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POINTWISE APPROXIMATION BY BÉZIER VARIANT OF AN OPERATOR BASED ON LAGUERRE POLYNOMIALS

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Abstract. In 2013, Öksüzer et al. [Math. Meth. Appl. Sci., Doi: 10.1002/mma.3705] defined the Bézier variant of an operator involving Laguerre polynomials of degree k and studied the rate of convergence of these operators. In the present paper, our aim is to study the degree of approximation of these operators by means of the first order Ditzian-Totik modulus of smoothness and also obtain a quantitative Voronovskaja type theorem and the error in the approximation of functions having derivatives of bounded variation.

1. Introduction

Following [18], the generating function of the associated Laguerre polynomials is given by

$$\sum_{k=0}^{\infty} L_k^{(n)}(t)x^k = \frac{1}{(1-x)^{n+1}} \exp\left(\frac{-xt}{1-x}\right), \quad 0 \leqslant x < 1,\ n \in \mathbb{Z}^+, \tag{1.1}$$

where $t \leqslant 0$, is a fixed parameter and $L_k^{(n)}$ is a Laguerre polynomial of degree k defined as

$$L_k^{(n)}(t) = \sum_{j=0}^{k} (-1)^j \binom{k+n}{k-j} \frac{t^j}{j!}, \quad t \in \mathbb{R},$$

where n is a non-negative integer.

Cheney and Sharma [11] introduced a sequence of linear positive operators based on the Laguerre polynomials as follows:

$$\begin{aligned}(P_n f(s))(x,t) &= (1-x)^{n+1} \exp\left(\frac{xt}{1-x}\right) \sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right) L_k^{(n)}(t)x^k,\\ (P_n f(s))(1,t) &= f(1).\end{aligned} \tag{1.2}$$

In particular, for $t = 0$, the operator given by (1.2), includes Meyer-König and Zeller operator [16].

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In [18], Öksüzer et al. defined the Bézier variant of the operator (1.2) as:

$$(P_{n,\alpha}f(s))(x,t) = \sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right) Q_{n,k}^{\alpha}(x,t) \quad (0 \leqslant x < 1), \tag{1.3}$$
$$(P_{n,\alpha}f(s))(1,t) = f(1),$$

where $\alpha \geqslant 1$, $Q_{n,k}^{\alpha}(x,t) = (J_{n,k}(x,t))^{\alpha} - (J_{n,k+1}(x,t))^{\alpha}$, $J_{n,k}(x,t) = \sum_{j=k}^{\infty} m_{n,j}(x,t)$ for $k = 0, 1 \cdots, n$, are the Bézier basis functions and

$$m_{n,k}(x,t) = (1-x)^{n+1} \exp\left(\frac{xt}{1-x}\right) L_k^{(n)}(t)x^k, \quad (0 \leqslant x < 1),$$

and estimated the rate of pointwise convergence of $P_{n,\alpha}$ by using the Chanturia modulus of variation, at those points $x \in (0,1)$ at which the one-sided limits exist. Öksüzer et al. [19] also studied the rate of convergence of operators given by (1.3) for the functions of bounded variation. Bojanic and Cheng [9], pioneered the study of the rate of convergence of Bernstein polynomials for functions with derivative of bounded variation. After that authors [10], discussed degree of approximation of Hermite-Fejér polynomials for the functions with derivatives of bounded variation. In 2003, Gupta et. al [15], discussed rate of convergence of summation integral type operators with derivatives of bounded variation. In 2005, Gupta et. al [14], studied same results for Beta operators of second kind. Subsequently, many researchers have contributed to this area of approximation theory (cf. [6], [8], [17] etc.) and the references therein.

The purpose of this paper is to determine the degree of approximation of the operators given by (1.3) in terms of first order Ditzian-Totik modulus of smoothness and a quantitative Voronovskaja type theorem. Also, we obtain the rate of convergence of these operators for functions with derivatives of bounded variation.

Throughout this paper, C denotes a positive constant not necessarily the same at each occurrence.

2. Preliminaries

Denote

$$A_{ij}^{(\nu)} \equiv A_{ij}^{(\nu)}(n,x,t) := (1-x)^{n+1+i} \exp\left(\frac{tx}{1-x}\right) \sum_{k=0}^{\infty} \frac{1}{(k+n+j)^{\nu}} L_k^{(n+i)}(t)x^k, \; i,j,\nu \in \mathbb{N}.$$

LEMMA 1. *For the functions $A_{ij}^{(\nu)}$, we have*

$$0 \leqslant A_{ij}^{(\nu)} \leqslant \frac{1}{(n+j)^{\nu}}.$$

Consequently, $\lim_{n\to\infty} A_{ij}^{(\nu)} = 0$.

Proof. Since $L_k^{(n)}(t) \geqslant 0$, $\forall\, n \in \mathbb{N} \cup \{0\}$ and $t \leqslant 0$, it follows that $A_{ij}^{(\nu)} \geqslant 0$, $\forall\, i, j, \nu \in \mathbb{N}$.

It is clear that

$$A_{ij}^{(\nu)} \leqslant \frac{1}{(n+j)^{\nu}}(1-x)^{n+1+i}\exp\left(\frac{tx}{1-x}\right)\sum_{k=0}^{\infty} L_k^{(n+i)}(t)x^k.$$

Using the following identity (see [20])

$$(1-x)^{n+1}\exp\left(\frac{xt}{1-x}\right)\sum_{k=0}^{\infty} L_k^{(n)}(t)x^k = 1,$$

we have

$$A_{ij}^{(\nu)} \leqslant \frac{1}{(n+j)^{\nu}}.$$

Hence, $\lim\limits_{n\to\infty} A_{ij}^{(\nu)} = 0$. $\square$

LEMMA 2. *Let $e_i(x) = x^i$, $i = 0, 1, 2 \ldots$. Then for each $x \in [0,1)$, $t \leqslant 0$, we have*

i) $(P_n e_0)(x,t) = 1$,

ii) $(P_n e_1)(x,t) = x - \dfrac{xt}{1-x}A_{11}^{(1)}$,

iii) $(P_n e_2)(x,t) = x^2 - x^2 A_{02}^{(1)} - \dfrac{2tx^2}{1-x}A_{12}^{(1)} + xA_{01}^{(1)} + \dfrac{t^2x^2}{(1-x)^2}A_{22}^{(2)} - \dfrac{tx}{1-x}A_{11}^{(2)}$,

iv) $(P_n e_3)(x,t)$

$$\begin{aligned}
&= xA_{01}^{(2)} - \frac{tx}{1-x}A_{11}^{(3)} - \frac{2tx^2}{1-x}A_{12}^{(2)} + \frac{2t^2x^2}{(1-x)^2}A_{22}^{(3)} + 3x^2\left(A_{02}^{(1)} - A_{02}^{(2)} - \frac{t}{1-x}A_{12}^{(2)}\right) \\
&\quad - tx^2\left(\frac{1}{1-x}A_{12}^{(2)} - \frac{t}{(1-x)^2}A_{22}^{(3)}\right) + x^3\left(1 - 3A_{03}^{(1)} + 2A_{03}^{(2)} - \frac{t}{1-x}A_{13}^{(1)} + \frac{t}{1-x}A_{13}^{(2)}\right) \\
&\quad - tx^3\left(A_{13}^{(1)} - \frac{2}{1-x}A_{13}^{(2)} - \frac{2t}{(1-x)^2}A_{23}^{(2)} + \frac{1}{1-x}A_{13}^{(1)}\right) + \frac{t^2x^3}{(1-x)^2}\left(A_{23}^{(2)} - \frac{t}{1-x}A_{33}^{(3)}\right),
\end{aligned}$$

v) $(P_n e_4)(x,t)$

$$\begin{aligned}
&= xA_{01}^{(3)} + \frac{tx}{1-x}A_{11}^{(4)} + 7x^2\left(A_{02}^{(2)} - A_{02}^{(3)} - \frac{t}{1-x}A_{12}^{(3)}\right) - \frac{7tx^2}{1-x}\left(A_{12}^{(3)} - \frac{t}{1-x}A_{22}^{(4)}\right) \\
&\quad + x^3\left(6A_{03}^{(1)} - 14A_{03}^{(2)} + 4A_{03}^{(3)} - \frac{12t}{1-x}A_{13}^{(2)} + \frac{8t}{1-x}A_{13}^{(3)} + \frac{6t^2}{(1-x)^2}A_{23}^{(3)}\right) \\
&\quad - \frac{6tx^2}{1-x}\left(A_{13}^{(2)} - A_{13}^{(3)} - \frac{2t}{1-x}A_{23}^{(3)} + \frac{t^2}{(1-x)^2}A_{33}^{(4)}\right) \\
&\quad + x^4\left(1 + 11A_{04}^{(2)} - 6A_{04}^{(1)} - 6A_{04}^{(3)} - \frac{6t}{1-x}A_{14}^{(3)} - \frac{3t}{1-x}A_{14}^{(1)}\right. \\
&\quad \left. + \frac{3t^2}{(1-x)^2}A_{24}^{(2)} + \frac{9t}{1-x}A_{14}^{(2)} - \frac{3t^2}{(1-x)^2}A_{24}^{(3)} - \frac{t^3}{(1-x)^3}A_{34}^{(3)}\right) \\
&\quad - \frac{tx^4}{1-x}\left(A_{14}^{(1)} - 3A_{14}^{(2)} - \frac{3t}{1-x}A_{24}^{(2)} + 2A_{14}^{(3)} + \frac{3t}{1-x}A_{24}^{(3)} + \frac{3t^2}{(1-x)^2}A_{34}^{(3)} - \frac{t^3}{(1-x)^3}A_{44}^{(4)}\right).
\end{aligned}$$

Proof. Using the following identity (see [20])

$$(1-x)^{n+1}\exp\left(\frac{xt}{1-x}\right)\sum_{k=0}^{\infty}L_k^{(n)}(t)x^k=1,$$

we get $(P_ne_0)(x,t)=1$.

In order to obtain the m^{th}-order moment of the operators based on the Laguerre polynomials we consider the known recurence formula (see [20])

$$tL_{k-1}^{(n+1)}(t)=(k+n)L_{k-1}^{(n)}(t)-kL_k^{(n)}(t). \tag{2.1}$$

Therefore, we get

$$\begin{aligned}
(P_ne_1)(x,t)&=(1-x)^{n+1}\exp\left(\frac{xt}{1-x}\right)\sum_{k=1}^{\infty}\frac{k}{k+n}L_k^{(n)}(t)x^k\\
&=(1-x)^{n+1}\exp\left(\frac{xt}{1-x}\right)\sum_{k=1}^{\infty}\frac{1}{k+n}\left\{(k+n)L_{k-1}^{(n)}(t)-tL_{k-1}^{(n+1)}(t)\right\}x^k\\
&=(1-x)^{n+1}\exp\left(\frac{xt}{1-x}\right)\sum_{k=1}^{\infty}\left(L_{k-1}^{(n)}(t)x^k-\frac{t}{k+n}L_{k-1}^{(n+1)}(t)x^k\right)\\
&=(1-x)^{n+1}\exp\left(\frac{xt}{1-x}\right)\left\{\sum_{k=0}^{\infty}L_k^{(n)}(t)x^{k+1}-\sum_{k=0}^{\infty}\frac{t}{k+n+1}L_k^{(n+1)}(t)x^{k+1}\right\}\\
&=x-\frac{tx}{1-x}A_{11}^{(1)}.
\end{aligned}$$

Using the recurrence formula (2.1) twice, we obtain

$$\begin{aligned}
(P_ne_2)(x,t)&=(1-x)^{n+1}\exp\left(\frac{xt}{1-x}\right)\sum_{k=0}^{\infty}\left(\frac{k}{k+n}\right)^2L_k^{(n)}(t)x^k\\
&=(1-x)^{n+1}\exp\left(\frac{xt}{1-x}\right)\sum_{k=1}^{\infty}\frac{k}{(k+n)^2}\left\{(k+n)L_{k-1}^{(n)}(t)-tL_{k-1}^{(n+1)}(t)\right\}x^k\\
&=(1-x)^{n+1}\exp\left(\frac{xt}{1-x}\right)\sum_{k=1}^{\infty}\left\{\frac{k}{k+n}L_{k-1}^{(n)}(t)-\frac{kt}{(k+n)^2}L_{k-1}^{(n+1)}(t)\right\}x^k\\
&=(1-x)^{n+1}\exp\left(\frac{xt}{1-x}\right)\sum_{k=0}^{\infty}\left\{\frac{k+1}{k+1+n}L_k^{(n)}(t)-\frac{(k+1)t}{(k+1+n)^2}L_k^{(n+1)}(t)\right\}x^{k+1}\\
&=(1-x)^{n+1}\exp\left(\frac{xt}{1-x}\right)\left\{\sum_{k=1}^{\infty}L_{k-1}^{(n)}(t)x^{k+1}-\sum_{k=1}^{\infty}\frac{1}{k+n+1}L_{k-1}^{(n)}(t)x^{k+1}\right.\\
&\quad-t\sum_{k=1}^{\infty}\frac{1}{k+n+1}L_{k-1}^{(n+1)}x^{(k+1)}+\sum_{k=0}^{\infty}\frac{1}{k+n+1}L_k^{(n)}(t)x^{k+1}\\
&\quad-t\sum_{k=1}^{\infty}\frac{1}{k+n+1}L_{k-1}^{(n+1)}(t)x^{k+1}+t^2\sum_{k=1}^{\infty}\frac{1}{(k+1+n)^2}L_{k-1}^{(n+2)}(t)x^{k+1}
\end{aligned}$$

$$-t\sum_{k=0}^{\infty}\frac{1}{(k+n+1)^2}L_k^{(n+1)}(t)x^{k+1}\Bigg\}$$

$$=x^2-x^2A_{02}^{(1)}-\frac{2tx^2}{1-x}A_{12}^{(1)}+xA_{01}^{(1)}+\frac{t^2x^2}{(1-x)^2}A_{22}^{(2)}-\frac{tx}{1-x}A_{11}^{(2)}.$$

In a similar way can be proved the next relations. □

LEMMA 3. *For $t\in[b,0]$, $b<0$, the operators based on the Laguerre polynomials (1.2) verify:*

i) $\lim\limits_{n\to\infty} n\left(P_n(s-x)\right)(x,t)\leqslant -\dfrac{xt}{1-x}$,

ii) $\lim\limits_{n\to\infty} n\left(P_n(s-x)^2\right)(x,t)\leqslant x$,

iii) $\lim\limits_{x\to\infty} n^2\left(P_n(s-x)^4\right)(x,t)\leqslant \dfrac{x^2}{1-x}\left(-12xt+13x(1-x)+3(1-x^3)\right)$.

Consequently, for sufficiently large n we have

$$\left(P_n(s-x)^2\right)(x,t)\leqslant\frac{Cx}{n}, \tag{2.2}$$

where C is a positive constant.

Proof. Using Lemma 2 we obtain

$$\lim_{n\to\infty} n\left(P_n(s-x)\right)(x,t)=\lim_{n\to\infty}-\frac{xtn}{1-x}A_{11}^{(1)}=-\frac{xt}{1-x},$$

$$\begin{aligned}
&\lim_{n\to\infty} n\left(P_n(s-x)^2\right)(x,t)\\
&=\lim_{n\to\infty} n\left\{-\frac{2x^2t}{1-x}A_{12}^{(1)}-x^2A_{02}^{(1)}+xA_{01}^{(1)}+\frac{t^2x^2}{(1-x)^2}A_{22}^{(2)}-\frac{tx}{1-x}A_{11}^{(2)}+\frac{2tx^2}{1-x}A_{11}^{(1)}\right\}\\
&\leqslant\lim_{n\to\infty} n\left\{xA_{02}^{(1)}+\frac{t^2x^2}{(1-x)^2}A_{22}^{(2)}-\frac{tx}{1-x}A_{11}^{(2)}\right\}\\
&\leqslant\lim_{n\to\infty}\left\{x+\frac{t^2x^2}{(1-x)^2n}-\frac{tx}{(1-x)n}\right\}=x,
\end{aligned}$$

$$\begin{aligned}
&\lim_{n\to\infty} n^2\left(P_n(s-x)^4\right)(x,t)\\
&=\lim_{n\to\infty}\frac{n^2x}{(1-x)^4}\Big\{6x^2(x+2)(1-x)^4\left(A_{03}^{(1)}-A_{02}^{(1)}\right)+6x^2(1-x)^4\left(A_{01}^{(1)}-A_{04}^{(1)}\right)\\
&\quad+4x^3t(1-x)^3\left(A_{11}^{(1)}-A_{14}^{(1)}\right)+12x^2t(1-x)^3\left(A_{13}^{(1)}-A_{12}^{(1)}\right)-4x(1-x)^4A_{01}^{(2)}
\end{aligned}$$

$$
\begin{aligned}
&+x(12x+7)(1-x)^4A_{02}^{(2)}-2x^2(4x+7)(1-x)^4A_{03}^{(2)}+24tx^2(1-x)^3A_{12}^{(2)}\\
&-6tx^2(2x+3)(1-x)^3A_{13}^{(2)}-12t^2x^3(1-x)^2A_{23}^{(2)}+12tx^3(1-x)^3A_{14}^{(2)}\\
&+6t^2x^3(1-x)^2A_{24}^{(2)}+11x^3(1-x)^4A_{04}^{(2)}+6t^2x^3(1-x)^2A_{22}^{(2)}-6tx^2(1-x)^3A_{11}^{(2)}\Big\}\\
\leqslant\ &\frac{x^2}{1-x}\left(-12xt+13x(1-x)+3(1-x^3)\right). \quad \square
\end{aligned}
$$

LEMMA 4. *Let $f \in C[0,1]$. Then the Bézier variant of the operators based on the Laguerre polynomials (1.3) verify:*

i) $\|P_{n,\alpha}\| \leqslant \|f\|$,

ii) $(P_{n,\alpha}f(s))(x,t) \leqslant \alpha\,(P_nf(s))(x,t)$, *where* $f \geqslant 0$ *on* $[0,1]$.

Proof. i) Since

$$
\begin{aligned}
(P_{n,\alpha}e_0)(x,t) &= \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x,t) = \sum_{k=0}^{\infty}\left\{\left[J_{n,k}(x,t)\right]^{\alpha}-\left[J_{n,k+1}(x,t)\right]^{\alpha}\right\}\\
&= [J_{n,0}(x,t)]^{\alpha} = \left(\sum_{j=0}^{\infty} m_{n,j}(x,t)\right)^{\alpha} = 1,
\end{aligned}
$$

it follows

$$
|(P_{n,\alpha}f(s))(x,t)| \leqslant \sum_{k=0}^{\infty}\left|f\left(\frac{k}{k+n}\right)\right| Q_{n,k}^{(\alpha)}(x,t) \leqslant \|f\|\sum_{k=0}^{n} Q_{n,k}^{(\alpha)}(x,t) = \|f\|.
$$

b) Using the inequality $|a^{\alpha}-b^{\alpha}| \leqslant \alpha|a-b|$, where $0 \leqslant a,b \leqslant 1$ and $\alpha \geqslant 1$, we get

$$
0 < \left[J_{n,k}(x,t)\right]^{\alpha}-\left[J_{n,k+1}(x,t)\right]^{\alpha} \leqslant \alpha(J_{n,k}(x,t)-J_{n,k+1}(x,t)) = \alpha m_{n,k}(x,t).
$$

Hence, in view of the definition of $P_{n,\alpha}$ and the positivity of f, we get $(P_{n,\alpha}f(s))(x,t) \leqslant \alpha\,(P_nf(s))(x,t)$. $\square$

3. Rate of convergence

In this section, we recall the definitions of the Ditzian-Totik first order modulus of smoothness and the K-functional [12]. Let $\phi(x) := \sqrt{x}$ and $f \in C[0,1]$. The first order modulus of smoothness is given by

$$
\omega_{\phi}(f;t) = \sup_{0<h\leqslant t}\left\{\left|f\left(x+\frac{h\phi(x)}{2}\right)-f\left(x-\frac{h\phi(x)}{2}\right)\right|,\ x \pm \frac{h\phi(x)}{2} \in [0,1]\right\}.
$$

Further, the appropriate K-functional is defined by

$$
K_{\phi}(f;t) = \inf_{g \in W_{\phi}[0,1]}\{||f-g||+t||\phi g'||+t^2||g'||\} \ \ (t>0),
$$

where $W_\phi[0,1]=\{g : g\in AC_{loc}[0,1],\ \|\phi g'\|<\infty\}$, $g\in AC_{loc}[0,1]$ denotes the class of all locally absolutely continuous function and $||.||$ is the sup norm on $C[0,1]$. It is well known [12, p.11] that there exists a constant $C>0$ such that

$$C^{-1}\omega_\phi(f;t)K_\phi(f;t)\leqslant C\omega_\phi(f;t).$$

THEOREM 1. *Let $f\in C[0,1]$ and $\phi(x)=\sqrt{x}$. For every $x\in[0,1)$, $t\in[b,0]$, $b<0$ and sufficiently large n, we have*

$$|(P_{n,\alpha}f(s))(x,t)-f(x)|\leqslant C\omega_\phi\left(f;\frac{1}{\sqrt{n}}\right).$$

Proof. By definition of $K_\phi(f,t)$, for fixed n,x, we can choose $g=g_{n,x}\in W_\phi[0,1]$ such that

$$||f-g||+\frac{1}{\sqrt{n}}||\phi g'||+\frac{1}{n}||g'||\leqslant\omega_\phi\left(f,\frac{1}{\sqrt{n}}\right). \tag{3.1}$$

$$\begin{aligned}|(P_{n,\alpha}f(s))(x,t)-f(x)|&\leqslant|(P_{n,\alpha}(f-g)(s))(x,t)|+|f-g|+|(P_{n,\alpha}g(s))(x,t)-g(x)|\\&\leqslant C||f-g||+|(P_{n,\alpha}g(s))(x,t)-g(x)|.\end{aligned} \tag{3.2}$$

Now, to estimate the second term in the above relation, we will split the domain into two parts, i.e $x\in I_n=[0,\frac{1}{n}]$ and $x\in I_n^c=(\frac{1}{n},1)$. Using the representation $g(s)=g(x)+\int_x^s g'(u)du$, we get

$$|(P_{n,\alpha}g(s))(x,t)-g(x)|\leqslant\left|\left(P_{n,\alpha}\left(\int_x^s g'(u)du\right)\right)(x,t)\right|. \tag{3.3}$$

Let $x\in I_n^c=(\frac{1}{n},1)$, we have

$$\begin{aligned}\left|\int_x^s g'(u)du\right|&\leqslant||\phi g'||\left|\int_x^s\frac{1}{\phi(u)}du\right|=2||\phi g'|||\sqrt{s}-\sqrt{x}|\\&\leqslant\frac{2||\phi g'|||s-x|}{\sqrt{x}}=\frac{2||\phi g'|||s-x|}{\phi(x)}.\end{aligned} \tag{3.4}$$

Now, combining (3.3) and (3.4), we get

$$\begin{aligned}|(P_{n,\alpha}g(s))(x,t)-g(x)|&\leqslant\frac{2||\phi g'||}{\phi(x)}|(P_{n,\alpha}|s-x|)(x,t)|\leqslant\frac{2||\phi g'||}{\phi(x)}|((P_{n,\alpha}(s-x)^2)(x,t))^{1/2}|\\&\leqslant\frac{2||\phi g'||\sqrt{\alpha}}{\phi(x)}\sqrt{\frac{Cx}{n}}=2\sqrt{C\alpha}||\phi g'||\sqrt{\frac{1}{n}}\leqslant C\frac{||\phi g'||}{\sqrt{n}}.\end{aligned}$$

Again, for $x\in I_n=[0,\frac{1}{n}]$, using Lemma 4(ii) and (2.2) we have

$$\begin{aligned}|(P_{n,\alpha}g(s))(x,t)-g(x)|&\leqslant||g'|||(P_{n,\alpha}|s-x|)(x,t)|\\&\leqslant||g'|||(P_{n,\alpha}(s-x)^2)(x,t)|^{1/2}\leqslant\alpha||g'||\{(P_{n,\alpha}(s-x)^2)(x,t)\}^{1/2}\\&\leqslant||g'||\sqrt{\alpha}\sqrt{\frac{Cx}{n}}\leqslant\frac{C}{n}||g'||.\end{aligned}$$

Therefore,

$$|(P_{n,\alpha}g(s))(x,t)-g(x)| \leqslant C\left(\frac{||\phi g'||}{\sqrt{n}}+\frac{||g'||}{n}\right). \tag{3.5}$$

Collecting (3.1), (3.2), (3.5), the proof of theorem is immediate. □

4. Voronovskaja type theorem

In this section we prove a quantitative Voronovskaja type theorem for the Bézier variant of the operators based on the Laguerre polynomials (1.3) in terms of the first order Ditzian-Totik modulus of smoothness. For some other researches in this direction we refer the reader to ([1]–[5], [7]) and the references therein.

THEOREM 2. *Let $f \in C^2[0,1]$, $x \in [0,1)$, $t \in [b,0]$, $b<0$. For sufficiently large n, the following inequalities hold*

i)
$$\left|n\left[\left(P_{n,\alpha}f(s)\right)(x,t)-f(x)-\mu_{n,\alpha}^{(1)}(x,t)f'(x)-\frac{1}{2}\mu_{n,\alpha}^{(2)}(x,t)f''(x)\right]\right| \leqslant C\omega_\phi\left(f'',\phi(x)n^{-1/2}\right),$$

ii)
$$\left|n\left[\left(P_{n,\alpha}f(s)\right)(x,t)-f(x)-\mu_{n,\alpha}^{(1)}(x,t)f'(x)-\frac{1}{2}\mu_{n,\alpha}^{(2)}(x,t)f''(x)\right]\right| \leqslant C\phi(x)\omega_\phi\left(f'',n^{-1/2}\right),$$

where

$$\mu_{n,\alpha}^{(m)}(x,t)=\left(P_{n,\alpha}(s-x)^m\right)(x,t),\ \ m\in\mathbb{N}.$$

Proof. Let $f \in C^2[0,1]$ be given and $s,x \in [0,1]$. Using Taylor's expansion

$$f(s)-f(x)=(s-x)f'(x)+\int_x^s(s-u)f''(u)du.$$

we get

$$\begin{aligned} f(s)-f(x)-(s-x)f'(x)-\frac{1}{2}(s-x)^2f''(x) &= \int_x^s(s-u)f''(u)du-\int_x^s(s-u)f''(x)du \\ &= \int_x^s(s-u)[f''(u)-f''(x)]du. \end{aligned}$$

Applying $P_{n,\alpha}$ to both sides of the above relation, we obtain

$$\begin{aligned} &\left|\left(P_{n,\alpha}f(s)\right)(x,t)-f(x)-\mu_{n,\alpha}^{(1)}(x,t)f'(x)-\frac{1}{2}\mu_{n,\alpha}^{(2)}(x,t)f''(x)\right| \\ &\leqslant P_{n,\alpha}\left(\left|\int_x^s|s-u||f''(u)-f''(x)|du\right|;x\right). \end{aligned} \tag{4.1}$$

The following estimation can be obtained (see [13, p. 337])

$$\left|\int_x^s |f''(u)-f''(x)||s-u|du\right| \leqslant 2\|f''-g\|(s-x)^2+2\|\phi g'\|\phi^{-1}(x)|s-x|^3, \quad (4.2)$$

where $g \in W_\phi$.

Using the relations (4.1)–(4.2), Lemma 3, Lemma 4 and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\left|(P_{n,\alpha}f(s))(x,t)-f(x)-\mu_{n,\alpha}^{(1)}(x,t)f'(x)-\frac{1}{2}\mu_{n,\alpha}^{(2)}(x,t)f''(x)\right|\\
&\leqslant 2\|f''-g\|\left(P_{n,\alpha}(s-x)^2\right)(x,t)+2\|\phi g'\|\phi^{-1}(x)\left(P_{n,\alpha}|t-x|^3\right)(x,t)\\
&\leqslant 2\|f''-g\|\alpha\left(P_n(s-x)^2\right)(x,t)+2\alpha\|\phi g'\|\phi^{-1}(x)\left\{\left(P_n(s-x)^2\right)(x,t)\right\}^{1/2}\\
&\quad\times\left\{\left(P_n(s-x)^4\right)(x,t)\right\}^{1/2}\\
&\leqslant 2\|f''-g\|\alpha\left(P_n(s-x)^2\right)(x,t)+2\alpha\frac{C}{n}\|\phi g'\|\left\{\left(P_n(s-x)^2\right)(x,t)\right\}^{1/2}\\
&\leqslant C\left\{\frac{\phi^2(x)}{n}\|f''-g\|+\frac{1}{n}\frac{\phi(x)}{\sqrt{n}}\|\phi g'\|\right\}\\
&\leqslant \frac{C}{n}\left\{\phi^2(x)\|f''-g\|+n^{-1/2}\phi(x)\|\phi g'\|\right\}.
\end{aligned}$$

The constant $C>0$ is not the same at each occurrence. Since $\phi^2(x)\leqslant\phi(x)\leqslant 1$, $x\in[0,1)$, we obtain

$$\begin{aligned}
&\left|(P_{n,\alpha}f(s))(x,t)-f(x)-\mu_{n,\alpha}^{(1)}(x,t)f'(x)-\frac{1}{2}\mu_{n,\alpha}^{(2)}(x,t)f''(x)\right|\\
&\leqslant\frac{C}{n}\left\{\|f''-g\|+n^{-1/2}\phi(x)\|\phi g'\|\right\}.
\end{aligned}$$

Also, the following inequality can be obtained

$$\begin{aligned}
&\left|(P_{n,\alpha}f(s))(x,t)-f(x)-\mu_{n,\alpha}^{(1)}(x,t)f'(x)-\frac{1}{2}\mu_{n,\alpha}^{(2)}(x,t)f''(x)\right|\\
&\leqslant\frac{C}{n}\phi(x)\left\{\|f''-g\|+n^{-1/2}\|\phi g'\|\right\}.
\end{aligned}$$

Taking the infimum on the right hand side of the above relations over $g\in W_\phi$, we get

$$\begin{aligned}
&\left|n\left[(P_{n,\alpha}f(s))(x,t)-f(x)-\mu_{n,\alpha}^{(1)}(x,t)f'(x)-\frac{1}{2}\mu_{n,\alpha}^{(2)}(x,t)f''(x)\right]\right|\\
&\leqslant\begin{cases} CK_\phi\left(f'';\phi(x)n^{-1/2}\right),\\ C\phi(x)K_\phi\left(f'';n^{-1/2}\right).\end{cases} \qquad \square
\end{aligned}$$

5. Approximation of functions with a derivative of bounded variation

Let $DBV[0,1]$ be the space of all absolutely continuous functions f defined on $[0,1]$ and having a derivative f' equivalent with a function of bounded variation on $[0,1]$. For $f \in DBV[0,1]$ we may write,

$$f(x) = \int_0^x g(t)dt + f(0).$$

We can rewrite the operator given by (1.3) as

$$(P_{n,\alpha} f(s))(x,t) = \int_0^1 f(w) \frac{\partial}{\partial w} \{M_n^{\alpha}(x,w,t)\} dw,$$

where

$$M_n^{\alpha}(x,w,t) = \begin{cases} \sum_{\frac{k}{k+n} \leqslant w} Q_{n,k}^{(\alpha)}(x,t), & 0 < w < 1 \\ 0, & w = 0 \end{cases}.$$

LEMMA 5. *Let $x \in [0,1)$, then for sufficiently large n, we have*

(i) $v_{n,\alpha}(x,s,t) = \int_0^s \frac{\partial}{\partial w}\{M_n^{\alpha}(x,w,t)\}dw \leqslant \frac{C\alpha x}{n(x-t)^2}, \quad 0 \leqslant s < x,$

(ii) $1 - v_{n,\alpha}(x,z,t) = \int_z^1 \frac{\partial}{\partial w}\{M_n^{\alpha}(x,w,t)\}dw \leqslant \frac{C\alpha x}{n(x-z)^2}, \quad x < z < 1.$

Proof. (i) Using Lemma 3, we have

$$\begin{aligned} v_{n,\alpha}(x,s,t) &= \int_0^s \frac{\partial}{\partial w}\{M_n^{\alpha}(x,w,t)\}dw \leqslant \int_0^s \left(\frac{x-w}{x-s}\right)^2 \frac{\partial}{\partial w}\{M_n^{\alpha}(x,w,t)\} dw \\ &= \frac{1}{(x-s)^2}(P_{n,\alpha}(x-w)^2)(x,w) \leqslant \frac{C\alpha x}{n(x-s)^2}. \end{aligned}$$

The proof of (ii) is similar, hence it is omitted. □

THEOREM 3. *Let $f \in DBV[0,1]$. Then,*

$$\begin{aligned} |(P_{n,\alpha} f(s))(x,t) - f(x)| \leqslant{}& \sqrt{\frac{C\alpha x}{n}} \left|\frac{f'(x+) + \alpha f'(x-)}{\alpha+1}\right| + 2\sqrt{\frac{C\alpha x}{n}} \left|\frac{f'(x+) - f'(x-)}{2}\right| \\ &+ \frac{C\alpha}{n} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{x-\frac{x}{k}}^{x} f_x'\right) + \frac{x}{\sqrt{n}} \left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f_x'\right) + \frac{1-x}{\sqrt{n}} \left(\bigvee_{x}^{x+\frac{1-x}{\sqrt{n}}} f_x'\right) \\ &+ \frac{C\alpha x}{n(1-x)} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{x}^{x+\frac{1-x}{k}} f_x'\right), \end{aligned}$$

where

$$f'_x(t) = \begin{cases} f'(t) - f'(x-), & 0 \leqslant t < x \\ 0, & t = x, \\ f'(t) - f'(x+) & x < t < \infty. \end{cases} \tag{5.1}$$

Proof. For any $f \in DBV[0,1]$, we may write

$$f'(s) = f'_x(s) + \frac{1}{\alpha+1}(f'(x+) + \alpha f'(x-)) + \frac{1}{2}(f'(x+) - f'(x-))\left(sgn(s-x) + \frac{\alpha-1}{\alpha+1}\right) + \delta_x(s)[f'(s) - \frac{1}{2}(f'(x+) + f'(x-))], \tag{5.2}$$

where

$$\delta_x(s) = \begin{cases} 1\,, & s = x \\ 0\,, & s \neq x \end{cases}.$$

Again, we have

$$\begin{aligned}
&(P_{n,\alpha} f(s))(x,t) - f(x) \\
&= \int_0^1 (f(s) - f(x)) \frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\} ds \\
&= \int_0^x (f(s) - f(x)) \frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\} ds + \int_x^1 (f(s) - f(x)) \frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\} ds \\
&= -\int_0^x \left(\int_s^x f'(u) du\right) \frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\} ds + \int_x^1 \left(\int_x^s f'(u) du\right) \frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\} ds \\
&= -K_1^\alpha(n,x,t) + K_2^\alpha(n,x,t) \qquad (say).
\end{aligned} \tag{5.3}$$

Now, from equation (5.2), we have

$$\begin{aligned}
K_1^\alpha(n,x,t) = &\frac{f'(x+) + \alpha f'(x-)}{\alpha+1} \int_0^x (x-s) \frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\} ds \\
&+ \int_0^x \left(\int_s^x f'_x(u) du\right) \frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\} ds \\
&- \frac{2}{\alpha+1} \frac{f'(x+) - f'(x-)}{2} \int_0^x (x-s) \frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\} ds.
\end{aligned} \tag{5.4}$$

Similarly,

$$\begin{aligned}
K_2^\alpha(n,x,t) = &\frac{f'(x+) + \alpha f'(x-)}{\alpha+1} \int_x^1 (s-x) \frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\} ds \\
&+ \int_x^1 \left(\int_x^s f'_x(u) du\right) \frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\} ds \\
&+ \frac{2\alpha}{\alpha+1} \frac{f'(x+) - f'(x-)}{2} \int_x^1 (s-x) \frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\} ds.
\end{aligned} \tag{5.5}$$

Using (5.4)–(5.5), from (5.2), we get

$$\begin{aligned}
&(P_{n,\alpha}f(s))(x,t)-f(x)\\
&=\frac{f'(x+)+\alpha f'(x-)}{\alpha+1}\int_0^1(s-x)\frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\}ds+\frac{2}{\alpha+1}\frac{f'(x+)-f'(x-)}{2}\\
&\quad\times\int_0^x(x-s)\frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\}ds+\frac{2\alpha}{\alpha+1}\frac{(f'(x+)-f'(x-))}{2}\int_x^1(s-x)\frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\}ds\\
&\quad-\int_0^x\left(\int_s^x f_x'(u)du\right)\frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\}ds+\int_x^1\left(\int_x^s f_x'(u)du\right)\frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\}ds.
\end{aligned}$$

Hence

$$\begin{aligned}
&|(P_{n,\alpha}f(s))(x,t)-f(x)|\\
&\leqslant\left|\frac{(f'(x+)+\alpha f'(x-))}{\alpha+1}\right|(P_{n,\alpha}|s-x|)(x,t)+\left|f'(x+)-f'(x-)\right||(P_{n,\alpha}|s-x|)(x,t)|\\
&\quad+\left|\int_0^x\left(\int_s^x f_x'(u)du\right)\frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\}ds\right|+\left|\int_x^1\left(\int_x^s f_x'(u)du\right)\frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\}ds\right|.
\end{aligned}$$

Applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&|(P_{n,\alpha}f(s))(x,t)-f(x)|\\
&\leqslant\left|\frac{f'(x+)+\alpha f'(x-)}{\alpha+1}\right|((P_{n,\alpha}(s-x)^2)(x,t))^{1/2}+2\left|\frac{f'(x+)-f'(x-)}{2}\right|((P_{n,\alpha}(s-x)^2)(x,t))^{1/2}\\
&\quad+\left|\int_0^x\left(\int_s^x f_x'(u)du\right)\frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\}ds\right|+\left|\int_x^1\left(\int_x^s f_x'(u)du\right)\frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\}ds\right|.
\end{aligned}\tag{5.6}$$

Now, using Lemma 2 and then integration by parts, we get

$$\begin{aligned}
\int_0^x\left(\int_s^x f_x'(u)du\right)\frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\}ds&=\int_0^x\left(\int_s^x f_x'(u)du\right)\frac{\partial}{\partial s}\{v_{n,\alpha}(x,s,t)\}ds\\
&=-\int_0^x f_x'(s)v_{n,\alpha}(x,s,t)ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\int_0^x\left(\int_s^x f_x'(u)du\right)\frac{\partial}{\partial s}\{M_n^\alpha(x,s,t)\}ds\right|\\
&\leqslant\int_0^x|f_x'(s)|v_{n,\alpha}(x,s,t)ds\\
&\leqslant\int_0^{x-\frac{x}{\sqrt{n}}}|f_x'(s)|v_{n,\alpha}(x,s,t)ds+\int_{x-\frac{x}{\sqrt{n}}}^x|f_x'(s)|v_{n,\alpha}(x,s,t)ds.
\end{aligned}$$

Using $f_x'(x)=0$ and $v_{n,\alpha}(x,s,t)\leqslant 1$, we get

$$\begin{aligned}\int_{x-\frac{x}{\sqrt{n}}}^{x}|f_x'(s)|v_{n,\alpha}(x,s,t)ds &= \int_{x-\frac{x}{\sqrt{n}}}^{x}|f_x'(s)-f_x'(x)|v_{n,\alpha}(x,s,t)ds\\ &\leqslant \int_{x-\frac{x}{\sqrt{n}}}^{x}\bigvee_s^x(f_x')ds \leqslant \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f_x' \int_{x-\frac{x}{\sqrt{n}}}^{x}\bigvee_s^x f_x' ds = \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f_x'.\end{aligned}$$

Again, using $v_{n,\alpha}(x,s,t)\leqslant \frac{C\alpha x}{n(x-s)^2}$ and putting $s=x-\frac{x}{u}$, we get

$$\begin{aligned}\int_0^{x-\frac{x}{\sqrt{n}}}|f_x'(s)|v_{n,\alpha}(x,s,t)ds &\leqslant \frac{C\alpha}{n}\int_1^{\sqrt{n}}\bigvee_{x-\frac{x}{u}}^{x} f_x' du\\ &\leqslant \frac{C\alpha}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x} f_x'.\end{aligned}$$

Hence,

$$\left|\int_0^x\left(\int_s^x f_x'(u)du\right)\frac{\partial}{\partial s}\{M_n^{\alpha}(x,s,t)\}ds\right|\leqslant \frac{C\alpha}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x} f_x' + \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f_x'. \tag{5.7}$$

Again,

$$\begin{aligned}&\left|\int_x^1\left(\int_x^s f_x'(u)du\right)\frac{\partial}{\partial s}\{M_n^{\alpha}(x,s,t)\}ds\right|\\ &=\left|\int_x^z\left(\int_x^s f_x'(u)du\right)\frac{\partial}{\partial s}(1-v_{n,\alpha}(x,s,t))ds\right.\\ &\quad\left.+\int_z^1\left(\int_x^s f_x'(u)du\right)\frac{\partial}{\partial s}(1-v_{n,\alpha}(x,s,t))ds\right|\\ &=\left|\left(\int_x^z f_x'(u)du\right)(1-v_{n,\alpha}(x,z,t))-\int_x^z f_x'(s)(1-v_{n,\alpha}(x,s,t))ds\right.\\ &\quad\left.-\left(\int_x^z f_x'(u)du\right)(1-v_{n,\alpha}(x,z,t))-\int_z^1 f_x'(s)(1-v_{n,\alpha}(x,s,t))ds\right|\\ &=\left|\int_x^z f_x'(s)(1-v_{n,\alpha}(x,s,t))ds+\int_z^1 f_x'(s)(1-v_{n,\alpha}(x,s,t))ds\right|\\ &\leqslant \int_x^z\bigvee_x^s f_x' ds+\frac{C\alpha x}{n}\int_z^1\left(\bigvee_x^s f_x'\right)(s-x)^{-2}ds.\end{aligned}$$

Now, let $z=x+\frac{1-x}{\sqrt{n}}$ and then putting $u=\frac{1-x}{s-x}$, we get

$$\begin{aligned}&\left|\int_x^1\left(\int_x^s f_x'(u)du\right)\frac{\partial}{\partial s}\{M_n^{\alpha}(x,s,t)\}ds\right|\\ &\leqslant \frac{1-x}{\sqrt{n}}\left(\bigvee_x^{x+\frac{1-x}{\sqrt{n}}} f_x'\right)+\frac{C\alpha x}{n}\int_{x+\frac{1-x}{\sqrt{n}}}^1\left(\bigvee_x^s f_x'\right)(s-x)^{-2}ds\end{aligned}$$

$$\leqslant \frac{1-x}{\sqrt{n}}\left(\bigvee_{x}^{x+\frac{1-x}{\sqrt{n}}} f_x'\right) + \frac{C\alpha x}{n}\int_1^{\sqrt{n}}\left(\bigvee_{x}^{x+\frac{1-x}{u}} f_x'\right)(1-x)^{-1}du$$

$$\leqslant \frac{1-x}{\sqrt{n}}\left(\bigvee_{x}^{x+\frac{1-x}{\sqrt{n}}} f_x'\right) + \frac{C\alpha x}{n(1-x)}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x}^{x+\frac{1-x}{k}} f_x'\right) \tag{5.8}$$

Collecting estimates from (5.6–5.8), we get required result. □

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