Journal of
Mathematical
Inequalities
Volume 12, Number 3 (2018), 709–717 doi:10.7153/jmi-2018-12-54

LYAPUNOV–TYPE INEQUALITIES FOR NONLINEAR DIFFERENTIAL EQUATION WITH HILFER FRACTIONAL DERIVATIVE OPERATOR

YOUYU WANG AND QICHAO WANG

(*Communicated by T. Burić*)

Abstract. In this work, we establish Lyapunov-type inequalities for the fractional boundary value problems with Hilfer fractional derivative operator, the results of this paper are new and generalize and improve some early results in the literature.

1. Introduction

The well-known result of Lyapunov [10] states that if $u(t)$ is a nontrivial solution of the differential system

$$\begin{aligned} &u''(t)+r(t)u(t)=0, \quad t\in(a,b),\\ &u(a)=0=u(b), \end{aligned} \tag{1.1}$$

where $r(t)$ is a continuous function defined in $[a,b]$, then

$$\int_a^b |r(t)|dt > \frac{4}{b-a}, \tag{1.2}$$

and the constant 4 cannot be replaced by a larger number.

Since the appearance of Lyapunov's fundamental paper, there are many improvements and generalizations of (1.2) in some literatures. A thorough literature review of continuous and discrete Lyapunov-type inequalities and their applications can be found in the survey articles by Cheng [2], Brown and Hinton [1] and Tiryaki [11].

The study of Lyapunov-type inequalities for the differential equation depends on a fractional differential operator was initiated by Rui A. C. Ferreira [4]. He first obtained a Lyapunov-type inequality when the differential equation depends on the Riemann-Liouville fractional derivative, the main result is as follows.

Mathematics subject classification (2010): 34A40, 26A33, 34B05.
Keywords and phrases: Fractional differential equation, Lyapunov-type inequalities, fractional integral boundary conditions, Hilfer fractional derivative operator.

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Paper JMI-12-54

THEOREM 1.1. *If the following fractional boundary value problem (FBVP)*

$$(D_{a^+}^{\alpha}u)(t)+q(t)u(t)=0, \quad a<t<b,\ 1<\alpha\leqslant 2, \tag{1.3}$$

$$u(a)=0=u(b), \tag{1.4}$$

has a nontrivial solution, where q is a real and continuous function, then

$$\int_a^b |q(s)|ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{1.5}$$

Meanwhile, a Lyapunov-type inequality when the differential equation depends on the Caputo fractional derivative was also obtained by Rui A. C. Ferreira [5].

THEOREM 1.2. *If a nontrivial continuous solution of the fractional boundary value problem (FBVP)*

$$({}^{C}D_{a^+}^{\alpha}u)(t)+q(t)u(t)=0, \quad a<t<b,\ 1<\alpha\leqslant 2, \tag{1.6}$$

$$u(a)=0=u(b), \tag{1.7}$$

exists, where q is a real and continuous function, then

$$\int_a^b |q(s)|ds > \frac{\Gamma(\alpha)\alpha^{\alpha}}{[(\alpha-1)(b-a)]^{\alpha-1}}. \tag{1.8}$$

Recently, M. Jleli and B. Samet [8] investigated Lyapunov-type inequalities for fractional differential equation involving the Caputo fractional derivative under two types of mixed boundary conditions. The results are as follows.

THEOREM 1.3. *If a nontrivial continuous solution of the fractional boundary value problem*

$$({}^{C}D_{a^+}^{\alpha}u)(t)+q(t)u(t)=0, \quad a<t<b,\ 1<\alpha\leqslant 2, \tag{1.9}$$

$$u(a)=u'(b)=0, \tag{1.10}$$

exists, where q is a real and continuous function in [a,b], then

$$\int_a^b (b-s)^{\alpha-2}|q(s)|ds \geqslant \frac{\Gamma(\alpha)}{\max\{\alpha-1,2-\alpha\}(b-a)}. \tag{1.11}$$

THEOREM 1.4. *If a nontrivial continuous solution of the fractional boundary value problem*

$$({}^{C}D_{a^+}^{\alpha}u)(t)+q(t)u(t)=0, \quad a<t<b,\ 1<\alpha\leqslant 2, \tag{1.12}$$

$$u'(a)=u(b)=0, \tag{1.13}$$

exists, where q is a real and continuous function in [a,b], then

$$\int_a^b (b-s)^{\alpha-1}|q(s)|ds \geqslant \Gamma(\alpha). \tag{1.14}$$

Very recently, S. Dhar et al. [3] investigate the equation (1.3) with the following fractional integral boundary conditions:

$$(I_{a^+}^{2-\alpha}u)(a) = 0 = (I_{a^+}^{2-\alpha}u)(b). \tag{1.15}$$

They obtain a series of Lyapunov-type inequalities.

Motivated by the above works, we establish in this paper Lyapunov-type inequalities for the fractional differential equation with Hilfer fractional derivative operator,

$$(D_{a^+}^{\alpha,\beta}u)(t) + q(t)u(t) = 0, \quad a < t < b, \ 1 < \alpha \leqslant 2, \ 0 \leqslant \beta \leqslant 1, \tag{1.16}$$

under the boundary condition

$$(I_{a^+}^{(2-\alpha)(1-\beta)}u)(a) = 0 = u'(b). \tag{1.17}$$

More Cauchy type problems with Hilfer fractional derivative can be found in the articles [13-16].

2. Preliminaries

In this section, we recall the concepts of the Riemann-Liouville fractional integral, the Riemann-Liouville fractional derivative, the Caputo fractional derivative of order $\alpha \geqslant 0$ and the Hilfer fractional derivative of order α $(n-1 < \alpha \leqslant n, \, n \in N)$, and type $0 \leqslant \beta \leqslant 1$.

DEFINITION 2.1. [9] Let $\alpha \geqslant 0$ and f be a real function defined on $[a,b]$. The Riemann-Liouville fractional integral of order α is defined by $(I_{a^+}^0 f) \equiv f$ and

$$(I_{a^+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0, \ t \in [a,b].$$

DEFINITION 2.2. [9] The Riemann-Liouville fractional derivative of order $\alpha \geqslant 0$ is defined by $(D_{a^+}^0 f) \equiv f$ and

$$(D_{a^+}^{\alpha} f)(t) = (D^m I_{a^+}^{m-\alpha} f)(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_a^t (t-s)^{m-\alpha-1} f(s)ds,$$

for $\alpha > 0$, where m is the smallest integer greater or equal to α.

DEFINITION 2.3. [9] The Caputo fractional derivative of order $\alpha \geqslant 0$ is defined by $({}^C D_{a^+}^0 f) \equiv f$ and

$$({}^C D_{a^+}^{\alpha} f)(t) = (I_{a^+}^{m-\alpha} D^m f)(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} f^{(m)}(s)ds,$$

for $\alpha > 0$, where m is the smallest integer greater or equal to α.

DEFINITION 2.4. [6, 7] The Hilfer fractional derivative or generalized Riemann-Liouville fractional derivative of order α $(n-1<\alpha\leqslant n,\ n\in N)$, and type $0\leqslant\beta\leqslant 1$ with respect to t, is defined as

$$(D_{a^+}^{\alpha,\beta}f)(t)=\left(I_{a^+}^{\beta(n-\alpha)}\frac{d^n}{dt^n}\left(I_{a^+}^{(1-\beta)(n-\alpha)}f\right)\right)(t),$$

whenever the right-hand side exists.

REMARK 2.5. In the above definition, type β allows $D_{a^+}^{\alpha,\beta}$ to interpolate continuously between the classical Riemann-Liouville fractional derivative and the Caputo fractional derivative. As in the case $\beta=0$, the definition reduces to the classical Riemann-Liouville fractional derivative and for $\beta=1$, it gives the Caputo fractional derivative.

In [12], the compositional property of Riemann-Liouville fractional integral operator with the Hilfer fractional derivative operator is obtained.

LEMMA 2.6. *Let $f\in L(a,b)$, $n-1<\alpha\leqslant n$, $n\in\mathbb{N}$, $0\leqslant\beta\leqslant 1$, $I_{a^+}^{(n-\alpha)(1-\beta)}f\in AC^k[a,b]$. Then the Riemann-Liouville fractional integral $I_{a^+}^{\alpha}$ and the Hilfer fractional derivative operator $D_{a^+}^{\alpha,\beta}$ are connected by the relation*

$$\left(I_{a^+}^{\alpha}D_{a^+}^{\alpha,\beta}f\right)(t)=f(t)-\sum_{k=0}^{n-1}\frac{(t-a)^{k-(n-\alpha)(1-\beta)}}{\Gamma(k-(n-\alpha)(1-\beta)+1)}\lim_{t\to a^+}\frac{d^k}{dt^k}\left(I_{a^+}^{(n-\alpha)(1-\beta)}f\right)(t).$$

LEMMA 2.7. *For $1<\alpha\leqslant 2$, $0\leqslant\beta\leqslant 1$, we have*

$$(I_{a^+}^{(2-\alpha)(1-\beta)}(s-a)^{-(2-\alpha)(1-\beta)})(t)=\Gamma(1-(2-\alpha)(1-\beta)),$$
$$(I_{a^+}^{(2-\alpha)(1-\beta)}(s-a)^{1-(2-\alpha)(1-\beta)})(t)=(t-a)\Gamma(2-(2-\alpha)(1-\beta)).$$

Proof. By definition, we have

$$\begin{aligned}(I_{a^+}^{(2-\alpha)(1-\beta)}(s-a)^{-(2-\alpha)(1-\beta)})(t)&=\int_a^t\frac{(t-s)^{(2-\alpha)(1-\beta)-1}(s-a)^{-(2-\alpha)(1-\beta)}}{\Gamma((2-\alpha)(1-\beta))}ds\\&=\int_0^1\frac{\gamma^{(2-\alpha)(1-\beta)-1}(1-\gamma)^{-(2-\alpha)(1-\beta)}}{\Gamma((2-\alpha)(1-\beta))}d\gamma\\&=\frac{B((2-\alpha)(1-\beta),1-(2-\alpha)(1-\beta))}{\Gamma((2-\alpha)(1-\beta))}\\&=\Gamma(1-(2-\alpha)(1-\beta)).\end{aligned}$$

Similarly, we also have

$$
\begin{aligned}
(I_{a^+}^{(2-\alpha)(1-\beta)}(s-a)^{1-(2-\alpha)(1-\beta)})(t) &= \int_a^t \frac{(t-s)^{(2-\alpha)(1-\beta)-1}(s-a)^{1-(2-\alpha)(1-\beta)}}{\Gamma((2-\alpha)(1-\beta))}ds \\
&= (t-a)\int_0^1 \frac{\gamma^{(2-\alpha)(1-\beta)-1}(1-\gamma)^{1-(2-\alpha)(1-\beta)}}{\Gamma((2-\alpha)(1-\beta))}d\gamma \\
&= (t-a)\frac{B((2-\alpha)(1-\beta),2-(2-\alpha)(1-\beta))}{\Gamma((2-\alpha)(1-\beta))} \\
&= (t-a)\Gamma(2-(2-\alpha)(1-\beta)). \quad \square
\end{aligned}
$$

3. Main results

We begin by writing problems (1.16)–(1.17) in its equivalent integral form.

LEMMA 3.1. *We have that $u \in C[a,b]$ is a solution to the boundary value problem (1.16)–(1.17) if and only if u satisfies the integral equation*

$$
u(t) = \int_a^b G(t,s)q(s)u(s)ds,
$$

where $G(t,s) = \frac{(\alpha-1)(b-s)^{\alpha-2}H(t,s)}{(\alpha-1+2\beta-\alpha\beta)\Gamma(\alpha)}$ and $H(t,s)$ is given by

$$
H(t,s) = \begin{cases} (b-a)^{(2-\alpha)(1-\beta)}(t-a)^{\alpha-1+2\beta-\alpha\beta} - \frac{\alpha-1+2\beta-\alpha\beta}{\alpha-1}(t-s)^{\alpha-1}(b-s)^{2-\alpha}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad a \leqslant s \leqslant t \leqslant b, \\ (b-a)^{(2-\alpha)(1-\beta)}(t-a)^{\alpha-1+2\beta-\alpha\beta}, \qquad\qquad a \leqslant t \leqslant s \leqslant b. \end{cases} \tag{3.1}
$$

Proof. From Lemma 2.6, $u \in C[a,b]$ is a solution to the boundary value problem (1.16)–(1.17) if and only if

$$
u(t) = c_0\frac{(t-a)^{-(2-\alpha)(1-\beta)}}{\Gamma(1-(2-\alpha)(1-\beta))} + c_1\frac{(t-a)^{1-(2-\alpha)(1-\beta)}}{\Gamma(2-(2-\alpha)(1-\beta))} - \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)u(s)ds, \tag{3.2}
$$

where c_0 and c_1 are some real constants. We apply the operator $I_{a^+}^{(2-\alpha)(1-\beta)}$ to both side of (3.2), we obtain

$$
(I_{a^+}^{(2-\alpha)(1-\beta)}u)(t) = c_0 + c_1(t-a) - \frac{1}{\Gamma(2-2\beta+\alpha\beta)}\int_a^t (t-s)^{1-2\beta+\alpha\beta}q(s)u(s)ds.
$$

By the boundary condition $(I_{a^+}^{(2-\alpha)(1-\beta)}u)(a) = 0$, we can obtain that $c_0 = 0$. Thus we get

$$
u(t) = c_1\frac{(t-a)^{1-(2-\alpha)(1-\beta)}}{\Gamma(2-(2-\alpha)(1-\beta))} - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}q(s)u(s)ds.
$$

The time derivative of the above equation gives

$$u'(t) = c_1[1-(2-\alpha)(1-\beta)]\frac{(t-a)^{-(2-\alpha)(1-\beta)}}{\Gamma(2-(2-\alpha)(1-\beta))} - \frac{\alpha-1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-2}q(s)u(s)ds.$$

The boundary condition $u'(b)=0$ yields

$$c_1 = \frac{(\alpha-1)\Gamma(2-(2-\alpha)(1-\beta))(b-a)^{(2-\alpha)(1-\beta)}}{[1-(2-\alpha)(1-\beta)]\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-2}q(s)u(s)ds.$$

Hence

$$\begin{aligned}
u(t) =& c_1\frac{(t-a)^{1-(2-\alpha)(1-\beta)}}{\Gamma(2-(2-\alpha)(1-\beta))} - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}q(s)u(s)ds \\
=& \frac{(\alpha-1)(b-a)^{(2-\alpha)(1-\beta)}(t-a)^{1-(2-\alpha)(1-\beta)}}{[1-(2-\alpha)(1-\beta)]\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-2}q(s)u(s)ds \\
& - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}q(s)u(s)ds \\
=& \int_a^b G(t,s)q(s)u(s)ds.
\end{aligned}$$

which concludes the proof. $\square$

LEMMA 3.2. *The function H defined in Lemma 3.1 satisfies the following property:*

$$|H(t,s)| \leqslant \frac{b-a}{\alpha-1}\max\{\alpha-1, 2\beta-\alpha\beta\}, \tag{3.3}$$

where $(t,s)\in[a,b]\times[a,b]$.

Proof. Obviously, $H(t,s)$ is an increasing function of t for $a\leqslant t\leqslant s\leqslant b$. For $a\leqslant s\leqslant t\leqslant b$, by the relation $(b-s)(t-a)-(b-a)(t-s)=(b-t)(s-a)\geqslant 0$, we have $\frac{b-a}{t-a}\leqslant\frac{b-s}{t-s}$ and

$$\left(\frac{b-a}{t-a}\right)^{(2-\alpha)(1-\beta)} \leqslant \left(\frac{b-s}{t-s}\right)^{(2-\alpha)(1-\beta)} \leqslant \left(\frac{b-s}{t-s}\right)^{2-\alpha},$$

therefore,

$$\frac{\partial H}{\partial t} = (\alpha-1+2\beta-\alpha\beta)\left[\left(\frac{b-a}{t-a}\right)^{(2-\alpha)(1-\beta)} - \left(\frac{b-s}{t-s}\right)^{2-\alpha}\right] \leqslant 0.$$

So, for a given s, $H(t,s)$ is a decreasing function of $t\in[s,b]$. Hence,

$$|H(t,s)| \leqslant \max\{H(s,s), |H(b,s)|\}.$$

While

$$H(s,s) = (b-a)^{(2-\alpha)(1-\beta)}(s-a)^{\alpha-1+2\beta-\alpha\beta} \leqslant b-a,$$
$$|H(b,s)| = \left|(b-a) - \frac{\alpha-1+2\beta-\alpha\beta}{\alpha-1}(b-s)\right| \leqslant \max\left\{b-a, \frac{2\beta-\alpha\beta}{\alpha-1}(b-a)\right\},$$

which concludes the proof. □

Now, we are ready to prove our Lyapunov-type inequality.

THEOREM 3.3. *If a nontrivial continuous solution of the fractional boundary value problem*

$$(D^{\alpha,\beta}_{a^+}u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \leqslant 2,\ 0 \leqslant \beta \leqslant 1,$$
$$(I^{(2-\alpha)(1-\beta)}_{a^+}u)(a) = 0 = u'(b),$$

exists, where q *is a real and continuous function in* $[a,b]$*, then*

$$\int_a^b (b-s)^{\alpha-2}|q(s)|ds \geqslant \frac{(\alpha-1+2\beta-\alpha\beta)\Gamma(\alpha)}{(b-a)\max\{\alpha-1, 2\beta-\alpha\beta\}}. \tag{3.4}$$

Proof. Let $B = C[a,b]$ be the Banach space endowed with norm $\|u\| = \sup_{t\in[a,b]} |u(t)|$. It follows from Lemma 3.1 that a solution u to the boundary value problem satisfies the integral equation

$$u(t) = \int_a^b G(t,s)q(s)u(s)ds, \quad t \in [a,b].$$

Now, an application Lemma 3.2 yields

$$\|u\| \leqslant \frac{\alpha-1}{(\alpha-1+2\beta-\alpha\beta)\Gamma(\alpha)} \cdot \frac{b-a}{\alpha-1} \max\{\alpha-1, 2\beta-\alpha\beta\} \int_a^b (b-s)^{\alpha-2}|q(s)|ds\|u\|,$$

which implies that (3.4) holds. □

Let $\beta = 0$ in Theorem 3.3, we have the following result.

COROLLARY 3.4. *If a nontrivial continuous solution of the fractional boundary value problem*

$$(D^{\alpha}_{a^+}u)(t) + q(t)u(t) = 0, \quad a < t < b,\ \ 1 < \alpha \leqslant 2,$$
$$(I^{2-\alpha}_{a^+}u)(a) = 0 = u'(b),$$

exists, where q *is a real and continuous function in* $[a,b]$*, then*

$$\int_a^b (b-s)^{\alpha-2}|q(s)|ds \geqslant \frac{\Gamma(\alpha)}{b-a}. \tag{3.5}$$

REMARK 3.5. Let $\beta = 1$ in Theorem 3.3, then we obtain Theorem 1.3.

REMARK 3.6. In the proof of Lemma 3.1, we find that if the boundary condition $(I_{a^+}^{(2-\alpha)(1-\beta)}u)(a) = 0$ in (1.17) changed as $u(a) = 0$, the conclusion is also holds.

THEOREM 3.7. *If a nontrivial continuous solution of the fractional boundary value problem*

$$\begin{aligned} &(D_{a^+}^{\alpha,\beta}u)(t) + q(t)u(t) = 0, \quad a < t < b, \ 1 < \alpha \leqslant 2, \ 0 \leqslant \beta \leqslant 1, \\ &u(a) = 0 = u'(b), \end{aligned}$$

exists, where q is a real and continuous function in $[a,b]$, then

$$\int_a^b (b-s)^{\alpha-2}|q(s)|ds \geqslant \frac{(\alpha - 1 + 2\beta - \alpha\beta)\Gamma(\alpha)}{(b-a)\max\{\alpha - 1, 2\beta - \alpha\beta\}}. \tag{3.6}$$

Let $\beta = 0$ in Theorem 3.7, then we obtain

COROLLARY 3.8. *If a nontrivial continuous solution of the fractional boundary value problem*

$$\begin{aligned} &(D_{a^+}^{\alpha}u)(t) + q(t)u(t) = 0, \quad a < t < b, \ 1 < \alpha \leqslant 2, \\ &u(a) = 0 = u'(b), \end{aligned}$$

exists, where q is a real and continuous function in $[a,b]$, then

$$\int_a^b (b-s)^{\alpha-2}|q(s)|ds \geqslant \frac{\Gamma(\alpha)}{b-a}. \tag{3.7}$$

REFERENCES

[1] R. C. BROWN, D. B. HINTON, *Lyapunov inequalities and their applications, in Survey on Classical Inequalities*, T. M. Rassias, Ed., Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000, 1–25.

[2] S. CHENG, *Lyapunov inequalities for differential and difference equations*, Fasc. Math. **23** (1991) 25–41.

[3] S. DHAR, Q. KONG AND M. MCCABE, *Fractional boundary value problems and Lyapunov-type inequalities with fractional integral boundary conditions*, Electron. J. Qual. Theory Differ. Equ. (2016), no. 43, 1–16.

[4] R. A. C. FERREIRA, *A Lyapunov-type inequality for a fractional boundary value problem*, Fract. Calc. Appl. Anal. **16**, no. 4 (2013), 978–984.

[5] R. A. C. FERREIRA, *On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function*, J. Math. Anal. Appl. **412**, no. 2 (2014), 1058–1063.

[6] R. HILFER, *Fractional calculus and regular variation in thermodynamics*, in: Applications of Fractional Calculus in Physics, (Ed. R. Hilfer), World Scientific, Singapore (2000).

[7] R. HILFER, Y. LUCHKO AND Z. TOMOVSKI, *Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives*, Fract. Calc. Appl. Anal. **3**, 12 (2009) 299–318.

[8] M. JLELI AND B. SAMET, *Lyapunov-type inequalities for a fractional differential equation with mixed boundary conditions*, Math. Inequal. Appl. **18**, no. 2 (2015), 443–451.

[9] A. A. KILBAS, H. M. SRIVASTAVA AND J. J. TRUJILLO, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies 204 Elsevier, Amsterdam, The Netherlands, 2006.

[10] A. M. LYAPUNOV, *Probleme général de la stabilité du mouvement*, (French Translation of a Russian paper dated 1893), Ann. Fac. Sci. Univ. Toulouse 2 (1907) 27–247 (Reprinted as Ann. Math. Studies, no. 17, Princeton Univ. Press, Princeton, NJ, USA, 1947).

[11] A. TIRYAKI, *Recent development of Lyapunov-type inequalities*, Adv. Dyn. Syst. Appl. **5** no. 2 (2010), 231–248.

[12] Z. TOMOVSKI, *Generalized Cauchy type problems for nonlinear fractional differential equations with composite fractional derivative operator*, Nonlinear Analysis, **7**, 75 (2012) 3364–3384.

[13] H. M. SRIVASTAVA, Z. TOMOVSKI, *Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel*, Appl. Math. Comput. **211** (2009) 198–210.

[14] T. SANDEV, R. METZLER, Z. TOMOVSKI, *Fractional diffusion equation with a generalized Riemann-Liouville time fractional derivative*, J. Phys. A: Math. Theor. **44** (25) (2011), ArticleID 255203.

[15] Z. TOMOVSKI, T. SANDEV, R. METZLER, J. L. A. DUBBELDAM, *Generalized space-time fractional diffusion equation with composite fractional time derivative*, Physica A Statistical Mechanics and Its Applications **391** (8) (2012) 2527–2542.

[16] Z. TOMOVSKI, R. HILFER, H. M. SRIVASTAVA, *Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions*, Integral Transf. Spec. Funct. **21** (11–12) (2010) 797–814.

(Received October 20, 2017)

Youyu Wang
Department of Mathematics
Tianjin University of Finance and Economics
Tianjin 300222, P. R. China
e-mail: wang_youyu@163.com

Qichao Wang
Department of Mathematics
Tianjin University of Finance and Economics
Tianjin 300222, P. R. China
e-mail: qichaowang@163.com