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GENERALIZATION OF MAJORIZATION THEOREM–II

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Abstract. This paper begins with a rigorous study of convex functions with the goal of developing the majorization theorems in the form of Taylor representation. In this paper, some new types of Green functions, introduced by Pečarić-Agarwal-Butt-Mehmood (2017) [11] and Taylor's formula, are used to obtain the identities related to majorization type inequalities. We present the monotonicity of the linear functionals deduced from our generalized results by using the family of $(n+1)$-convex functions at a point. We give upper bounds and mean value theorems for obtained generalized identities. At the end, we explore some applications.

1. Introduction and preliminaries

In [1], Pečarić et al. (2015) gave the results about majorization theorem for the classical Green function and in this paper, we give the results of majorization theorem for the newly defined four different Green functions (introduced in [11]) which are continuous as well as convex. One can see that this is the version-II which is more generalize version of majorization theorem discussed as in [1].

Today inequalities play a significant role in all fields of mathematics and they present a very active and attractive field of research. Majorization theorem for convex functions and the classical concept of majorization, due to Hardy et al. [7], have numerous applications in different fields of applied sciences (see the monograph [10]). In recent times, Majorization type results has attracted the interest of several mathematicians which resulting with interesting generalizations and applications (see for example [1]–[3], [12]–[14]). The main purpose of this paper is to extend majorization theorem to convex function for higher order, i.e. to n-convex functions which are in a special case convex in the usual sense.

Taylor's formula may be viewed as being an extended form of the Mean Value Theorem. The method involving the Taylors's formula is one of the most widely used methods for approximating a function by polynomials and provides an estimate of the error involved in the approximation. The techniques that we use are based on the classical real analysis and an application of Taylor's formula with the integral remainder which we introduce in the sequel.

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THEOREM 1. ([17]) (Taylor's Formula) *Let $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and n be a positive integer. Then for all $x \in [\vartheta_1, \vartheta_2]$,*

$$f(x) = P_{n-1}(f;a,x) + R_{n-1}(f;a,x), \tag{1}$$

is called Taylor's formula at the point $a \in [\vartheta_1, \vartheta_2]$. Here $P_{n-1}(f;a,x)$ is Taylor's polynomial of degree $n-1$, which is given by

$$P_{n-1}(f;a,x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k, \tag{2}$$

and the remainder is given by the formula

$$R_{n-1}(f;a,x) = \frac{(-1)^{n-1}}{(n-1)!}\int_a^x f^{(n)}(t)(t-x)^{n-1}dt = \frac{1}{(n-1)!}\int_a^x f^{(n)}(t)(x-t)^{n-1}dt. \tag{3}$$

Due to absolute continuity of $f^{(n-1)}$ on $[\vartheta_1, \vartheta_2]$, its derivative $f^{(n)}$ exists as an L^1-function. Two other important expressions for the remainder term $R_{n-1}(f;a,x)$ are given by:

$$R_{n-1}(f;a,x) = \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}(x-a), \tag{4}$$

which is the Cauchy form of the remainder and

$$R_{n-1}(f;a,x) = \frac{f^{(n)}(t)}{n!}(x-a)^n, \tag{5}$$

is the Lagrange form of the remainder.

Majorization. For fixed $l \geqslant 2$, let $\mathbf{x} = (x_1, \ldots, x_l)$, $\mathbf{y} = (y_1, \ldots, y_l)$ be non-increasing sequences of real numbers. Then [15, p. 319] we say that $\mathbf{y}$ is majorized by $\mathbf{x}$ or $\mathbf{x}$ majorizes $\mathbf{y}$, in symbol, $\mathbf{x} \succ \mathbf{y}$, if we have

$$\sum_{i=1}^{j} y_i \leqslant \sum_{i=1}^{j} x_i, \tag{6}$$

for $j = 1, 2, \ldots, l-1$ and

$$\sum_{i=1}^{l} x_i = \sum_{i=1}^{l} y_i. \tag{7}$$

The following theorem is the Classical Majorization Theorem given in the monograph by Marshall-Olkin-Arnold [10, p. 11] (see also [15, p. 320]):

THEOREM 2. (Classical Majorization Theorem) *Let $\boldsymbol{x} = (x_1, \ldots, x_l)$, $\boldsymbol{y} = (y_1, \ldots, y_l)$ be two non-increasing real l-tuples such that $x_i, y_i \in [\vartheta_1, \vartheta_2] \subset \mathbb{R}$ for $i = 1, \ldots, l$. Then $\mathbf{x}$ majorizes $\mathbf{y}$ if and only if for every continuous convex function $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$, the following inequality holds*

$$\sum_{i=1}^{l} f(y_i) \leqslant \sum_{i=1}^{l} f(x_i). \tag{8}$$

The following theorem is a generalization of Theorem 2 known as Weighted Majorization Theorem and is proved by Fuchs in [6] (see also [15, p. 323]):

THEOREM 3. (Weighted Majorization Theorem) *Let* $\boldsymbol{x} = (x_1, \ldots, x_l)$, $\boldsymbol{y} = (y_1, \ldots, y_l)$ *be two non-increasing real* l*-tuples such that* x_i, $y_i \in [\vartheta_1, \vartheta_2]$ *for* $i = 1, \ldots, l$. *Let* $\mathbf{p} = (p_1, \ldots, p_l)$ *be a real* l*-tuple such that*

$$\sum_{i=1}^{j} p_i y_i \leqslant \sum_{i=1}^{j} p_i x_i, \tag{9}$$

for $j = 1, 2, \ldots, l-1$ *and*

$$\sum_{i=1}^{l} p_i y_i = \sum_{i=1}^{l} p_i x_i. \tag{10}$$

Then for every continuous convex function $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$, *we have the following inequality*

$$\sum_{i=1}^{l} p_i f(y_i) \leqslant \sum_{i=1}^{l} p_i f(x_i). \tag{11}$$

REMARK 1. If the assumptions of the Theorem 3 are satisfied, then

$$\mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{p}, f(.)) := \sum_{i=1}^{l} p_i f(x_i) - \sum_{i=1}^{l} p_i f(y_i) \geqslant 0, \tag{12}$$

for f be continuous and convex function. Also $\mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{p}, f(.)) = 0$ when $f(x) =$ constant or $f(x)$ is a linear function.

This paper arrange in this manner: in Section 2, we present some technical lemmas. In Section 3, we give several generalized majorization type identities via using Taylor's formula and newly defined Green functions. In Section 4, we present the monotonicity of the linear functionals deduced from our generalized results by using the family of $(n+1)$-convex functions at a point. In Section 5, we give upper bounds like Grüss and Ostrowski-type inequalities for obtained generalized identities. In Section 6, we present the Cauchy mean value theorems and n-exponential convexity for positive linear functionals deduced from our results. At the end in Section 7, we give some applications for Ostrowski-type upper bounds.

2. Some technical lemmas

In this section we present two technical lemmas, first lemma gives us identities which will be very useful for us to obtain main results and the second one gives the equivalent statements of majorization inequality between continuous convex functions and newly defined Green functions.

Let $[\vartheta_1, \vartheta_2] \subset \mathbb{R}$ and $d = 1,2,3,4$. Recently (2017), Pečarić *et al.* introduce some new types of Green functions, $G_d : [\vartheta_1, \vartheta_2] \times [\vartheta_1, \vartheta_2] \to \mathbb{R}$, which are defined as follows:

$$G_1(u,v) = \begin{cases} (\vartheta_1 - v), & \vartheta_1 \leqslant v \leqslant u, \\ (\vartheta_1 - u), & u \leqslant v \leqslant \vartheta_2. \end{cases} \tag{13}$$

$$G_2(u,v) = \begin{cases} (u - \vartheta_2), & \vartheta_1 \leqslant v \leqslant u, \\ (v - \vartheta_2), & u \leqslant v \leqslant \vartheta_2. \end{cases} \tag{14}$$

$$G_3(u,v) = \begin{cases} (u - \vartheta_1), & \vartheta_1 \leqslant v \leqslant u, \\ (v - \vartheta_1), & u \leqslant v \leqslant \vartheta_2. \end{cases} \tag{15}$$

$$G_4(u,v) = \begin{cases} (\vartheta_2 - v), & \vartheta_1 \leqslant v \leqslant u, \\ (\vartheta_2 - u), & u \leqslant v \leqslant \vartheta_2. \end{cases} \tag{16}$$

LEMMA 1. *Let $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ such that $f \in C^2([\vartheta_1, \vartheta_2])$ and G_d $(d = 1,2,3,4)$ be Green functions as defined in (13), (14), (15) and (16). Then we have the following identities.*

$$f(u) = f(\vartheta_1) + (u - \vartheta_1) f'(\vartheta_2) + \int_{\vartheta_1}^{\vartheta_2} G_1(u,v) f''(v) dv, \tag{17}$$

$$f(u) = f(\vartheta_2) + (u - \vartheta_2) f'(\vartheta_1) + \int_{\vartheta_1}^{\vartheta_2} G_2(u,v) f''(v) dv, \tag{18}$$

$$f(u) = f(\vartheta_2) - (\vartheta_2 - \vartheta_1) f'(\vartheta_2) + (u - \vartheta_1) f'(\vartheta_1) + \int_{\vartheta_1}^{\vartheta_2} G_3(u,v) f''(v) dv, \tag{19}$$

$$f(u) = f(\vartheta_1) + (\vartheta_2 - \vartheta_1) f'(\vartheta_1) - (\vartheta_2 - u) f'(\vartheta_2) + \int_{\vartheta_1}^{\vartheta_2} G_4(u,v) f''(v) dv. \tag{20}$$

LEMMA 2. *Let $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ be a continuous convex function on the interval $[\vartheta_1, \vartheta_2]$ and $\boldsymbol{x} = (x_1, \ldots, x_l)$, $\boldsymbol{y} = (y_1, \ldots, y_l)$ and $\boldsymbol{p} = (p_1, \ldots, p_l)$ be l-tuples such that $x_i, y_i \in [\vartheta_1, \vartheta_2]$ and $p_i \in \mathbb{R}$ for $i = 1,2,\ldots,l$, which satisfy the condition*

$$\sum_{i=1}^{l} p_i y_i = \sum_{i=1}^{l} p_i x_i. \tag{21}$$

If we define G_d $(d = 1,2,3,4)$ as in (13), (14), (15) and (16), then we have following equivalent statements.

(i) For every continuous convex function $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$, we have

$$\sum_{i=1}^{l} p_i f(y_i) \leqslant \sum_{i=1}^{l} p_i f(x_i). \tag{22}$$

(ii) For all $v \in [\vartheta_1, \vartheta_2]$*, we have*

$$\sum_{i=1}^{l} p_i G_d(y_i, v) \leqslant \sum_{i=1}^{l} p_i G_d(x_i, v). \tag{23}$$

Moreover, if we change the sign of inequality in both inequalities (22) *and* (23)*, then the above result still holds.*

Proof. The scheme of proof is similar for each $d = 1,2,3,4$, therefore we will only give the proof for $d = 4$.

$(i) \Rightarrow (ii)$: Let statement (i) holds. As the function $G_4 : [\vartheta_1, \vartheta_2] \times [\vartheta_1, \vartheta_2] \to \mathbb{R}$ is convex and continuous, so it will satisfy the condition (22), i.e.,

$$\sum_{i=1}^{l} p_i G_4(y_i, v) \leqslant \sum_{i=1}^{l} p_i G_4(x_i, v). \tag{24}$$

$(ii) \Rightarrow (i)$: Let $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ be a convex function such that $f \in C^2([\vartheta_1, \vartheta_2])$. Further, assume that the statement (ii) holds. Then by Lemma 1, we have

$$f(x_i) = f(\vartheta_1) + (\vartheta_2 - \vartheta_1) f'(\vartheta_1) - (\vartheta_2 - x_i) f'(\vartheta_2) + \int_{\vartheta_1}^{\vartheta_2} G_4(x_i, v) f''(v) dv, \tag{25}$$

$$f(y_i) = f(\vartheta_1) + (\vartheta_2 - \vartheta_1) f'(\vartheta_1) - (\vartheta_2 - y_i) f'(\vartheta_2) + \int_{\vartheta_1}^{\vartheta_2} G_4(y_i, v) f''(v) dv. \tag{26}$$

From (25) and (26), we get

$$\begin{aligned} \sum_{i=1}^{l} p_i f(x_i) - \sum_{i=1}^{l} p_i f(y_i) = & - \sum_{i=1}^{l} p_i (\vartheta_2 - x_i) f'(\vartheta_2) + \sum_{i=1}^{l} p_i (\vartheta_2 - y_i) f'(\vartheta_2) \\ & + \int_{\vartheta_1}^{\vartheta_2} \left[\sum_{i=1}^{l} p_i G_4(x_i, v) - \sum_{i=1}^{l} p_i G_4(y_i, v) \right] f''(v) dv. \end{aligned} \tag{27}$$

Using (21), we have

$$\sum_{i=1}^{l} p_i f(x_i) - \sum_{i=1}^{l} p_i f(y_i) = \int_{\vartheta_1}^{\vartheta_2} \left[\sum_{i=1}^{l} p_i G_4(x_i, v) - \sum_{i=1}^{l} p_i G_4(y_i, v) \right] f''(v) dv. \tag{28}$$

As f is convex function, therefore $f''(v) \geqslant 0$ for all $v \in [\vartheta_1, \vartheta_2]$. Hence using (23) in (28), we get (22).

Note that the condition for the existence of second derivative of f is not necessary ([15, p. 172]). As it is possible to approximate uniformly a continuous convex function by convex polynomials, so we can directly eliminate this differentiablity condition. □

3. Majorized identities via Taylor's formula

We start this section with generalization of majorization theorem in the sense of Taylor representation which seem interesting and useful:

THEOREM 4. *Let $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geqslant 3$ and let $\boldsymbol{x} = (x_1, \ldots, x_l)$, $\boldsymbol{y} = (y_1, \ldots, y_l)$ and $\boldsymbol{p} = (p_1, \ldots, p_l)$ be l-tuples such that $x_i, y_i \in [\vartheta_1, \vartheta_2]$ and $p_i \in \mathbb{R}$ for $i = 1, 2, \ldots, l$. If we define $G_d\,(d = 1, 2, 3, 4)$ as in (13), (14), (15) and (16), then*

$$
\begin{aligned}
&\mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{p}, f(.)) \\
&= f'(\xi_d) \sum_{i=1}^{l} p_i(x_i - y_i) + \sum_{k=0}^{n-3} \frac{f^{(k+2)}(\vartheta_1)}{k!} \int_{\vartheta_1}^{\vartheta_2} \mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{p}, G_d(., v))\,(v - \vartheta_1)^k dv \\
&\quad + \frac{1}{(n-3)!} \int_{\vartheta_1}^{\vartheta_2} \left(\int_{u}^{\vartheta_2} \mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{p}, G_d(., v))\,(v - u)^{n-3} dv \right) f^{(n)}(u) du,
\end{aligned} \tag{29}
$$

and

$$
\begin{aligned}
&\mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{p}, f(.)) \\
&= f'(\xi_d) \sum_{i=1}^{l} p_i(x_i - y_i) + \sum_{k=0}^{n-3} \frac{(-1)^k f^{(k+2)}(\vartheta_2)}{k!} \int_{\vartheta_1}^{\vartheta_2} \mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{p}, G_d(., v))\,(\vartheta_2 - v)^k dv \\
&\quad - \frac{1}{(n-3)!} \int_{\vartheta_1}^{\vartheta_2} \left(\int_{\vartheta_1}^{u} \mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{p}, G_d(., v))\,(v - u)^{n-3} dv \right) f^{(n)}(u) du,
\end{aligned} \tag{30}
$$

where $\xi_1, \xi_4 = \vartheta_2$, $\xi_2, \xi_3 = \vartheta_1$ and $G_d\,(d = 1, 2, 3, 4)$.

Proof. The scheme of proof is similar for each $d = 1, 2, 3, 4$, therefore we will only give the proof for $d = 4$. From (27), we have

$$
\mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{p}, f(.)) = \sum_{i=1}^{l} p_i(x_i - y_i) f'(\vartheta_2) + \int_{\vartheta_1}^{\vartheta_2} \mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{p}, G_4(., v))\, f''(v) dv. \tag{31}
$$

Now using Taylor's formula (1) for the function f'' at point ϑ_1 and replacing n by $n-2$ $(n \geqslant 3)$, we have

$$
f''(v) = \sum_{k=0}^{n-3} \frac{f^{(k+2)}(\vartheta_1)}{k!} (v - \vartheta_1)^k + \frac{1}{(n-3)!} \int_{\vartheta_1}^{v} f^{(n)}(u)(v - u)^{n-3} du. \tag{32}
$$

Similarly, Taylor's formula on the function f'' at point ϑ_2 and replacing n by $n-2$ $(n \geqslant 3)$, we get

$$
f''(v) = \sum_{k=0}^{n-3} \frac{f^{(k+2)}(\vartheta_2)}{k!} (v - \vartheta_2)^k - \frac{1}{(n-3)!} \int_{v}^{\vartheta_2} f^{(n)}(u)(v - u)^{n-3} du. \tag{33}
$$

Using (32) in (31), we have

$$\begin{aligned}
&\mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},f(.))\\
&= f'(\vartheta_2)\sum_{i=1}^{l} p_i(x_i-y_i) + \sum_{k=0}^{n-3}\frac{f^{(k+2)}(\vartheta_1)}{k!}\int_{\vartheta_1}^{\vartheta_2}\mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},G_4(.,v))(v-\vartheta_1)^k dv\\
&\quad + \frac{1}{(n-3)!}\int_{\vartheta_1}^{\vartheta_1}\mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},G_4(.,v))\left(\int_{\vartheta_1}^{v} f^{(n)}(u)(v-u)^{n-3}du\right)dv. \qquad (34)
\end{aligned}$$

Now by using Fubini's theorem in (34), we get (29). In similar way, we can find (30), by using (33) in (31). □

The following is an application of the previous theorem which is in fact generalization of majorization inequality for n-convex functions.

COROLLARY 1. *Let $f:[\vartheta_1,\vartheta_2]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\geqslant 3$ and let $\boldsymbol{x}=(x_1,\ldots,x_l)$, $\boldsymbol{y}=(y_1,\ldots,y_l)$ and $\boldsymbol{p}=(p_1,\ldots,p_l)$ be l-tuples such that $x_i,y_i\in[\vartheta_1,\vartheta_2]$ and $p_i\in\mathbb{R}$ for $i=1,2,\ldots,l$. Define G_d $(d=1,2,3,4)$ as in (13), (14), (15) and (16).*

(i) If f is n-convex and

$$\int_{u}^{\vartheta_2}\mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},G_d(.,v))(v-u)^{n-3}dv \geqslant 0, \quad u\in[\vartheta_1,\vartheta_2], \qquad (35)$$

then

$$\begin{aligned}
&\mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},f(.)) - f'(\xi_d)\sum_{i=1}^{l}p_i(x_i-y_i)\\
&\geqslant \sum_{k=0}^{n-3}\frac{f^{(k+2)}(\vartheta_2)}{k!}\int_{\vartheta_1}^{\vartheta_2}\mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},G_d(.,v))(v-\vartheta_1)^k dv. \qquad (36)
\end{aligned}$$

(ii) If f is n-convex and

$$\int_{\vartheta_1}^{u}\mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},G_d(.,v))(v-u)^{n-3}dv \leqslant 0, \quad u\in[\vartheta_1,\vartheta_2], \qquad (37)$$

then

$$\begin{aligned}
&\mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},f(.)) - f'(\xi_d)\sum_{i=1}^{l}p_i(x_i-y_i)\\
&\geqslant \sum_{k=0}^{n-3}\frac{(-1)^k f^{(k+2)}(\vartheta_2)}{k!}\int_{\vartheta_1}^{\vartheta_2}\mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},G_d(.,v))(\vartheta_2-v)^k dv, \qquad (38)
\end{aligned}$$

where $\xi_1,\xi_4=\vartheta_2$ and $\xi_2,\xi_3=\vartheta_1$.

Proof. By the n-convexity of the function f, we can assume without loss of generality that f is n-times differentiable and $f^{(n)} \geqslant 0$ (see [15, p. 16 and p. 293]). So using (29) and (30), we can have (36) and (38) respectively. $\square$

The following corollary gives the generalization of classical majorization theorem.

COROLLARY 2. *Let $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geqslant 3$ and let $\boldsymbol{x} = (x_1, \ldots, x_l)$, $\boldsymbol{y} = (y_1, \ldots, y_l)$ be l-tuples such that $x_i, y_i \in [\vartheta_1, \vartheta_2]$ for $i = 1, 2, \ldots, l$ and $\boldsymbol{x} \succ \boldsymbol{y}$. Define G_d $(d = 1, 2, 3, 4)$ as in (13), (14), (15) and (16).*

(i) If f is n-convex and then

$$\mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{1}, f(.)) \geqslant \sum_{k=0}^{n-3} \frac{f^{(k+2)}(\vartheta_1)}{k!} \int_{\vartheta_1}^{\vartheta_2} \mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{1}, G_d(., v))(v - \vartheta_1)^k dv. \tag{39}$$

(ii) If inequality (39) is satisfied and

$$\digamma_1(.) = \sum_{k=0}^{n-3} \frac{f^{(k+2)}(\vartheta_1)}{k!} \int_{\vartheta_1}^{\vartheta_2} G_d(., v)(v - \vartheta_1)^k dv, \tag{40}$$

is convex then the right hand side of (39) is non negative, i.e., (8) is satisfied.

(iii) If f is n-convex, where n is even, then

$$\mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{1}, f(.)) \geqslant \sum_{k=0}^{n-3} \frac{(-1)^k f^{(k+2)}(\vartheta_2)}{k!} \int_{\vartheta_1}^{\vartheta_2} \mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{1}, G_d(., v))(\vartheta_2 - v)^k dv. \tag{41}$$

(iv) If inequality (41) is satisfied and

$$\digamma_2(.) = \sum_{k=0}^{n-3} \frac{(-1)^k f^{(k+2)}(\vartheta_2)}{k!} \int_{\vartheta_1}^{\vartheta_2} G_d(., v)(\vartheta_2 - v)^k dv, \tag{42}$$

is convex then the right hand side of (41) is non negative, i.e., (8) is satisfied.

(v) If f is n-convex, where n is odd, then

$$\mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{1}, f(.)) \leqslant \sum_{k=0}^{n-3} \frac{(-1)^k f^{(k+2)}(\vartheta_2)}{k!} \int_{\vartheta_1}^{\vartheta_2} \mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{1}, G_d(., v))(\vartheta_2 - v)^k dv. \tag{43}$$

(vi) If the function $\digamma_2$ which is defined in (42), is concave and the inequality (43) is satisfied, then the right hand side of (43) is non positive, i.e., reverse inequality in (8) is satisfied.

where $\mathbf{1} = (1,1,\ldots,1)$ *is l-tuple.*

Proof. (i): Note that for $v \in [u, \vartheta_2]$, we have $(v-u)^{n-3} \geqslant 0$. Given that $\mathbf{x}$ majorizes $\mathbf{y}$, so (7) holds. Moreover G_d is continuous as well as convex, for $d = 1,2,3,4$, therefore by using Theorem 2, we can write

$$\mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{1}, G_d(.,v)) \geqslant 0.$$

Thus (35) holds for $p_i = 1\,(i = 1,2,\ldots,l)$. Hence by using Theorem 1, we can deduce inequality (39).

(ii): Right hand side of the inequality (39) can be written as

$$\sum_{i=1}^{l} \mathcal{F}_1(x_i) - \sum_{i=1}^{l} \mathcal{F}_1(y_i).$$

As $\mathcal{F}_1$ is convex, so by applying majorization theorem we note that the right hand side of (39) is non negative.

Remaining parts can also be proved in similar way. □

The following corollary gives the generalization of Fuchs's majorization theorem.

COROLLARY 3. *Let* $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ *be such that* $f^{(n-1)}$ *is absolutely continuous for some* $n \geqslant 3$. *Let* $\boldsymbol{x} = (x_1,\ldots,x_l)$, $\boldsymbol{y} = (y_1,\ldots,y_l)$ *be non increasing l-tuples and* $\boldsymbol{p} = (p_1,\ldots,p_l)$ *be l-tuple such that* $x_i, y_i \in [\vartheta_1, \vartheta_2]$ *and* $p_i \in \mathbb{R}$ *for* $i = 1,2,\ldots,l$, *which satisfy conditions* (9) *and* (10). *Define* $G_d\,(d = 1,2,3,4)$ *as in* (13), (14), (15) *and* (16).

(i) If f *is* n*-convex and then*

$$\mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{p}, f(.)) \geqslant \sum_{k=0}^{n-3} \frac{f^{(k+2)}(\vartheta_1)}{k!} \int_{\vartheta_1}^{\vartheta_2} \mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{p}, G_d(.,v))(v-\vartheta_1)^k dv. \tag{44}$$

(ii) If the inequality (44) *is satisfied and the function* $\mathcal{F}_1$ *which defined in* (40), *is convex then the right hand side of* (44) *is non negative, i.e.,* (11) *is satisfied.*

(iii) If f *is* n*-convex, where* n *is even, then*

$$\mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{p}, f(.)) \geqslant \sum_{k=0}^{n-3} \frac{(-1)^k f^{(k+2)}(\vartheta_2)}{k!} \int_{\vartheta_1}^{\vartheta_2} \mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{p}, G_d(.,v))(\vartheta_2 - v)^k dv. \tag{45}$$

(iv) If the inequality (45) *is satisfied and the function* $\mathcal{F}_2$ *which defined in* (42), *is convex then the right hand side of* (45) *is non negative, i.e.,* (11) *is satisfied.*

(v) If f is n-convex, where n is odd, then

$$\mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},f(.)) \leqslant \sum_{k=0}^{n-3} \frac{(-1)^k f^{(k+2)}(\vartheta_2)}{k!} \int_{\vartheta_1}^{\vartheta_2} \mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},G_d(.,v))\,(\vartheta_2-v)^k dv. \tag{46}$$

(vi) If the function F_2 which is defined in (42), is concave and the inequality (46) is satisfied, then the right hand side of (46) is non positive, i.e., reverse inequality in (11) is satisfied.

Proof. Following the proof of Corollary 2, one can prove the result easily. □

We are ending this section with the following remark:

REMARK 2. We can give the majorization theorems in the integral version of Theorem 4, Corollary 1, Corollary 2 and Corolary 3 like as given in [1].

4. Applications to $(n+1)$-convex functions at a point

In this section, we prove that the linear functionals deduce from the generalized identity (36) in the previous section constructed in two different intervals are monotonic by using the family of $(n+1)$-convex functions at a point, recently introduced by Peĉarić-Praljok-Witkowski (2015) in [16].

DEFINITION 1. Let $J \subseteq \mathbb{R}$ be an interval, $\tau \in J^0$ and $n \in \mathbb{N}$. A function $f : J \to \mathbb{R}$ is said to be $(n+1)$-convex at point τ if there exits a constant W_τ such that the function

$$\mathbb{F}(w) = f(w) - \frac{W_\tau}{n!} w^n \tag{47}$$

is n-concave on $I \cap (-\infty, \tau]$ and n-convex on $I \cap [\tau, \infty)$. A function f is said to be $(n+1)$-concave at a point τ if the function $-f$ is $(n+1)$-convex at a point τ.

It is the usual sense that a function is $(n+1)$-convex on an interval iff it is $(n+1)$-convex at every point of the interval (see [16, 11]). Peĉarić et al. in [16] described necessary and sufficient conditions on two linear functionals $\Psi : C([\vartheta_1, \tau]) \to \mathbb{R}$ and $\Omega : C([\tau, \vartheta_2]) \to \mathbb{R}$ so that the inequality $\Psi(f) \leqslant \Omega(f)$ holds for every function f that is $(n+1)$-convex at point τ.

Now we define linear functionals $\Psi_d(f)$ and $\Omega_d(f)$ for fix $d = 1,2,3,4$ whose are deduced from the difference of left and right sides of identity (36), constructed on the intervals $[\vartheta_1, \tau]$ and $[\tau, \vartheta_2]$ respectively, i.e., for $\mathbf{x},\mathbf{y} \in [\vartheta_1, \tau]^l$, $\mathbf{p} \in \mathbb{R}^l$, $\mathbf{r},\mathbf{s} \in [\tau, \vartheta_2]^{\bar{l}}$ and $\bar{\mathbf{p}} \in \mathbb{R}^{\bar{l}}$ let

$$\Psi_d(f) := \mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},f(.)) - f'(\xi_d) \sum_{i=1}^{l} p_i(x_i - y_i) \\ - \sum_{k=0}^{n-3} \frac{f^{(k+2)}(\vartheta_1)}{k!} \int_{\vartheta_1}^{\tau} \mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},G_d(.,v))\,(v-\vartheta_1)^k dv, \tag{48}$$

where $\xi_1, \xi_4 = \tau$ and $\xi_2, \xi_3 = \vartheta_1$ and

$$\Omega_d(f) := \mathbb{M}(\mathbf{r}, \mathbf{s}, \bar{\mathbf{p}}, f(.)) - f'(\xi_d) \sum_{i=1}^{l} \bar{p}_i (r_i - s_i)$$
$$- \sum_{k=0}^{n-3} \frac{f^{(k+2)}(\tau)}{k!} \int_{\tau}^{\vartheta_2} \mathbb{M}(\mathbf{r}, \mathbf{s}, \bar{\mathbf{p}}, G_d(., v))(v - \tau)^k dv, \tag{49}$$

where $\xi_1, \xi_4 = \vartheta_2$ and $\xi_2, \xi_3 = \tau$.

When we apply identity (29) to the linear functionals $\Psi_d(f)$ and $\Omega_d(f)$ for fix $d = 1,2,3,4$ on the intervals $[\vartheta_1, \tau]$ and $[\tau, \vartheta_2]$ respectively, we get

$$\Psi_d(f) = \frac{1}{(n-3)!} \int_{\vartheta_1}^{\tau} \left(\int_{u}^{\tau} \mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{p}, G_d(., v))(v - u)^{n-3} dv \right) f^{(n)}(u) du, \tag{50}$$

and

$$\Omega_d(f) = \frac{1}{(n-3)!} \int_{\tau}^{\vartheta_2} \left(\int_{u}^{\vartheta_2} \mathbb{M}(\mathbf{r}, \mathbf{s}, \bar{\mathbf{p}}, G_d(., v))(v - u)^{n-3} dv \right) f^{(n)}(u) du. \tag{51}$$

Now, we are in that position to state the monotonicity of linear functionals $\Psi_p(f)$ and $\Omega_p(f)$ involving $(n+1)$-convex function at a point:

THEOREM 5. *Let $\mathbf{x}, \mathbf{y} \in [\vartheta_1, \tau]^l$, $\mathbf{p} \in \mathbb{R}^l$, $\mathbf{r}, \mathbf{s} \in [\tau, \vartheta_2]^{\bar{l}}$ and $\bar{\mathbf{p}} \in \mathbb{R}^{\bar{l}}$ in such a way that for $d = 1,2,3,4$*

$$\int_{u}^{\tau} \mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{p}, G_d(., v))(v - u)^{n-3} dv \geqslant 0, \ \ u \in [\vartheta_1, \tau], \tag{52}$$

$$\int_{u}^{\vartheta_2} \mathbb{M}(\mathbf{r}, \mathbf{s}, \bar{\mathbf{p}}, G_d(., v))(v - u)^{n-3} dv \geqslant 0, \ \ u \in [\tau, \vartheta_2], \tag{53}$$

$$\int_{\vartheta_1}^{\tau} \left(\int_{u}^{\tau} \mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{p}, G_d(., v))(v - u)^{n-3} dv \right) du$$
$$= \int_{\tau}^{\vartheta_2} \left(\int_{u}^{\vartheta_2} \mathbb{M}(\mathbf{r}, \mathbf{s}, \bar{\mathbf{p}}, G_d(., v))(v - u)^{n-3} dv \right) du, \tag{54}$$

where G_d $(d = 1,2,3,4)$ defined as in (13), (14), (15) and (16) respectively and let linear functionals $\Psi_d(f)$ and $\Omega_d(f)$ be defined in (48) and (49). If $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ is $(n+1)$-convex at point τ, then the monotonicity of these linear functionals is

$$\Psi_d(f) \leqslant \Omega_d(f), \ \ \text{for d} = 1,2,3,4. \tag{55}$$

If the inequalities in (52) and (53) are reversed, then the reverse inequality in (55) holds.

Proof. From Definition 1, we can define a function $\mathbb{F}(w)$ in such manner that $\mathbb{F}(w)$ is n-concave on $[\vartheta_1, \tau]$ and n-convex on $[\tau, \vartheta_2]$. Now by using Theorem 1 to the function $\mathbb{F}(w)$ on $[\vartheta_1, \tau]$ and $[\tau, \vartheta_2]$ respectively, we get

$$\Psi_d(\mathbb{F}) = \Psi_d(f) - \frac{W_\tau}{n!}\Psi_d(w^n) \leqslant 0 \text{ and } \Omega_{\mathrm{d}}(\mathbb{F}) = \Omega_{\mathrm{d}}(\mathrm{f}) - \frac{\mathrm{W}_\tau}{\mathrm{n}!}\Omega_{\mathrm{d}}(\mathrm{w}^{\mathrm{n}}) \geqslant 0, \quad (56)$$

by fixing $d = 1,2,3,4$. By putting $f = w^n$ in the identities (50) and (51) we have

$$\Psi_d(w^n) = \left(n^3 - 3n^2 + 2n\right) \int_{\vartheta_1}^{\tau} \left(\int_u^{\tau} \mathbb{M}\left(\mathbf{x}, \mathbf{y}, \mathbf{p}, G_d\left(., v\right)\right)(v-u)^{n-3} dv \right) du \quad (57)$$

and

$$\Omega_d(w^n) = \left(n^3 - 3n^2 + 2n\right) \int_{\tau}^{\vartheta_2} \left(\int_u^{\vartheta_2} \mathbb{M}\left(\mathbf{r}, \mathbf{s}, \bar{\mathbf{p}}, G_d\left(., v\right)\right)(v-u)^{n-3} dv \right) du. \quad (58)$$

So (54) implies that

$$\Psi_d(w^n) = \Omega_d(w^n).$$

Therefore the required result follows from (56). □

REMARK 3. As in [11], in the proof of above theorem we have shown that for $d = 1,2,3,4$

$$\Psi_d(f) \leqslant \frac{W_\tau}{n!}\Psi_d(w^n) = \frac{W_\tau}{n!}\Omega_d(w^n) \leqslant \Omega_d(f).$$

Moreover, if we replace condition (54) with the weaker condition that is $W_\tau(\Omega_d(w^n) - \Psi_d(w^n)) \geqslant 0$, the inequality (55) still holds.

Finally, we are ending this section with the following remark:

REMARK 4. We can also give the results of this section by defining the linear functionals via using inequality (38) and the newly defined Green functions G_d for $d = 1,2,3,4$.

We can also give the results of this section by defining the linear functionals via using the identities deduced from the integral version of the theorems in the previous section and the newly defined Green functions G_d for $d = 1,2,3,4$.

5. Grüss and Ostrowski-type inequalities as new upper bounds

Consider the Čebyšev functional

$$\Lambda(f,h) = \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)h(t)dt - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)dt \cdot \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)dt,$$

where $f,h : [\alpha,\beta] \to \mathbb{R}$ are two Lebesgue integrable functions.

Following theorems are proved by Cerone and Dragomir in [5].

THEOREM 6. *Let $f:[\alpha,\beta]\to\mathbb{R}$ be Lebesgue integrable function and $h:[\alpha,\beta]\to\mathbb{R}$ be an absolutely continuous function such that $(\cdot-\alpha)(\beta-\cdot)[h']^2\in L[\alpha,\beta]$. Then we have*

$$|\Lambda(f,h)|\leqslant\frac{1}{\sqrt{2}}[\Lambda(f,f)]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left(\int_\alpha^\beta(x-\alpha)(\beta-x)[h'(x)]^2dx\right)^{\frac{1}{2}}. \tag{59}$$

The constant $\frac{1}{\sqrt{2}}$ in (59) is the best possible.

THEOREM 7. *Let $f:[\alpha,\beta]\to\mathbb{R}$ be an absolutely continuous function with $f'\in L_\infty[\alpha,\beta]$ and $h:[\alpha,\beta]\to\mathbb{R}$ be monotonic nondecreasing function on $[\alpha,\beta]$. Then we have*

$$|\Lambda(f,h)|\leqslant\frac{1}{2(\beta-\alpha)}\|f'\|_\infty\int_\alpha^\beta(x-\alpha)(\beta-x)dh(x). \tag{60}$$

The constant $\frac{1}{2}$ in (60) is the best possible.

In this section, we give the upper bounds like Grüss-type and Ostrowski-type for our generalized results.

Let $\mathbf{x}=(x_1,\ldots,x_l)$, $\mathbf{y}=(y_1,\ldots,y_l)$ and $\mathbf{p}=(p_1,\ldots,p_l)$ be l-tuples such that $x_i,y_i\in[\vartheta_1,\vartheta_2]$ and $p_i\in\mathbb{R}$ for $i=1,2,\ldots,l$. Let G_d $(d=1,2,3,4)$ be Green functions as defined in (13), (14), (15) and (16). For $d=1,2,3,4$, we define

$$\Theta_d(u)=\int_u^{\vartheta_2}\mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},G_d(.,v))(v-u)^{n-3}dv,\quad u\in[\vartheta_1,\vartheta_2], \tag{61}$$

$$\Upsilon_d(u)=\int_{\vartheta_1}^{u}\mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},G_d(.,v))(v-u)^{n-3}dv,\quad u\in[\vartheta_1,\vartheta_2]. \tag{62}$$

Now consider the following Čebyšev functionals for $d=1,2,3,4$

$$\Lambda(\Theta_d,\Theta_d)=\frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\Theta_d^2(u)du-\left(\frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\Theta_d(u)du\right)^2, \tag{63}$$

$$\Lambda(\Upsilon_d,\Upsilon_d)=\frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\Upsilon_d^2(u)du-\left(\frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\Upsilon_d(u)du\right)^2. \tag{64}$$

THEOREM 8. *Consider that all the suppositions of Theorem 4 are true. Let $f:[\vartheta_1,\vartheta_2]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\geqslant 3$ and $(\cdot-\vartheta_1)(\vartheta_2-\cdot)[f^{(n+1)}]^2\in L[\vartheta_1,\vartheta_2]$. If Θ_d and Υ_d are functions, defined in (61) and (62) respectively, then the following identities hold for $d=1,2,3,4$.*

(i)

$$\begin{aligned}
&\mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},f(.))\\
&= f'(\xi_d)\sum_{i=1}^{l} p_i(x_i-y_i)+\sum_{k=0}^{n-3}\frac{f^{(k+2)}(\vartheta_1)}{k!}\int_{\vartheta_1}^{\vartheta_2}\mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},G_d(.,v))(v-\vartheta_1)^k dv\\
&\quad+\frac{f^{(n-1)}(\vartheta_2)-f^{(n-1)}(\vartheta_1)}{(\vartheta_2-\vartheta_1)(n-3)!}\int_{\vartheta_1}^{\vartheta_2}\Theta_d(u)du+\mathbb{REM}(f^{(n)},\Theta_d,\vartheta_1,\vartheta_2), \qquad (65)
\end{aligned}$$

where $\mathbb{REM}(f^{(n)},\Theta_d,\vartheta_1,\vartheta_2)$ *is the remainder which satisfies the following inequality*

$$\begin{aligned}
&\left|\mathbb{REM}(f^{(n)},\Theta_d,\vartheta_1,\vartheta_2)\right|\\
&\leqslant \frac{\sqrt{\vartheta_2-\vartheta_1}}{\sqrt{2}(n-3)!}[\Lambda(\Theta_d,\Theta_d)]^{\frac{1}{2}}\left|\int_{\vartheta_1}^{\vartheta_2}(u-\vartheta_1)(\vartheta_2-u)[f^{(n+1)}(u)]^2du\right|^{\frac{1}{2}}. \qquad (66)
\end{aligned}$$

(ii)

$$\begin{aligned}
&\mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},f(.))\\
&= f'(\xi_d)\sum_{i=1}^{l} p_i(x_i-y_i)+\sum_{k=0}^{n-3}\frac{(-1)^k f^{(k+2)}(\vartheta_2)}{k!}\int_{\vartheta_1}^{\vartheta_2}\mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},G_d(.,v))(\vartheta_2-v)^k dv\\
&\quad+\frac{f^{(n-1)}(\vartheta_2)-f^{(n-1)}(\vartheta_1)}{(\vartheta_1-\vartheta_2)(n-3)!}\int_{\vartheta_1}^{\vartheta_2}\Upsilon(u)du-\mathbb{REM}(f^{(n)},\Upsilon_d,\vartheta_1,\vartheta_2), \qquad (67)
\end{aligned}$$

where $\mathbb{REM}(f^{(n)},\Upsilon_d,\vartheta_1,\vartheta_2)$ *is the remainder which satisfies the following inequality*

$$\begin{aligned}
&\left|\mathbb{REM}(f^{(n)},\Upsilon_d,\vartheta_1,\vartheta_2)\right|\\
&\leqslant \frac{\sqrt{\vartheta_2-\vartheta_1}}{\sqrt{2}(n-3)!}[\Lambda(\Upsilon_d,\Upsilon_d)]^{\frac{1}{2}}\left|\int_{\vartheta_1}^{\vartheta_2}(u-\vartheta_1)(\vartheta_2-u)[f^{(n+1)}(u)]^2du\right|^{\frac{1}{2}}. \qquad (68)
\end{aligned}$$

Moreover, $\xi_1,\ \xi_4=\vartheta_2$ *and* $\xi_2,\ \xi_3=\vartheta_1$.

Proof. (i): From (29) and (65), we have for fix $d=1,2,3,4$,

$$\begin{aligned}
&\frac{1}{(n-3)!}\int_{\vartheta_1}^{\vartheta_2}\Theta_d(u)f^{(n)}(u)du\\
&=\frac{f^{(n-1)}(\vartheta_2)-f^{(n-1)}(\vartheta_1)}{(\vartheta_2-\vartheta_1)(n-3)!}\int_{\vartheta_1}^{\vartheta_2}\Theta_d(u)du+\mathbb{REM}(f^{(n)},\Theta_d,\vartheta_1,\vartheta_2).
\end{aligned}$$

This implies

$$\begin{aligned}
&\mathbb{REM}(f^{(n)},\Theta_d,\vartheta_1,\vartheta_2)\\
&=\frac{1}{(n-3)!}\int_{\vartheta_1}^{\vartheta_2}\Theta_d(u)f^{(n)}(u)du-\frac{1}{(\vartheta_2-\vartheta_1)(n-3)!}\int_{\vartheta_1}^{\vartheta_2}\Theta_d(u)du\int_{\vartheta_1}^{\vartheta_2}f^{(n)}(u)du,
\end{aligned}$$

which can be written in terms of Čebyšev functional as

$$\mathbb{REM}(f^{(n)},\Theta_d,\vartheta_1,\vartheta_2)=\frac{(\vartheta_2-\vartheta_1)}{(n-3)!}\Lambda(\Theta_d,f^{(n)}). \tag{69}$$

Using Theorem 6, we get (66).

(ii): Similarly, we can prove the part (ii) by comparing (30) and (67). □

Following theorem gives Grüss-type inequalities.

THEOREM 9. *Let $f:[\vartheta_1,\vartheta_2]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous for some $n\geqslant 3$ and $f^{(n+1)}\geqslant 0$ on $[\vartheta_1,\vartheta_2]$. If Θ_d and Υ_d are functions, defined in (61) and (62) respectively, then for $d=1,2,3,4$,*

(i) the remainder $\mathbb{REM}(f^{(n)},\Theta_d,\vartheta_1,\vartheta_2)$ in (65) is satisfied the following bound

$$\begin{aligned}&\left|\mathbb{REM}(f^{(n)},\Theta_d,\vartheta_1,\vartheta_2)\right|\\ &\leqslant\frac{1}{(n-3)!}\|\Theta_d'\|_\infty\left\{\frac{f^{(n-1)}(\vartheta_2)+f^{(n-1)}(\vartheta_1)}{2}-\frac{f^{(n-2)}(\vartheta_2)-f^{(n-2)}(\vartheta_1)}{\vartheta_2-\vartheta_1}\right\}.\end{aligned} \tag{70}$$

(ii) the remainder $\mathbb{REM}(f^{(n)},\Upsilon_d,\vartheta_1,\vartheta_2)$ in (67) is satisfied the following bound

$$\begin{aligned}&\left|\mathbb{REM}(f^{(n)},\Upsilon_d,\vartheta_1,\vartheta_2)\right|\\ &\leqslant\frac{1}{(n-3)!}\|\Upsilon_d'\|_\infty\left\{\frac{f^{(n-1)}(\vartheta_2)+\phi^{(n-1)}(\vartheta_1)}{2}-\frac{f^{(n-2)}(\vartheta_2)-f^{(n-2)}(\vartheta_1)}{\vartheta_2-\vartheta_1}\right\}.\end{aligned} \tag{71}$$

Proof. From (69), we have for fix $d=1,2,3,4$,

$$\mathbb{REM}(f^{(n)},\Theta_d,\vartheta_1,\vartheta_2)=\frac{(\vartheta_2-\vartheta_1)}{(n-3)!}\Lambda(\Theta_d,f^{(n)}). \tag{72}$$

Using Theorem 7 on right hand side, we deduce (70). □

We now define q-norm of a function $f:[\vartheta_1,\vartheta_2]\to\mathbb{R}$ by:

$$\|f\|_q=\begin{cases}(\int_{\vartheta_1}^{\vartheta_2}|f(u)|^q du)^{\frac{1}{q}}, & \text{for } 1\leqslant q<\infty,\text{ if } |f|^q \text{ is } R\text{-integrable function},\\ \text{essential supremum of } f, & \text{for } q=\infty,\text{ if } \phi \text{ is essential bounded.}\end{cases}$$

Next theorem gives the Ostrowski-type inequalities related to generalized majorization inequality.

THEOREM 10. *Consider that all the suppositions of Theorem 4 are true. Let $f:[\vartheta_1,\vartheta_2]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\geqslant 3$. Let (q,q') be a pair of conjugate exponents, that is $1\leqslant q,q'\leqslant\infty$ and $\frac{1}{q}+\frac{1}{q'}=1$. If $|f^n|^q:[\vartheta_1,\vartheta_2]\to\mathbb{R}\,(n\geqslant 3)$ is R-integrable function, then we have the following identities for $d=1,2,3,4$.*

(i)

$$\Big|\mathbb{M}\left(\mathbf{x},\mathbf{y},\mathbf{p},f(.)\right)-f'(\xi_d)\sum_{i=1}^{l}p_i(x_i-y_i)$$
$$-\sum_{k=0}^{n-3}\frac{f^{(k+2)}(\vartheta_1)}{k!}\int_{\vartheta_1}^{\vartheta_2}\mathbb{M}\left(\mathbf{x},\mathbf{y},\mathbf{p},G_d(.,v)\right)(v-\vartheta_1)^k dv\Big|$$
$$\leqslant \frac{1}{(n-3)!}\|f^{(n)}\|_q\|f\|_{q'}, \tag{73}$$

where

$$f(u)=\int_u^{\vartheta_2}\mathbb{M}\left(\mathbf{x},\mathbf{y},\mathbf{p},G_d(.,v)\right)(v-u)^{n-3}dv.$$

Right hand side of (73) is constant which is sharp for $1<q\leqslant\infty$ and the best possible for $q=1$.

(ii)

$$\Big|\mathbb{M}\left(\mathbf{x},\mathbf{y},\mathbf{p},f(.)\right)-f'(\xi_d)\sum_{i=1}^{l}p_i(x_i-y_i)$$
$$-\sum_{k=0}^{n-3}\frac{(-1)^k f^{(k+2)}(\vartheta_2)}{k!}\int_{\vartheta_1}^{\vartheta_2}\mathbb{M}\left(\mathbf{x},\mathbf{y},\mathbf{p},G_d(.,v)\right)(\vartheta_2-v)^k dv\Big|$$
$$\leqslant \frac{1}{(n-3)!}\|f^{(n)}\|_q\|\bar{f}\|_{q'}, \tag{74}$$

where

$$\bar{f}(u)=\int_{\vartheta_1}^{u}\mathbb{M}\left(\mathbf{x},\mathbf{y},\mathbf{p},G_d(.,v)\right)(v-u)^{n-3}dv.$$

Right hand side of (74) is constant which is sharp for $1<q\leqslant\infty$ and the best possible for $q=1$.

Moreover, $\xi_1,\xi_4=\vartheta_2$ and $\xi_2,\xi_3=\vartheta_1$.

Proof. By the arrangement of identity (29) for fix $d=1,2,3,4,$ we have the following identity:

$$\Big|\mathbb{M}\left(\mathbf{x},\mathbf{y},\mathbf{p},f(.)\right)-f'(\xi_d)\sum_{i=1}^{l}p_i(x_i-y_i)$$
$$-\sum_{k=0}^{n-3}\frac{f^{(k+2)}(\vartheta_1)}{k!}\int_{\vartheta_1}^{\vartheta_2}\mathbb{M}\left(\mathbf{x},\mathbf{y},\mathbf{p},G_d(.,v)\right)(v-\vartheta_1)^k dv\Big|$$
$$=\frac{1}{(n-3)!}\left|\int_{\vartheta_1}^{\vartheta_2}\Theta_d(u)f^{(n)}(u)du\right|, \tag{75}$$

classical Hölder's inequality apply to the right hand side of (75) implies (73). For the proof of the sharpness of the constant $\|f\|_{q'}$ is analog to one in proof of Theorem 19 in [1]. $\square$

We are ending this section with the following remark:

REMARK 5. We can give the integral version of the upper bound theorems like Theorem 6, Theorem 9 and Theorem 10 as given in [1].

6. Mean value theorems and n-exponential convexity

Since the general convex functions are defined by a functional inequality, it is not surprising that this notion will lead to a number of interesting and fundamental inequalities. Now we give some essential results for general convex functions.

Supppose all the assumptions of Corollary 1 are satisfied. Making use of inequalities (36) and (38) we now define following linear functionals:

$$\begin{aligned}\Re_1(f) = \mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},f(.)) - f'(\xi_d)\sum_{i=1}^{l} p_i(x_i-y_i)\\ -\sum_{k=0}^{n-3}\frac{f^{(k+2)}(\vartheta_1)}{k!}\int_{\vartheta_1}^{\vartheta_2}\mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},G_d(.,v))\,(v-\alpha)^k dv,\end{aligned} \tag{76}$$

and

$$\begin{aligned}\Re_2(f) = \mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},f(.)) - f'(\xi_d)\sum_{i=1}^{l} p_i(x_i-y_i)\\ -\sum_{k=0}^{n-3}\frac{(-1)^k f^{(k+2)}(\vartheta_2)}{k!}\int_{\vartheta_1}^{\vartheta_2}\mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},G_d(.,v))\,(\beta-v)^k dv,\end{aligned} \tag{77}$$

where $\xi_1,\xi_4=\vartheta_2$ and $\xi_2,\xi_3=\vartheta_1$.

REMARK 6. Let all the assumptions of Corollary 1 are satisfied. Then $\Re_i(f)\geqslant 0$, $i=1,2$ for all n-convex functions f.

Following theorems give the Lagrange and Cauchy type mean value theorems for the functionals defined in (76) and (77).

THEOREM 11. *Let $f:[\vartheta_1,\vartheta_2]\to\mathbb{R}$ be such that $f\in C^n[\vartheta_1,\vartheta_2]$. Consider the inequalities (35) and (37) hold. Let $\Re_i(f)$, $i=1,2$ be functionals defined in (76) and (77) and also $\psi(x)=\frac{x^n}{n!}$. Then there exists $\lambda_i\in[\vartheta_1,\vartheta_2]$ such that*

$$\Re_i(f) = f^{(n)}(\lambda_i)\Re_i(\psi), \quad i=1,2. \tag{78}$$

Proof. Since $f^{(n)}$ is continuous on $[\vartheta_1,\vartheta_2]$, so $m\leqslant f^{(n)}(x)\leqslant M$ for $x\in[\vartheta_1,\vartheta_2]$, where $m=\min_{x\in[\vartheta_1,\vartheta_2]} f^{(n)}(x)$ and $M=\max_{x\in[\vartheta_1,\vartheta_2]} f^{(n)}(x)$ (see for example Theorem 4.1 in [8] and also in [2, 4]). Consider the functions f_1 and f_2 defined on I as

$$f_1(x)=\frac{Mx^n}{n!}-f(x) \quad \text{and} \quad f_2(x)=f(x)-\frac{mx^n}{n!} \text{ for } x\in[\vartheta_1,\vartheta_2].$$

It is easily seen that

$$f_1^{(n)}(x)=M-f^{(n)}(x) \quad \text{and} \quad f_2^{(n)}(x)=f^{(n)}(x)-m \text{ for } x\in I.$$

So, f_1 and f_2 are n-convex functions.

Now by applying f_1 for f in Corollary 1, we have

$$\begin{aligned}&\mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},f_1(.))-f'(\xi_d)\sum_{i=1}^{l}p_i(x_i-y_i)\\ &\geqslant\sum_{k=0}^{n-3}\frac{f_1^{(k+2)}(\vartheta_2)}{k!}\int_{\vartheta_1}^{\vartheta_2}\mathbb{M}(\mathbf{x},\mathbf{y},\mathbf{p},G_d(.,v))(v-\vartheta_1)^k dv,\end{aligned} \tag{79}$$

where ξ_1, $\xi_4=\vartheta_2$, ξ_2, $\xi_3=\vartheta_1$ and $d=1,2,3,4$. Hence, we get after some simplification

$$\mathfrak{R}_1(f)\leqslant M\mathfrak{R}_1(\psi). \tag{80}$$

Now by applying f_2 for f in Corollary 1 and some simplification we get

$$m\mathfrak{R}_1(\psi)\leqslant\mathfrak{R}_1(f). \tag{81}$$

If $\mathfrak{R}_1(\psi)=0$, then from (80) and (81) follow that for any $\lambda_1\in[\vartheta_1,\vartheta_2]$, (78) is satisfied.

If $\mathfrak{R}_1(\psi)>0$, it follows from (80) and (81) that

$$m\leqslant\frac{\mathfrak{R}_1(f)}{\mathfrak{R}_1(\psi)}\leqslant M. \tag{82}$$

Now using the fact that for $m\leqslant\rho\leqslant M$ there exists $\lambda_1\in[\vartheta_1,\vartheta_2]$ such that $f^{(n)}(\lambda_1)=\rho$, we get (78). □

COROLLARY 4. *Let $f,g:[\vartheta_1,\vartheta_2]\to\mathbb{R}$ be such that $f,g\in C^n[\vartheta_1,\vartheta_2]$. Consider the inequalities (35) and (37) hold. Let $\mathfrak{R}_i(f)$, $i=1,2$ be functionals defined in (76) and (77). Then there exists $\lambda_i\in[\vartheta_1,\vartheta_2]$ such that*

$$\frac{\mathfrak{R}_i(f)}{\mathfrak{R}_i(g)}=\frac{f^{(n)}(\lambda_i)}{g^{(n)}(\lambda_i)},\quad i=1,2, \tag{83}$$

provided that the denominators are non-zero.

Proof. See Corollary 4.2 in [8] (see also in [2, 4]). $\square$

We can define Cauchy means for $i=1,2$ by using generalized Cauchy second mean value theorem i.e., Corollary 4 as

$$\lambda_i = \left(\frac{f^{(n)}}{g^{(n)}}\right)^{-1} \frac{\mathfrak{R}_i(f)}{\mathfrak{R}_i(g)},$$

which shows that λ_i is a mean of ϑ_1, ϑ_2 for given functions f and g.

REMARK 7. We can give the n-exponential convexity, exponential convexity as well as log-convexity from the above defined positive linear functionals $\mathfrak{R}_i(f)$, $i=1,2$ for both discrete as well as continuous case by using the interesting method introduced by Pečarć et al. (2013) [8, 9] (see also [1, 11]). We can also construct a large families of functions which are exponentially convex as given in [1]. From the log-convexity, we can get the Dresher's inequality from which we find the Cauchy means and investigate their monotonicity.

7. Applications

The purpose of this section will be to explore applications of our generalized identities. We will obtain the Ostrowski-type upper bounds for the generalized identity in discrete case for different well-known convex functions. In fact, in first two applications we will discuss the relationship between the components of both vectors $\mathbf{x}$ and $\mathbf{y}$. We can also give the beautiful examples of our generalized result like as n-exponential convexity method in [9] (one can also see examples in [1]).

APPLICATION 1. Let $f:(0,\infty)\to\mathbb{R}$ be function defined by $f(x)=-\log x$. Let us consider that $\mathbf{x}=(x_1,x_2,\ldots,x_l)$ and $\mathbf{y}=(y_1,y_2,\ldots,y_l)$ be positive l-tuples. Then the Ostrowski-type inequality (73) for $n=3$ as an upper bound of our generalized result becomes

$$\left|\sum_{i=1}^{l} p_i(-log x_i) - \sum_{i=1}^{l} p_i(-\log y_i) + \frac{1}{\vartheta_2}\sum_{i=1}^{l} p_i(x_i-y_i) - \frac{1}{\vartheta_1^2}\mathbb{G}_d\right| \\ \leqslant \frac{2}{(1-3q)^{\frac{1}{q}}}\left(\vartheta_2^{1-3q}-\vartheta_1^{1-3q}\right)^{\frac{1}{q}}\|f\|_{q'},$$

if the majorization condition $\sum_{i=1}^{l} p_i x_i = \sum_{i=1}^{l} p_i y_i$ holds and $p_i=1$, $(i=1,2,\ldots,l)$ then

$$\left|\log\left(x_1^{-1}\cdot x_2^{-1}\cdots x_l^{-1}\right)+\log\left(y_1\cdot y_2\cdots y_l\right)-\frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d\right| \leqslant \frac{2}{(1-3q)^{\frac{1}{q}}}\left(\vartheta_2^{1-3q}-\vartheta_1^{1-3q}\right)^{\frac{1}{q}}\|f\|_{q'},$$

$$\Rightarrow \left|\log\left(\frac{\Pi_{i=1}^{l} y_i}{\Pi_{i=1}^{l} x_i}\right)-\frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d\right| \leqslant \frac{2}{(1-3q)^{\frac{1}{q}}}\left(\vartheta_2^{1-3q}-\vartheta_1^{1-3q}\right)^{\frac{1}{q}}\|f\|_{q'},$$

if $\left|\log\left(\frac{\Pi_{i=1}^{l}y_i}{\Pi_{i=1}^{l}x_i}\right)\right| \geqslant \left|\frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d\right|$, then

$$\Rightarrow \left|\log\left(\frac{\Pi_{i=1}^{l}y_i}{\Pi_{i=1}^{l}x_i}\right)\right| - \left|\frac{1}{x^2}\tilde{\mathbb{G}}_d\right| \leqslant \left|\ln\left(\frac{\Pi_{i=1}^{l}y_i}{\Pi_{i=1}^{l}x_i}\right) - \frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d\right|$$
$$\leqslant \frac{2}{(1-3q)^{\frac{1}{q}}}\left(\vartheta_2^{1-3q} - \vartheta_1^{1-3q}\right)^{\frac{1}{q}} \|f\|_{q'},$$
$$\Rightarrow 0 \leqslant \left|\log\left(\frac{\Pi_{i=1}^{l}y_i}{\Pi_{i=1}^{l}x_i}\right)\right| \leqslant \frac{2}{(1-3q)^{\frac{1}{q}}}\left(\vartheta_2^{1-3q} - \vartheta_1^{1-3q}\right)^{\frac{1}{q}} \|f\|_{q'} + \frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d,$$

if the quotient in the L. H. S. is greater than equal to 1, then

$$0 \leqslant \Pi_{i=1}^{l}y_i \leqslant e^{\frac{2}{(1-3q)^{\frac{1}{q}}}\left(\vartheta_2^{1-3q}-\vartheta_1^{1-3q}\right)^{\frac{1}{q}}\|f\|_{q'}+\frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d}\,\Pi_{i=1}^{l}x_i,$$

is the relation between the elements of $\mathbf{y}$ and the elements of $\mathbf{x}$, here

$$\mathbb{G}_d := \int_{\vartheta_1}^{\vartheta_2}\left(\sum_{i=1}^{l} p_i G_d(x_i,v) - \sum_{i=1}^{l} p_i G_d(y_i,v)\right) dv,$$
$$\tilde{\mathbb{G}}_d := \int_{\vartheta_1}^{\vartheta_2}\left(\sum_{i=1}^{l} G_d(x_i,v) - \sum_{i=1}^{l} G_d(y_i,v)\right) dv,$$
$$f(t) = \int_{u}^{\vartheta_2}\left(\sum_{i=1}^{l} p_i G_d(x_i,v) - \sum_{i=1}^{l} p_i G_d(y_i,v)\right)(v-u)^{n-3} dv.$$

APPLICATION 2. Let $f:(0,\infty)\to\mathbb{R}$ be function defined by $f(x) = x\log x$. Let us consider that $\mathbf{x} = (x_1,x_2,\ldots,x_l)$ and $\mathbf{y} = (y_1,y_2,\ldots,y_l)$ be positive l-tuples. Then the Ostrowski-type inequality (73) for $n = 3$ as an upper bound of our generalized result becomes

$$\left|\sum_{i=1}^{l} p_i x_i log x_i - \sum_{i=1}^{l} p_i y_i \log y_i + (\log\vartheta_2 + 1)\sum_{i=1}^{l} p_i (x_i - y_i) + \frac{1}{\vartheta_1^2}\mathbb{G}_d\right|$$
$$\leqslant \frac{1}{(2q-1)^{\frac{1}{q}}}\left(\vartheta_2^{1-2q} - \vartheta_1^{1-2q}\right)^{\frac{1}{q}} \|f\|_{q'},$$

if the majorization condition $\sum_{i=1}^{l} p_i x_i = \sum_{i=1}^{l} p_i y_i$ holds and $p_i = 1, (i = 1,2,\ldots,l)$ then

$$\left|\log\left(x_1^{x_1}\cdot x_2^{x_2}\cdots x_l^{x_l}\right) + \log\left(y_1^{-y_1}\cdot y_2^{-y_2}\cdots y_l^{-y_l}\right) + \frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d\right|$$
$$\leqslant \frac{1}{(2q-1)^{\frac{1}{q}}}\left(\vartheta_2^{1-2q} - \vartheta_1^{1-2q}\right)^{\frac{1}{q}} \|f\|_{q'},$$
$$\Rightarrow \left|\log\left(\frac{\Pi_{i=1}^{l}x_i^{x_i}}{\Pi_{i=1}^{l}y_i^{y_i}}\right) + \frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d\right| \leqslant \frac{1}{(2q-1)^{\frac{1}{q}}}\left(\vartheta_2^{1-2q} - \vartheta_1^{1-2q}\right)^{\frac{1}{q}} \|f\|_{q'},$$

if $\left|\log\left(\frac{\Pi_{i=1}^{l}x_i^{x_i}}{\Pi_{i=1}^{l}y_i^{y_i}}\right)\right| \geqslant \left|\frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d\right|$, then

$$\Rightarrow \left|\log\left(\frac{\Pi_{i=1}^{l}x_i^{x_i}}{\Pi_{i=1}^{l}y_i^{y_i}}\right)\right| - \left|\frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d\right| \leqslant \left|\log\left(\frac{\Pi_{i=1}^{l}x_i^{x_i}}{\Pi_{i=1}^{l}y_i^{y_i}}\right) + \frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d\right|$$

$$\leqslant \frac{1}{(2q-1)^{\frac{1}{q}}}\left(\vartheta_2^{1-2q} - \vartheta_1^{1-2q}\right)^{\frac{1}{q}} \|f\|_{q'},$$

$$\Rightarrow 0 \leqslant \left|\log\left(\frac{\Pi_{i=1}^{l}x_i^{x_i}}{\Pi_{i=1}^{l}y_i^{y_i}}\right)\right| \leqslant \frac{1}{(2q-1)^{\frac{1}{q}}}\left(\vartheta_2^{1-2q} - \vartheta_1^{1-2q}\right)^{\frac{1}{q}} + \frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d,$$

if the quotient in the L. H. S. is greater than equal to 1, then

$$0 \leqslant \Pi_{i=1}^{l}x_i^{x_i} \leqslant e^{\frac{2}{(1-3q)^{\frac{1}{q}}}\left(\vartheta_2^{1-3q}-\vartheta_1^{1-3q}\right)^{\frac{1}{q}} + \frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d} \Pi_{i=1}^{l}y_i^{y_i},$$

this is the another relation between the elements of $\mathbf{x}$ and the elements of $\mathbf{y}$, here, $\mathbb{G}_d$, $\tilde{\mathbb{G}}_d$ and $f(t)$ be defined as in Application 1.

APPLICATION 3. Let us consider that $\mathbf{x} = (x_1, x_2, \ldots, x_l)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_l)$ be l-tuples such that $x_i, y_i \in [\vartheta_1, \vartheta_2]$ and $\mathbf{p} = (p_1, p_2, \ldots, p_l)$ such that $p_i \in \mathbb{R}$. Then the Ostrowski-type inequality (73) for $n = 3$ as an upper bound of our generalized result is as follows:

- let $f(x) = e^x$, $x \in \mathbb{R}$, then

$$0 \leqslant \left|\sum_{i=1}^{l} p_i e^{x_i} - \sum_{i=1}^{l} p_i e^{y_i} - e^{\vartheta_2}\sum_{i=1}^{l} p_i (x_i - y_i) - e^{\vartheta_1}\mathbb{G}_d\right| \leqslant \frac{1}{q}(e^{q\vartheta_2} - e^{q\vartheta_1})^{\frac{1}{q}} \parallel f \parallel_{q'},$$

- let $f(x) = x^r$, $[0, \infty)$ for $r > 1$, then

$$0 \leqslant \left|\sum_{i=1}^{l} p_i x_i^r - \sum_{i=1}^{l} p_i y_i^r - r\vartheta_2^{r-1} - r(r-1)\vartheta_1^{r-2}\mathbb{G}_d\right|$$

$$\leqslant \frac{r(r-1)(r-2)}{(rq-3q+1)^{\frac{1}{q}}}\left(\vartheta_2^{q(r-3)+1} - \vartheta_1^{q(r-3)+1}\right)^{\frac{1}{q}} \parallel f \parallel_{q'}.$$

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