)}{\psi(M)-\psi(m)}\right)\\
&\quad - \frac{A\left([M_\psi \mathbf{1} - \psi(f)][\psi(f) - m_\psi \mathbf{1}]\right)}{M_\psi - m_\psi}\left(\frac{\chi(M)-\chi(m)}{\psi(M)-\psi(m)} - [\chi\circ\psi^{-1}]'_+(m_\psi)\right)
\end{aligned}$$

for every $f \in L$ such that $m \leqslant f(t) \leqslant M$ for $t \in [m,M]$, where $[m_\psi, M_\psi] = \psi([m,M])$. If $-\phi$ is 3-convex, then the inequalities in (12) are reversed.

Proof. The inequalities (12) are obtained by making the same substitutions in the relation (7) from Theorem 4 as in the proof of the previous theorem. □

REMARK 1. With notations as in Theorems 5 and 6, suppose that the function $\chi \circ \psi^{-1}$ is differentiable in points ψ_m and ψ_M. In this case, expressions ψ_m and ψ_M can respectively be replaced by $\psi(m)$ and $\psi(M)$, due to the symmetry. In addition, utilizing the chain rule, the expressions

$$[\chi \circ \psi^{-1}]'_-(\psi(M)) \text{ and } [\chi \circ \psi^{-1}]'_+(\psi(m))$$

can be rewritten in a more suitable form, that is,

$$[\chi \circ \psi^{-1}]'_-(\psi(M)) = \frac{\chi'(M)}{\psi'(M)} \text{ and } (\chi \circ \psi^{-1})'_+(\psi(m)) = \frac{\chi'(m)}{\psi'(m)}.$$

Examples with power means

Suppose that L and A satisfy the conditions $L1$, $L2$ and $A1$, $A2$ with $A(\mathbf{1}) = 1$, on a non-empty set E. The power mean of a function $f \in L$ with respect to the operator A is a special case of the generalized mean, and it is defined for $r \in \mathbb{R}$ with:

$$M^{[r]}(f,A) = \begin{cases} (A(f^r))^{1/r} & : r \neq 0 \\ \exp(A(\log f)) & : r = 0 \end{cases} \tag{13}$$

where $f(t) > 0$ for $t \in E$, $f^r \in L$ and $\log f \in L$.

Following two results are simple consequences of Theorem 5 and Theorem 6, that is, the series of inequalities in (11) and (12) with particular choices of functions χ and ψ respectively. The first result is a Edmundson-Lah-Ribarič type inequality for power means.

COROLLARY 1. *Let $I \subset \mathbb{R}$ be such that its interior contains the interval $[m,M]$. Suppose that L and A satisfy the conditions $L1$, $L2$ and $A1$, $A2$ with $A(\mathbf{1}) = 1$ on a non-empty set E, and let $f \in L$ be such that $0 < m \leqslant f(t) \leqslant M$ for $t \in E$, $f^r, f^s \in L$ for $r,s \in \mathbb{R}$ and $\log f \in L$.*

- *If any of the relations $0 \leqslant s \leqslant r$ or $0 \leqslant 2r \leqslant s$ or $r < 0 < s$ or $2r < s < r < 0$ hold, then*

$$\begin{aligned} &\frac{A([M^r\mathbf{1} - f^r][f^r - m^r\mathbf{1}])}{M^r - m^r}\left(\frac{M^s - m^s}{M^r - m^r} - m^{s-r}\right) \\ &\leqslant \frac{M^r - M^{[r]}(f,A)^r}{M^r - m^r} m^s + \frac{M^{[r]}(f,A)^r - m^r}{M^r - m^r} M^s - M^{[s]}(f,A)^s \\ &\leqslant \frac{A\left([M_\psi\mathbf{1} - \psi(f)][\psi(f) - m_\psi\mathbf{1}]\right)}{M^r - m^r}\left(M^{s-r} - \frac{M^s - m^s}{M^r - m^r}\right). \end{aligned} \tag{14}$$

If $r \leqslant s \leqslant 0$ or $s \leqslant 2r \leqslant 0$ or $s < 0 < r$ or $0 < r < s < 2r$, then the inequalities in (14) are reversed.

- *If $r \neq 0$, then*

$$\begin{aligned}
&\frac{A\left([M^r\mathbf{1} - f^r][f^r - m^r\mathbf{1}]\right)}{M^r - m^r}\left(\frac{\log M - \log m}{M^r - m^r} - \frac{1}{rm^r}\right)\\
\leqslant&\frac{M^r - M^{[r]}(f,A)^r}{M^r - m^r}\log m + \frac{M^{[r]}(f,A)^r - m^r}{M^r - m^r}\log M - \log[M^{[0]}(f,A)] \qquad (15)\\
\leqslant&\frac{A\left([M^r\mathbf{1} - f^r][f^r - m^r\mathbf{1}]\right)}{M^r - m^r}\left(\frac{1}{rM^r} - \frac{\log M - \log m}{M^r - m^r}\right).
\end{aligned}$$

- *If $s > 0$, then*

$$\begin{aligned}
&\frac{A\left([\log M\mathbf{1} - \log f][\log f - \log m\mathbf{1}]\right)}{\log M - \log m}\left(\frac{M^s - m^s}{\log M - \log m} - sm^s\right)\\
\leqslant&\frac{\log M - \log[M^{[0]}(f,A)]}{\log M - \log m}m^s + \frac{\log[M^{[0]}(f,A)] - \log m}{\log M - \log m}M^s - M^{[s]}(f,A)^s \qquad (16)\\
\leqslant&\frac{A\left([\log M\mathbf{1} - \log f][\log f - \log m\mathbf{1}]\right)}{\log M - \log m}\left(sM^s - \frac{M^s - m^s}{\log M - \log m}\right),
\end{aligned}$$

and if $s < 0$, the inequality signs in (16) are reversed.

Proof. Let us set $\chi(t) = t^s$ and $\psi(t) = t^r$, where s and r are mutually different real parameters not equal to zero. Then the function $\left(\chi \circ \psi^{-1}\right)(t) = t^{\frac{s}{r}}$ is 3-convex on $\mathbb{R}_+$ if $0 \leqslant \frac{s}{r} \leqslant 1$ or $\frac{s}{r} \geqslant 2$. It is possible in each of the following four cases: $0 \leqslant s \leqslant r$ or $r \leqslant s \leqslant 0$ or $0 \leqslant 2r \leqslant s$ or $s \leqslant 2r \leqslant 0$. We calculate $\left(\chi \circ \psi^{-1}\right)'(t) = \frac{s}{r}t^{\frac{s-r}{r}}$. Since the function $\psi(t) = t^r$ is increasing for $r > 0$ we have $m_\psi = \psi(m)$ and $M_\psi = \psi(M)$. Now, considering (11) with the above functions χ and ψ on the interval $[m, M]$, we obtain (14). For $r < 0$ the function $\psi(t) = t^r$ is decreasing, which means that $m_\psi = \psi(M)$ and $M_\psi = \psi(m)$, so those inequalities are reversed.

On the other hand, the function $-\left(\chi \circ \psi^{-1}\right)(t) = -t^{\frac{s}{r}}$ is 3-convex on $\mathbb{R}_+$ if $0 \leqslant \frac{s}{r} < 0$ or $1 < \frac{s}{r} < 2$, which is possible in any of the following cases: $r < 0 < s$ or $s < 0 < r$ or $0 < r < s < 2r$ or $2r < s < r < 0$. Again, if $r > 0$ the function $\psi(t) = t^r$ is increasing, so we get the inequalities (14) with the reversed sign of inequality by setting $\chi(t) = t^s$ and $\psi(t) = t^r$ in the reversed inequalities (11), and if $r < 0$, we get exactly inequalities (14).

It remains to consider the cases when one of the parameters r and s is equal to zero. If $s = 0$, then setting $\chi(t) = \log t$ and $\psi(t) = t^r$, it follows that $\left(\chi \circ \psi^{-1}\right)(t) = \frac{1}{r}\log t$. Clearly, this function is 3-convex for $r > 0$, while $-\chi \circ \psi^{-1}$ is 3-convex for $r < 0$. Moreover, since $\left(\chi \circ \psi^{-1}\right)'(t) = \frac{1}{rt}$, after a straightforward computation and taking into account that the function $\psi(t) = t^r$ is increasing for $r > 0$ and decreasing for $r < 0$, we obtain (15).

Finally, if $r=0$, then setting $\chi(t)=t^s$ and $\psi(t)=\log t$, it follows that the function $\left(\chi\circ\psi^{-1}\right)(t)=\exp(st)$ is 3-convex for $s>0$. The function $\psi(t)=\log t$ is increasing, so after calculating $\left(\chi\circ\psi^{-1}\right)'(t)=s\exp(st)$, from (11) we get (16). $\square$

Next result is a Jensen type inequality for power means, and it is obtained from Theorem 6 in an analogue way as described in the proof of the previous corollary.

COROLLARY 2. *Let $I\subset\mathbb{R}$ be such that its interior contains the interval $[m,M]$. Suppose that L and A satisfy the conditions $L1$, $L2$ and $A1$, $A2$ with $A(\mathbf{1})=1$ on a non-empty set E, and let $f\in L$ be such that $0<m\leqslant f(t)\leqslant M$ for $t\in E$, $f^r,f^s\in L$ for $r,s\in\mathbb{R}$ and $\log f\in L$.*

- *If any of the relations $0\leqslant s\leqslant r$ or $0\leqslant 2r\leqslant s$ or $r<0<s$ or $2r<s<r<0$ hold, then*

$$\begin{aligned}
&\frac{(M^r-M^{[r]}(f,A)^r)(M^{[r]}(f,A)^r-m^r)}{M^r-m^r}\left(\frac{M^s-m^s}{M^r-m^r}-\frac{s}{r}m^{s-r}\right)\\
&\quad-\frac{A\left[(M^r\mathbf{1}-f^r)(f^r-m^r\mathbf{1})\right]}{M^r-m^r}\left(\frac{s}{r}M^{s-r}-\frac{M^s-m^s}{M^r-m^r}\right)\\
&\leqslant M^{[s]}(f,A)^s-M^{[r]}(f,A)^s \qquad (17)\\
&\leqslant\frac{(M^r-M^{[r]}(f,A)^r)(M^{[r]}(f,A)^r-m^r)}{M^r-m^r}\left(\frac{s}{r}M^{s-r}-\frac{M^s-m^s}{M^r-m^r}\right)\\
&\quad-\frac{A\left[(M^r\mathbf{1}-f^r)(f^r-m^r\mathbf{1})\right]}{M^r-m^r}\left(\frac{M^s-m^s}{M^r-m^r}-\frac{s}{r}m^{s-r}\right)
\end{aligned}$$

  If $r\leqslant s\leqslant 0$ or $s\leqslant 2r\leqslant 0$ or $s<0<r$ or $0<r<s<2r$, then the inequalities in (17) are reversed.

- *If $r\neq 0$, then*

$$\begin{aligned}
&\frac{(M^r-M^{[r]}(f,A)^r)(M^{[r]}(f,A)^r-m^r)}{M^r-m^r}\left(\frac{\log M-\log m}{M^r-m^r}-\frac{1}{rm^r}\right)\\
&\quad-\frac{A\left([M^r\mathbf{1}-f^r][f^r-m^r\mathbf{1}]\right)}{M^r-m^r}\left(\frac{1}{rM^r}-\frac{\log M-\log m}{M^r-m^r}\right)\\
&\leqslant\log[M^{[0]}(f,A)]-\log[M^{[r]}(f,A)] \qquad (18)\\
&\leqslant\frac{(M^r-M^{[r]}(f,A)^r)(M^{[r]}(f,A)^r-m^r)}{M^r-m^r}\left(\frac{1}{rM^r}-\frac{\log M-\log m}{M^r-m^r}\right)\\
&\quad-\frac{A\left([M^r\mathbf{1}-f^r][f^r-m^r\mathbf{1}]\right)}{M^r-m^r}\left(\frac{\log M-\log m}{M^r-m^r}-\frac{1}{rm^r}\right).
\end{aligned}$$

- *If $s>0$, then*

$$\begin{aligned}
&\frac{(\log M-\log[M^{[0]}(f,A))(\log[M^{[0]}(f,A)-\log m)}{\log M-\log m}\left(\frac{M^s-m^s}{\log M-\log m}-sm^s\right)\\
&\quad-\frac{A\left([\log M\mathbf{1}-\log f][\log f-\log m\mathbf{1}]\right)}{\log M-\log m}\left(sM^s-\frac{M^s-m^s}{\log M-\log m}\right)
\end{aligned}$$

$$\begin{aligned}
&\leqslant M^{[s]}(f,A)^s - M^{[0]}(f,A)^s \qquad (19)\\
&\leqslant \frac{(\log M - \log[M^{[0]}(f,A))(\log[M^{[0]}(f,A) - \log m)}{\log M - \log m}\left(sM^s - \frac{M^s - m^s}{\log M - \log m}\right)\\
&\quad - \frac{A\left([\log M\mathbf{1} - \log f][\log f - \log m\mathbf{1}]\right)}{\log M - \log m}\left(\frac{M^s - m^s}{\log M - \log m} - sm^s\right)
\end{aligned}$$

and if $s < 0$, the inequality signs in (19) are reversed.

4. Applications to Csiszár f-divergence

Let us denote the set of all probability distributions by $\mathbb{P}$, that is we say $\boldsymbol{p} = (p_1,\dots,p_n) \in \mathbb{P}$ if $p_i \in [0,1]$ for $i = 1,\dots,n$ and $\sum_{i=1}^n p_i = 1$.

Many theoretic divergence measures between two probability distributions (such as Kullback-Leibler divergence, Hellinger divergence, Renyi divergence, Bhattacharyya divergence, harmonic divergence, Jeffreys divergence, triangular divergence etc.) have been introduced and extensively studied.

The applications of these measures can be found in the analysis of contingency tables [11], in approximation of probability distributions [7], [22], in signal processing [18], and in pattern recognition [3], [5].

Csiszár [8]–[9] introduced the $f-$divergence functional as

$$D_f(\boldsymbol{p},\boldsymbol{q}) = \sum_{i=1}^n q_i f\left(\frac{p_i}{q_i}\right), \qquad (20)$$

where $f\colon [0,+\infty\rangle$ is a convex function, and it represent a "distance function" on the set of probability distributions $\mathbb{P}$.

All of the mentioned divergences are special cases of Csiszár f-divergence for different choices of the function f.

As in Csiszár [9], we interpret undefined expressions by

$$f(0) = \lim_{t\to 0^+} f(t), \quad 0\cdot f\left(\frac{0}{0}\right) = 0,$$

$$0\cdot f\left(\frac{a}{0}\right) = \lim_{\varepsilon\to 0^+} f\left(\frac{a}{\varepsilon}\right) = a\cdot \lim_{t\to\infty}\frac{f(t)}{t}.$$

In this paper we will study a generalization of the f-divergence functional for a different class of functions. Throughout this section, when mentioning the interval $[m,M]$, we assume that $[m,M] \subseteq \mathbb{R}_+$. For a 3-convex function $f\colon [m,M]\to\mathbb{R}$ we define generalized f-divergence functional

$$\tilde{D}_f(\boldsymbol{p},\boldsymbol{q}) = \sum_{i=1}^n q_i f\left(\frac{p_i}{q_i}\right). \qquad (21)$$

As an application of Theorem 3 we get an Edmundson-Lah-Ribarič type inequality for the above defined generalized f-divergence functional.

THEOREM 7. *Let $[m,M] \subset \mathbb{R}$ be an interval such that $m \leqslant 1 \leqslant M$ and let $f\colon [m,M] \to \mathbb{R}$ be a 3-convex function. Let $\boldsymbol{p} = (p_1,\dots,p_n)$ and $\boldsymbol{p} = (q_1,\dots,q_n)$ be probability distributions such that $p_i/q_i \in [m,M]$ for every $i = 1,\dots,n$. Then we have*

$$\begin{aligned}
&\frac{1}{M-m}\sum_{i=1}^{n} q_i\left(M-\frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i}-m\right)\left(\frac{f(M)-f(m)}{M-m}-f'_+(m)\right)\\
\leqslant &\frac{M-1}{M-m}f(m)+\frac{1-m}{M-m}f(M)-\tilde{D}_f(\boldsymbol{p},\boldsymbol{q}) \qquad (22)\\
\leqslant &\frac{1}{M-m}\sum_{i=1}^{n} q_i\left(M-\frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i}-m\right)\left(f'_-(M)-\frac{f(M)-f(m)}{M-m}\right)
\end{aligned}$$

Proof. Let $\boldsymbol{x} = (x_1,\dots,x_n)$ such that $x_i \in [m,M]$ for $i = 1,\dots,n$. For a 3-convex function ϕ, in the relation (3) we can replace

$$f \longleftrightarrow \boldsymbol{x}, \text{ and } A(\boldsymbol{x}) = \sum_{i=1}^{n} p_i x_i.$$

In that way we get

$$\begin{aligned}
&\frac{\sum_{i=1}^n p_i(M-x_i)(x_i-m)}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m}-\phi'_+(m)\right)\\
\leqslant &\frac{M-\bar{x}}{M-m}\phi(m)+\frac{\bar{x}-m}{M-m}\phi(M)-\sum_{i=1}^{n} p_i\phi(x_i)\\
\leqslant &\frac{\sum_{i=1}^n p_i(M-x_i)(x_i-m)}{M-m}\left(\phi'_-(M)-\frac{\phi(M)-\phi(m)}{M-m}\right),
\end{aligned}$$

where $\bar{x} = \sum_{i=1}^n p_i x_i$. Since the function f is 3-convex, in the previous relation we can set

$$\phi = f, \quad p_i = q_i \text{ and } x_i = \frac{p_i}{q_i},$$

and after calculating

$$\bar{x} = \sum_{i=1}^{n} q_i \frac{p_i}{q_i} = \sum_{i=1}^{n} p_i = 1$$

we get (22). □

In a similar way, as an application of Theorem 4, an inequality of the Jensen type for the generalized f-divergence functional follows.

THEOREM 8. *Let $[m,M] \subset \mathbb{R}$ be an interval such that $m \leqslant 1 \leqslant M$ and let $f\colon [m,M] \to \mathbb{R}$ be a 3-convex function. Let $\boldsymbol{p} = (p_1,\dots,p_n)$ and $\boldsymbol{p} = (q_1,\dots,q_n)$ be probability distributions such that $p_i/q_i \in [m,M]$ for every $i = 1,\dots,n$. Then we have*

$$\begin{aligned}
&\frac{(M-1)(1-m)}{M-m}\left(\frac{f(M)-f(m)}{M-m}-f'_+(m)\right)\\
&-\frac{1}{M-m}\sum_{i=1}^{n} q_i\left(M-\frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i}-m\right)\left(f'_-(M)-\frac{f(M)-f(m)}{M-m}\right) \qquad (23)
\end{aligned}$$

$$\leqslant \tilde{D}_f(\boldsymbol{p},\boldsymbol{q}) - f(1) \leqslant \frac{(M-1)(1-m)}{M-m}\left(f'_-(M) - \frac{f(M)-f(m)}{M-m}\right)$$
$$- \frac{1}{M-m}\sum_{i=1}^n q_i\left(M-\frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i}-m\right)\left(\frac{f(M)-f(m)}{M-m} - f'_+(m)\right).$$

Proof. As in the proof of the previous theorem, let $\boldsymbol{x} = (x_1,\dots,x_n)$ such that $x_i \in [m,M]$ for $i = 1,\dots,n$. For a 3-convex function ϕ, in the relation (7) we can replace

$$f \longleftrightarrow \boldsymbol{x}, \text{ and } A(\boldsymbol{x}) = \sum_{i=1}^n p_i x_i$$

and obtain the following discrete sequence of inequalities:

$$\frac{(M-\bar{x})(\bar{x}-m)}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m} - \phi'_+(m)\right)$$
$$- \frac{\sum_{i=1}^n p_i(M-x_i)(x_i-m)}{M-m}\left(\phi'_-(M) - \frac{\phi(M)-\phi(m)}{M-m}\right)$$
$$\leqslant \sum_{i=1}^n p_i\phi(x_i) - \phi(\bar{x}) \leqslant \frac{(M-\bar{x})(\bar{x}-m)}{M-m}\left(\phi'_-(M) - \frac{\phi(M)-\phi(m)}{M-m}\right)$$
$$- \frac{\sum_{i=1}^n p_i(M-x_i)(x_i-m)}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m} - \phi'_+(m)\right).$$

The function f is 3-convex, so in the previous relation we can set

$$\phi = f, \;\; p_i = q_i \text{ and } x_i = \frac{p_i}{q_i},$$

and after calculating

$$\bar{x} = \sum_{i=1}^n q_i \frac{p_i}{q_i} = \sum_{i=1}^n p_i = 1$$

we get (23). □

EXAMPLE 1. Let $\boldsymbol{p} = (p_1,\dots,p_n)$ and $\boldsymbol{p} = (q_1,\dots,q_n)$ be probability distributions and let $[m,M] \subset \mathbb{R}$ be an interval such that $m \leqslant 1 \leqslant M$ and $p_i/q_i \in [m,M]$ for every $i = 1,\dots,n$.

▷ *Kullback-Leibler divergence* of the probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined as

$$D_{KL}(\boldsymbol{p},\boldsymbol{q}) = \sum_{i=1}^n q_i \log\frac{q_i}{p_i},$$

and the corresponding generating function is $f(t) = t\log t$, $t > 0$. We can calculate $f'''(t) = -\frac{1}{t^2} < 0$, so the function $-f(t) = -t\log t$ is 3-convex. Now it is obvious that for the Kullback-Leibler divergence the inequalities (22) and (23) hold with reversed signs of inequality, with

$$f'_+(m) = \log m + 1 \text{ and } f'_-(M) = \log M + 1.$$

▷ *Hellinger divergence* of the probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined as

$$D_H(\boldsymbol{p},\boldsymbol{q}) = \frac{1}{2}\sum_{i=1}^{n}(\sqrt{q_i}-\sqrt{p_i})^2,$$

and the corresponding generating function is $f(t)=\frac{1}{2}(1-\sqrt{t})^2$, $t>0$. We see that $f'''(t)=-\frac{3}{8}t^{-\frac{5}{2}}<0$, so the function $-f(t)=-\frac{1}{2}(1-\sqrt{t})^2$ is 3-convex. It is clear that for the Hellinger divergence the inequalities (22) and (23) hold with reversed signs of inequality, with

$$f'_+(m) = -\frac{1}{2\sqrt{m}}+\frac{1}{2} \text{ and } f'_-(M) = -\frac{1}{2\sqrt{M}}+\frac{1}{2}.$$

▷ *Renyi divergence* of the probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined as

$$D_\alpha(\boldsymbol{p},\boldsymbol{q}) = \sum_{i=1}^{n} q_i^{\alpha-1}p_i^{\alpha},\ \alpha\in\mathbb{R},$$

and the corresponding generating function is $f(t)=t^{\alpha}$, $t>0$. We calculate that $f'''(t)=\alpha(\alpha-1)(\alpha-2)t^{\alpha-3}$ and see that the function $f(t)=t^{\alpha}$ is 3-convex for $0\leqslant\alpha\leqslant1$ and $\alpha\geqslant2$, and $-f(t)=-t^{\alpha}$ is 3-convex for $\alpha\leqslant0$ and $1<\alpha<2$, and we have

$$f'_+(m)=\alpha m^{\alpha-1} \text{ and } f'_-(M)=\alpha M^{\alpha-1}.$$

As regards the Renyi divergence, the inequalities (22) and (23) hold for $0\leqslant\alpha\leqslant1$ and $\alpha\geqslant2$, and if $\alpha\leqslant0$ or $1<\alpha<2$ the signs of inequality are reversed.

▷ *Harmonic divergence* of the probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined as

$$D_{Ha}(\boldsymbol{p},\boldsymbol{q}) = \sum_{i=1}^{n}\frac{2p_iq_i}{p_i+q_i},$$

and the corresponding generating function is $f(t)=\frac{2t}{1+t}$. We can calculate $f'''(t)=\frac{12}{(1+t)^4}>0$, so the function f is 3-convex. Now it is obvious that for the harmonic divergence the inequalities (22) and (23) hold with

$$f'_+(m)=\frac{2}{(1+m)^2} \text{ and } f'_-(M)=\frac{2}{(1+M)^2}.$$

▷ *Jeffreys divergence* of the probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined as

$$D_J(\boldsymbol{p},\boldsymbol{q}) = \frac{1}{2}\sum_{i=1}^{n}(q_i-p_i)\log\frac{q_i}{p_i},$$

and the corresponding generating function is $f(t)=(1-t)\log\frac{1}{t}$, $t>0$. We see that $f'''(t)=-\frac{1}{t^2}-\frac{2}{t^3}<0$, so the function $-f(t)=(1-t)\log t$ is 3-convex. Instantly we get that for the Jeffreys divergence the inequalities (22) and (23) hold with reversed signs of inequality, with

$$f'_+(m)=\log M-\frac{1}{M}+1 \text{ and } f'_-(M)=\log m-\frac{1}{m}+1.$$

5. Applications to Zipf-Mandelbrot law

Benoit Mandelbrot in 1966 gave improvement of Zipf law for the count of the low-rank words. Various scientific fields use this law for different purposes, for example information sciences use it for indexing [10, 27], ecological field studies in predictability of ecosystem [24], in music is used to determine aesthetically pleasing music [23].

Zipf-Mandelbrot law is a discrete probability distribution with parameters $N \in \mathbb{N}$, $q, s \in \mathbb{R}$ such that $q \geqslant 0$ and $s > 0$, possible values $\{1, 2, \dots, N\}$ and probability mass function

$$f(i; N, q, s) = \frac{1/(i+q)^s}{H_{N,q,s}}, \quad \text{where } H_{N,q,s} = \sum_{i=1}^{N} \frac{1}{(i+q)^s}. \tag{24}$$

Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be Zipf-Mandelbrot laws with parameters $N \in \mathbb{N}$, $q_1, q_2 \geqslant 0$ and $s_1, s_2 > 0$ respectively and let us denote

$$\begin{aligned} m_{\boldsymbol{p},\boldsymbol{q}} &:= \min\left\{\frac{p_i}{q_i}\right\} = \frac{H_{N,q_2,s_2}}{H_{N,q_1,s_1}} \min\left\{\frac{(i+q_2)^{s_2}}{(i+q_1)^{s_1}}\right\} \\ M_{\boldsymbol{p},\boldsymbol{q}} &:= \max\left\{\frac{p_i}{q_i}\right\} = \frac{H_{N,q_2,s_2}}{H_{N,q_1,s_1}} \max\left\{\frac{(i+q_2)^{s_2}}{(i+q_1)^{s_1}}\right\} \end{aligned} \tag{25}$$

The results from the previous section can be utilized in obtaining different inequalities for the Zipf-Mandelbrot law. The first result that follows is a special case of Theorem 7, and it gives us Edmundson-Lah-Ribarič type inequality for the generalized f-divergence of the Zipf-Mandelbrot law.

COROLLARY 3. *Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be Zipf-Mandelbrot laws with parameters $N \in \mathbb{N}$, $q_1, q_2 \geqslant 0$ and $s_1, s_2 > 0$ respectively, and let $m_{\boldsymbol{p},\boldsymbol{q}}$ and $M_{\boldsymbol{p},\boldsymbol{q}}$ be defined in (25). Let $f\colon [m_{\boldsymbol{p},\boldsymbol{q}}, M_{\boldsymbol{p},\boldsymbol{q}}] \to \mathbb{R}$ be a 3-convex function. Then we have*

$$\begin{aligned} &\frac{1}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}} \left(\frac{f(M_{\boldsymbol{p},\boldsymbol{q}}) - f(m_{\boldsymbol{p},\boldsymbol{q}})}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}} - f'_+(m_{\boldsymbol{p},\boldsymbol{q}}) \right) \\ &\quad \times \sum_{i=1}^{n} \frac{1}{(i+q_2)^{s_2} H_{N,q_2,s_2}} \left(M_{\boldsymbol{p},\boldsymbol{q}} - \frac{(i+q_2)^{s_2} H_{N,q_2,s_2}}{(i+q_1)^{s_1} H_{N,q_1,s_1}} \right) \left(\frac{(i+q_2)^{s_2} H_{N,q_2,s_2}}{(i+q_1)^{s_1} H_{N,q_1,s_1}} - m_{\boldsymbol{p},\boldsymbol{q}} \right) \\ &\leqslant \frac{M_{\boldsymbol{p},\boldsymbol{q}} - 1}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}} f(m_{\boldsymbol{p},\boldsymbol{q}}) + \frac{1 - m_{\boldsymbol{p},\boldsymbol{q}}}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}} f(M_{\boldsymbol{p},\boldsymbol{q}}) - \tilde{D}_f(\boldsymbol{p}, \boldsymbol{q}) \\ &\leqslant \frac{1}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}} \left(f'_-(M_{\boldsymbol{p},\boldsymbol{q}}) - \frac{f(M_{\boldsymbol{p},\boldsymbol{q}}) - f(m_{\boldsymbol{p},\boldsymbol{q}})}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}} \right) \\ &\quad \times \sum_{i=1}^{n} \frac{1}{(i+q_2)^{s_2} H_{N,q_2,s_2}} \left(M_{\boldsymbol{p},\boldsymbol{q}} - \frac{(i+q_2)^{s_2} H_{N,q_2,s_2}}{(i+q_1)^{s_1} H_{N,q_1,s_1}} \right) \left(\frac{(i+q_2)^{s_2} H_{N,q_2,s_2}}{(i+q_1)^{s_1} H_{N,q_1,s_1}} - m_{\boldsymbol{p},\boldsymbol{q}} \right). \end{aligned} \tag{26}$$

Our next result follows directly from Theorem 8, and it represents a Jensen type inequality for the generalized f-divergence of the Zipf-Mandelbrot law.

COROLLARY 4. *Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be Zipf-Mandelbrot laws with parameters $N \in \mathbb{N}$, $q_1, q_2 \geqslant 0$ and $s_1, s_2 > 0$ respectively, and let $m_{\boldsymbol{p},\boldsymbol{q}}$ and $M_{\boldsymbol{p},\boldsymbol{q}}$ be defined in (25). Let $f\colon [m_{\boldsymbol{p},\boldsymbol{q}}, M_{\boldsymbol{p},\boldsymbol{q}}] \to \mathbb{R}$ be a 3-convex function. Then we have*

$$
\begin{aligned}
&\frac{(M_{\boldsymbol{p},\boldsymbol{q}}-1)(1-m_{\boldsymbol{p},\boldsymbol{q}})}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}\left(\frac{f(M_{\boldsymbol{p},\boldsymbol{q}})-f(m_{\boldsymbol{p},\boldsymbol{q}})}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}-f'_{+}(m_{\boldsymbol{p},\boldsymbol{q}})\right)\\
&\quad-\frac{1}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}\left(f'_{-}(M_{\boldsymbol{p},\boldsymbol{q}})-\frac{f(M_{\boldsymbol{p},\boldsymbol{q}})-f(m_{\boldsymbol{p},\boldsymbol{q}})}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}\right)\\
&\quad\times\sum_{i=1}^{n}\frac{1}{(i+q_2)^{s_2}H_{N,q_2,s_2}}\left(M_{\boldsymbol{p},\boldsymbol{q}}-\frac{(i+q_2)^{s_2}H_{N,q_2,s_2}}{(i+q_1)^{s_1}H_{N,q_1,s_1}}\right)\left(\frac{(i+q_2)^{s_2}H_{N,q_2,s_2}}{(i+q_1)^{s_1}H_{N,q_1,s_1}}-m_{\boldsymbol{p},\boldsymbol{q}}\right)\\
&\leqslant\tilde{D}_f(\boldsymbol{p},\boldsymbol{q})-f(1) \qquad (27)\\
&\leqslant\frac{(M_{\boldsymbol{p},\boldsymbol{q}}-1)(1-m_{\boldsymbol{p},\boldsymbol{q}})}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}\left(f'_{-}(M_{\boldsymbol{p},\boldsymbol{q}})-\frac{f(M_{\boldsymbol{p},\boldsymbol{q}})-f(m_{\boldsymbol{p},\boldsymbol{q}})}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}\right)\\
&\quad-\frac{1}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}\left(\frac{f(M_{\boldsymbol{p},\boldsymbol{q}})-f(m_{\boldsymbol{p},\boldsymbol{q}})}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}-f'_{+}(m_{\boldsymbol{p},\boldsymbol{q}})\right)\\
&\quad\times\sum_{i=1}^{n}\frac{1}{(i+q_2)^{s_2}H_{N,q_2,s_2}}\left(M_{\boldsymbol{p},\boldsymbol{q}}-\frac{(i+q_2)^{s_2}H_{N,q_2,s_2}}{(i+q_1)^{s_1}H_{N,q_1,s_1}}\right)\left(\frac{(i+q_2)^{s_2}H_{N,q_2,s_2}}{(i+q_1)^{s_1}H_{N,q_1,s_1}}-m_{\boldsymbol{p},\boldsymbol{q}}\right).
\end{aligned}
$$

REMARK 2. Corollary 3 and Corollary 4 can easily be applied to Kullback-Leibler divergence, Hellinger divergence, Renyi divergence, harmonic divergence or Jeffreys divergence considering Example 1.

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