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THE MEAN CONSISTENCY OF THE WEIGHTED ESTIMATOR IN THE FIXED DESIGN REGRESSION MODELS BASED ON m–END ERRORS

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Abstract. In this paper, some moment inequalities for m-extended negatively dependent (m-END, for short) random variables are established which can be applied to investigate the nonparametric regression models based on m-END errors. Some results on mean consistency for the estimator in nonparametric regression models are presented. As an application, the consistency for the nearest neighbor estimator is obtained.

1. Introduction

Firstly, let us recall the concepts of extended negatively dependent random variables and m-extended negatively dependent random variables.

DEFINITION 1.1. A finite collection of random variables $X_1, X_2, \ldots, X_n$ is said to be extended negatively dependent (END, for short) if there exists a constant $M > 0$ such that both

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leqslant M \prod_{i=1}^{n} P(X_i > x_i)$$

and

$$P(X_1 \leqslant x_1, X_2 \leqslant x_2, \ldots, X_n \leqslant x_n) \leqslant M \prod_{i=1}^{n} P(X_i \leqslant x_i)$$

hold for all real numbers $x_1, x_2, \ldots, x_n$. An infinite sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be END if every finite subcollection is END.

An array $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ of random variables is called rowwise END random variables if for every $n \geqslant 1, \{X_{ni}, 1 \leqslant i \leqslant n\}$ is a sequence of END random variables.

The concept of END sequence was introduced by Liu (2009). Negatively associated (NA, for short) random variables, negatively orthant dependent (NOD, for short)

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random variables and some other positively dependent structures are its special cases. Since the concept of END sequence was introduced by Liu (2009), many authors were devoted to studying the probability limit theorem and statistical large sample properties for END random variables, including the probability inequalities, moment inequalities and applications. For example, Chen et al. (2010) established the strong law of large numbers for extend negatively dependent random variables and showed applications to risk theory and renewal theory; Shen (2011) presented some probability inequalities and gave some applications; Wu and Guan (2012) presented some convergence properties for the partial sums of END random variables; Qiu et al. (2013) and Wang et al. (2013) provided some results on complete convergence for weighted sums of arrays of rowwise END random variables; Wu et al. (2014) obtained some results on complete convergence and complete moment convergence for arrays of rowwise END random variables; Shen (2014) studied the asymptotic approximation of inverse moments for a class of nonnegative random variables including END random variables; Shen (2016) established some results on complete convergence for weighted sums of END random variables and gave applications to nonparametric regression models, and so on.

Based on END random variables, Wang et al. (2016) introduced the concept of m-END random variables as follows.

DEFINITION 1.2. Let $m \geqslant 1$ be a fixed integer. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be m-extended negatively dependent (m-END, for short) if for any $n \geqslant 2$ and any $i_1, i_2, \ldots, i_n$ such that $|i_k - i_j| \geqslant m$ for all $1 \leqslant k \neq j \leqslant n$, we have that $X_{i_1}, X_{i_2}, \ldots, X_{i_n}$ are END.

An array $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ of random variables is called rowwise m-END if for every $n \geqslant 1, \{X_{ni}, i \geqslant 1\}$ is a sequence of m-END random variables.

The concept of m-END sequence was introduced by Wang et al. (2016). When $m = 1$, the concept of m-END random variables reduces to the so-called END random variables. Hence, the concept of m-END random variables is a natural extension of END random variables. Besides, it is well known that if for any $n \geqslant 2$ and any $i_1, i_2, \ldots, i_n$ such that $|i_k - i_j| \geqslant m$ for all $1 \leqslant k \neq j \leqslant n$, we have that $X_{i_1}, X_{i_2}, \ldots, X_{i_n}$ are independent (or NA), then we can say that $\{X_n, n \geqslant 1\}$ are m-dependence (or m-NA). Hence, m-END is weaker than m-dependence, m-NA and END. So, it is of great interest to study the probability inequalities, moment inequalities and probability limit theorems of m-END random variables and their applications in many stochastic models. We are devoted to studying the asymptotic properties for the estimator in nonparametric regression models based on m-END errors.

Now we consider the following fixed design regression model:

$$Y_{ni} = g(x_{ni}) + \varepsilon_{ni}, \quad i = 1, 2, \ldots, n, \quad n \geqslant 1, \tag{1.1}$$

where x_{ni} are design points on a set A in $\mathbb{R}^q$ for some $q \geqslant 1$, $g(\cdot)$ is an unknown regression function on A and ε_{ni} are random errors. Assume that for each n, $\{\varepsilon_{ni}, 1 \leqslant i \leqslant n\}$ has the same distribution as that of $\{\varepsilon_i, 1 \leqslant i \leqslant n\}$. As an estimator of $g(\cdot)$, the following weighted regression estimator is considered:

$$g_n(x) = \sum_{i=1}^{n} W_{ni}(x) Y_{ni}, x \in A \subset \mathbb{R}^q, \tag{1.2}$$

where $W_{ni}(x) = W_{ni}(x, x_{n1}, \ldots, x_{nn})$ are weight functions.

The above estimator was first proposed by Georgiev (1985) and has been studied by many authors subsequently. For example, when ε_{ni} are independent, consistency and asymptotic normality have been studied by Georgiev and Greblicki (1986), Georgiev (1988) and Müller (1987) and so on. As for the case when ε_{ni} are dependent, it has been studied by various authors in recent years as well. Roussas (1989) discussed strong consistency and quadratic mean consistency for $g_n(x)$ under mixing conditions. Roussas et al. (1992) established asymptotic normality of $g_n(x)$ assuming that the errors are from a strictly stationary stochastic process and satisfying the strong mixing condition. Tran et al. (1996) discussed asymptotic normality of $g_n(x)$ assuming that the errors form a linear time series, more precisely, a weakly stationary linear process based on martingale difference sequences. Hu et al. (2003) gave the mean consistency, complete consistency and asymptotic normality of regression models with linear process errors. Liang and Jing (2005) presented some asymptotic properties for estimator of nonparametric regression models based on negatively associated sequences. Yang et al. (2012) has investigated the consistency for the estimator of nonparametric regression model based on NOD errors. Shen (2013) presented the Bernstein-type inequality for widely dependent sequence and applied it to nonparametric regression models. Shen et al. (2015) established the Rosenthal-type inequality for negatively superadditive dependent random variables and gave its application in nonparametric regression models. Yang et al. (2016) established the complete consistency of estimators for regression models based on END errors, and so on.

The main aim of this work is to investigate the mean consistency and uniformly mean consistency for the estimator of nonparametric regression models based on m-END errors. The results obtained in the paper will generalize some corresponding ones for independent errors and some dependent errors.

For any function $g(x)$, we use $c(g)$ to denote all continuity points of function g on the set A in $\mathbb{R}^q$ for some $q \geqslant 1$. Let $C, C_1, C_2, \ldots$, denote the positive constants whose values may vary at each occurrence. $\lfloor x \rfloor$ denotes the largest integer not exceeding x, $I(B)$ is the indicator function of set B, $x^+ = xI(x \geqslant 0)$, $x^- = -xI(x < 0)$ and $\|x\|$ denotes Euclidean norm of x. In this article, some lemmas are presented in Section 2, main results and their proofs are presented in Section 3 and Section 4, respectively.

2. Some lemmas

In this section, we will present some lemmas which will be used to prove our main results. The first one is a basic property for END random variables, which was presented by Liu (2010).

LEMMA 2.1. *Let random variables $X_1, X_2, \ldots, X_n$ be END, $f_1, f_2, \ldots, f_n$ are all nondecreasing (or all nonincreasing) functions, then random variables $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$ are END.*

The following lemma is the Marcinkiewicz-Zygmund type inequality and Rosenthal type inequality for END random variables. The first one can be obtained by the method used in Chen et al. (2014), and the second one can be found in Shen (2011).

LEMMA 2.2. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of END random variables with* $EX_n = 0$ *and* $E|X_n|^p < \infty$ *for all* $n \geqslant 1$ *and some* $p \geqslant 1$. *Then*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leqslant c_p \sum_{i=1}^{n} E|X_i|^p, \quad 1 \leqslant p < 2, \tag{2.1}$$

and

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leqslant d_p \left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}, \quad p \geqslant 2, \tag{2.2}$$

where c_p *and* d_p *depend only on* p.

By Lemmas 2.1 and 2.2, we can obtain the following moment inequalities for m-END random variables which are indispensable in proving our main results.

LEMMA 2.3. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of* m*-END random variables with* $EX_n = 0$ *and* $E|X_n|^p < \infty$ *for all* $n \geqslant 1$ *and some* $p \geqslant 1$. *Then*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leqslant c_{m,p} \sum_{i=1}^{n} E|X_i|^p, \quad 1 \leqslant p < 2,$$

and

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leqslant d_{m,p} \left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}, \quad p \geqslant 2,$$

where $c_{m,p}$, $d_{m,p}$ *depend only on* p *and* m.

Proof. For any fixed n sufficiently large, if $n \geqslant m$, there exist positive integers i and $1 \leqslant j \leqslant m$ such that $n = mi + j$. Let $r = \lfloor n/m \rfloor$. Define

$$Y_i = \begin{cases} X_i, \ 1 \leqslant i \leqslant n, \\ 0, \quad i > n. \end{cases}$$

Noting that $\sum_{i=1}^{n} X_i = \sum_{j=1}^{m} \sum_{i=0}^{r} Y_{mi+j}$, we have

$$\left|\sum_{i=1}^{n} X_i\right| \leqslant \sum_{j=1}^{m} \sum_{i=0}^{r} |Y_{mi+j}|.$$

By the inequality above and C_r-inequality, we can get that

$$\begin{aligned} E\left|\sum_{i=1}^{n} X_i\right|^p &= E\left|\sum_{j=1}^{m} \sum_{i=0}^{r} Y_{mi+j}\right|^p \leqslant m^{p-1} \sum_{j=1}^{m} E\left(\sum_{i=0}^{r} |Y_{mi+j}|\right)^p \\ &\leqslant (2m)^{p-1} \sum_{j=1}^{m} E\left(\sum_{i=0}^{r} Y_{mi+j}^{+}\right)^p + (2m)^{p-1} \sum_{j=1}^{m} E\left(\sum_{i=0}^{r} Y_{mi+j}^{-}\right)^p. \end{aligned} \tag{2.3}$$

By the definition of m-END random variables, it can be easily seen that $Y_j, Y_{m+j}, \ldots, Y_{mr+j}$ are END random variables for each $j = 1,2,\ldots,m$. Hence, $Y_j^+, Y_{m+j}^+, \ldots, Y_{mr+j}^+$ and $Y_j^-, Y_{m+j}^-, \ldots, Y_{mr+j}^-$ are both END random variables for each $j = 1,2,\ldots,m$ by Lemma 2.1.

For $1 \leqslant p < 2$, we have by (2.1) and (2.3) that for any $n \geqslant m$,

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leqslant (2m)^{p-1} c_p \sum_{j=1}^{m}\sum_{i=0}^{r} E|Y_{mi+j}^+|^p + (2m)^{p-1} c_p \sum_{j=1}^{m}\sum_{i=0}^{r} E|Y_{mi+j}^-|^p$$
$$= (2m)^{p-1} c_p \sum_{j=1}^{m}\sum_{i=0}^{r} E|Y_{mi+j}|^p \triangleq c_{m,p} \sum_{i=1}^{n} E|X_i|^p.$$

For $p \geqslant 2$, we have by (2.2) and (2.3) that for any $n \geqslant m$,

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leqslant (2m)^{p-1} d_p \sum_{j=1}^{m}\left[\sum_{i=0}^{r} E|Y_{mi+j}^+|^p + (\sum_{i=0}^{r} E|Y_{mi+j}^+|^2)^{p/2}\right]$$
$$+(2m)^{p-1} d_p \sum_{j=1}^{m}\left[\sum_{i=0}^{r} E|Y_{mi+j}^-|^p + (\sum_{i=0}^{r} E|Y_{mi+j}^-|^2)^{p/2}\right]$$
$$\leqslant 2^p m^{p-1} d_p \sum_{j=1}^{m}\left[\sum_{i=0}^{r} E|Y_{mi+j}|^p + (\sum_{i=0}^{r} E|Y_{mi+j}|^2)^{p/2}\right]$$
$$\triangleq d_{m,p}\left(\sum_{i=1}^{n} E|X_i|^p + (\sum_{i=1}^{n} EX_i^2)^{p/2}\right).$$

The proof is completed. □

3. Main results

Under the nonparametric regression model (1.1), for any fixed point $x \in A \subset \mathbb{R}^q$, some hypotheses on weight functions $W_{ni}(x) = W_{ni}(x, x_{n1}, \ldots, x_{nn})$ are given as follows:

$(H_1)\ \sum_{i=1}^n |W_{ni}(x)|^p \to 0$ as $n \to \infty$ for some $p \in (1,2]$;
$(H_2)\ \sum_{i=1}^n W_{ni}(x) \to 1$ as $n \to \infty$;
$(H_3)\ \sum_{i=1}^n |W_{ni}(x)| \leqslant C$ for all n;
$(H_4)\ \sum_{i=1}^n W_{ni}^2(x) \to 0$ as $n \to \infty$;
$(H_5)\ \sum_{i=1}^n |W_{ni}(x)| \cdot |g(x_{ni}) - g(x)| I(\|x_{ni} - x\| > a) \to 0$ as $n \to \infty$ for all $a > 0$.

Based on the conditions above, we can get the following results on mean consistency for the nonparametric regression estimator $g_n(x)$.

THEOREM 3.1. *Let $\{\varepsilon_n, n \geqslant 1\}$ be a sequence of mean zero m-END random variables. Assume that the conditions (H_2)–(H_5) are satisfied. If $\sup_{n\geqslant 1} E\varepsilon_n^2 < \infty$, then for any $x \in c(g)$ and any $r \in (0,2]$,*

$$E|g_n(x) - g(x)|^r \to 0, \quad \textit{as } n \to \infty. \tag{3.1}$$

If $\sup_{n\geqslant 1} E|\varepsilon_n|^r < \infty$ for some $r > 2$, then (3.1) also holds true.

THEOREM 3.2. *Let $\{\varepsilon_n, n \geqslant 1\}$ be a sequence of mean zero m-END random variables. Assume that the conditions (H_1), (H_2), (H_3) and (H_5) are satisfied. If* $\sup_{n\geqslant 1} E|\varepsilon_n|^p < \infty$ *for some* $p \in (1,2]$, *then*

$$E|g_n(x) - g(x)|^p \to 0, \quad as\ n \to \infty. \tag{3.2}$$

In order to obtain the uniform convergence for the estimator of $g(x)$, some uniform version of assumptions on $W_{ni}(x) = W_{ni}(x, x_{n1}, \ldots, x_{nn})$ are needed as follows:

(H_1') $\sup_{x\in A} \sum_{i=1}^n |W_{ni}(x)|^p \to 0$ as $n \to \infty$ for some $p \in (1,2]$;
(H_2') $\sup_{x\in A} |\sum_{i=1}^n W_{ni}(x) - 1| \to 0$ as $n \to \infty$;
(H_3') $\sup_{x\in A} |\sum_{i=1}^n W_{ni}(x)| \leqslant C$ for all n;
(H_4') $\sup_{x\in A} \sum_{i=1}^n W_{ni}^2(x) \to 0$ as $n \to \infty$;
(H_5') $\sup_{x\in A} \sum_{i=1}^n |W_{ni}(x)| \cdot |g(x_{ni}) - g(x)| I(\|x_{ni} - x\| > a) \to 0$ as $n \to \infty$ for all $a > 0$.

Based on the conditions of (H_1')–(H_5'), we can get the uniform convergence for the estimator of $g(x)$ as follows.

THEOREM 3.3. *Let $\{\varepsilon_n, n \geqslant 1\}$ be a sequence of mean zero m-END random variables. Suppose that the conditions (H_2')–(H_5') hold true and g is continuous on the compact set A. If* $\sup_{n\geqslant 1} E\varepsilon_n^2 < \infty$, *then for any* $r \in (0,2]$,

$$\sup_{x\in A} E|g_n(x) - g(x)|^r \to 0, \quad as\ n \to \infty. \tag{3.3}$$

If $\sup_{n\geqslant 1} E|\varepsilon_n|^r < \infty$ *for some* $r > 2$, *then* (3.3) *also holds true.*

THEOREM 3.4. *Let $\{\varepsilon_n, n \geqslant 1\}$ be a sequence of mean zero m-END random variables. Assume that the conditions (H_1'), (H_2'), (H_3') and (H_5') hold true and g is continuous on the compact set A. If* $\sup_{n\geqslant 1} E|\varepsilon_n|^p < \infty$ *for some* $p \in (1,2]$, *then*

$$\sup_{x\in A} E|g_n(x) - g(x)|^p \to 0, \quad as\ n \to \infty. \tag{3.4}$$

As an application of the above results, we give an example for the nearest neighbor estimator of $g(x)$. Without loss of generality, let A=[0,1] and $x_{ni} = i/n, i = 1, 2, \ldots, n$. For any $x \in A \subset \mathbb{R}^q$, we rewrite $|x_{n1} - x|, |x_{n2} - x|, \ldots, |x_{nn} - x|$ as follows:

$$|x^{(n)}_{R_1(x)} - x| \leqslant |x^{(n)}_{R_2(x)} - x| \leqslant \ldots \leqslant |x^{(n)}_{R_n(x)} - x|,$$

if $|x_{ni} - x| = |x_{nj} - x|$, then $|x_{ni} - x|$ will be put in front for $i < j$. Define the nearest neighbor estimator as follows:

$$g_n(x) = \sum_{i=1}^n W_{ni}(x) Y_{ni}, x \in A \subset \mathbb{R}^q, \tag{3.5}$$

where $W_{ni}(x) = W_{ni}(x, x_{n1}, \ldots, x_{nn})$ are weight functions and

$$W_{ni}(x) = W_{ni}(x, x_{n1}, \ldots, x_{nn}) = \begin{cases} 1/k_n, \text{ if } |x_{ni} - x| \leqslant |x^{(n)}_{R_{k_n}(x)} - x|, \\ 0, \qquad \text{otherwise.} \end{cases}$$

Let $k_n = \lfloor n^{1/p} \rfloor$ for some $p > 1$ and g is continuous on the compact set A. Consequently, for every $x \in [0,1]$, we can obtain by the definition of $R_i(x)$ and the choice of x_{ni} that

$$\sum_{i=1}^{n} |W_{ni}(x)|^p = \sum_{i=1}^{n} |W_{nR_i(x)}(x)|^p = \sum_{i=1}^{k_n} \frac{1}{k_n^p} = k_n^{1-p};$$

$$\sum_{i=1}^{n} W_{ni}(x) = \sum_{i=1}^{n} W_{nR_i(x)}(x) = \sum_{i=1}^{k_n} \frac{1}{k_n} = 1;$$

$$\sum_{i=1}^{n} |W_{ni}(x)| = \sum_{i=1}^{n} W_{ni}(x) = \sum_{i=1}^{n} W_{nR_i(x)}(x) = \sum_{i=1}^{k_n} \frac{1}{k_n} = 1;$$

$$\sum_{i=1}^{n} W_{ni}^2(x) = \sum_{i=1}^{n} W_{nR_i(x)}^2(x) = \sum_{i=1}^{k_n} \frac{1}{k_n^2} = \frac{1}{k_n};$$

$$\begin{aligned}\sum_{i=1}^{n} W_{ni}(x) I(\|x_{ni} - x\| > a) &\leqslant \sum_{i=1}^{n} W_{ni}(x) \frac{(x_{ni} - x)^2}{a^2} = \sum_{i=1}^{k_n} \frac{(x_{R_i(x)}^{(n)} - x)^2}{k_n a^2} \\ &\leqslant \sum_{i=1}^{k_n} \frac{(i/n)^2}{k_n a^2} \leqslant (\frac{k_n}{na})^2 \leqslant \frac{C}{n^{2-2/p}}.\end{aligned}$$

Hence, the conditions (H_1)–(H_5) are satisfied. Similarly, the conditions (H_1')–(H_5') are also satisfied. Therefore, by Theorems 3.3 and 3.4, we can get the following results on mean consistency for the nearest neighbor estimator.

COROLLARY 3.1. *Let $\{\varepsilon_n, n \geqslant 1\}$ be a sequence of mean zero m-END random variables. Assume that g is continuous on the compact set A and $k_n = \lfloor n^{1/p} \rfloor$ for some $p > 1$.*

(i) If $\sup_{n \geqslant 1} E\varepsilon_n^2 < \infty$, then (3.3) holds for any $r \in (0,2]$;

(ii) If $\sup_{n \geqslant 1} E|\varepsilon_n|^p < \infty$, then (3.4) holds true.

4. Proofs of the main results

Proof of Theorem 3.1. By C_r-inequality, we have

$$E|g_n(x) - g(x)|^r \leqslant C[E|g_n(x) - Eg_n(x)|^r + |Eg_n(x) - g(x)|^r]. \tag{4.1}$$

For $x \in c(g)$ and $a > 0$, we can see that

$$\begin{aligned}|Eg_n(x) - g(x)| \leqslant &\sum_{i=1}^{n} |W_{ni}(x)| \cdot |g(x_{ni}) - g(x)| I(\|x_{ni} - x\| \leqslant a) \\ &+ \sum_{i=1}^{n} |W_{ni}(x)| \cdot |g(x_{ni}) - g(x)| I(\|x_{ni} - x\| > a) \\ &+ |g(x)| \cdot |\sum_{i=1}^{n} |W_{ni}(x) - 1|.\end{aligned}$$

Since $x \in c(g)$, we can get that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|g(x') - g(x)| < \varepsilon$ when $|x' - x| < \delta$. Setting $a \in (0, \delta)$, we can get

$$
\begin{aligned}
|Eg_n(x) - g(x)| \leqslant \varepsilon \sum_{i=1}^{n} |W_{ni}(x)| \; &+ \sum_{i=1}^{n} |W_{ni}(x)| \cdot |g(x_{ni}) - g(x)| I(\|x_{ni} - x\| > a) \\
&+ |g(x)| \cdot |\sum_{i=1}^{n} W_{ni}(x) - 1|.
\end{aligned}
$$

By conditions (H_2), (H_3) and (H_5), we have that

$$
|Eg_n(x) - g(x)| \to 0, \text{ as } n \to \infty, \quad x \in c(g). \tag{4.2}
$$

On the other hand, for the fixed x, by Lemma 2.1, we can see that $\{W_{ni}^{+}(x)\varepsilon_i, 1 \leqslant i \leqslant n\}$ and $\{W_{ni}^{-}(x)\varepsilon_i, 1 \leqslant i \leqslant n\}$ are both m-END sequences. Noting that $W_{ni}(x)\varepsilon_i = W_{ni}^{+}(x)\varepsilon_i - W_{ni}^{-}(x)\varepsilon_i$, without loss of generality, we assume that $W_{ni}(x) \geqslant 0$ in what follows. If $0 < r \leqslant 2$, by Jensen's inequality, Lemma 2.2, (H_4) and $\sup_{n \geqslant 1} E\varepsilon_n^2 < \infty$, we have

$$
\begin{aligned}
E|g_n(x) - Eg_n(x)|^r &= E\left|\sum_{i=1}^{n} W_{ni}(x)\varepsilon_{ni}\right|^r \\
&= E\left|\sum_{i=1}^{n} W_{ni}(x)\varepsilon_i\right|^r \leqslant \left[E\left(\sum_{i=1}^{n} W_{ni}(x)\varepsilon_i\right)^2\right]^{r/2} \\
&\leqslant C_1 \left[\sum_{i=1}^{n} W_{ni}^2(x) E\varepsilon_i^2\right]^{r/2} \\
&\leqslant C_2 \left[\sum_{i=1}^{n} W_{ni}^2(x)\right]^{r/2} \to 0, \text{ as } n \to \infty,
\end{aligned} \tag{4.3}
$$

following from that $\{\varepsilon_{ni}, 1 \leqslant i \leqslant n\}$ has the same distribution as that of $\{\varepsilon_i, 1 \leqslant i \leqslant n\}$ for each n. If $r > 2$, by Lemma 2.2, $\sup_{n \geqslant 1} E|\varepsilon_n|^r < \infty$ and (H_4) again, we obtain

$$
\begin{aligned}
E\,|g_n(x) - Eg_n(x)|^r &= E\left|\sum_{i=1}^{n} W_{ni}(x)\varepsilon_{ni}\right|^r = E\left|\sum_{i=1}^{n} W_{ni}(x)\varepsilon_i\right|^r \\
&\leqslant C_3 \left\{\sum_{i=1}^{n} W_{ni}^r(x) E|\varepsilon_i|^r + \left[\sum_{i=1}^{n} W_{ni}^2(x) E\varepsilon_i^2\right]^{r/2}\right\} \\
&\leqslant C_4 \left\{\left[\sum_{i=1}^{n} W_{ni}^2(x)\right]^{r/2} + \left[\sum_{i=1}^{n} W_{ni}^2(x)\right]^{r/2}\right\} \\
&\to 0, \text{ as } n \to \infty,
\end{aligned} \tag{4.4}
$$

since $(\sum_{i=1}^{n} a_i^{\alpha})^{1/\alpha} \geqslant (\sum_{i=1}^{n} a_i^{\beta})^{1/\beta}$ holds for any positive number sequence $\{a_i, 1 \leqslant i \leqslant n\}$ and $1 \leqslant \alpha \leqslant \beta$. Therefore, the desired result (3.1) follows from (4.1)–(4.4) immediately. This completes the proof of the theorem. $\square$

Proof of Theorem 3.2. For $p \in (1,2]$, by (H_1), Lemma 2.2 and $\sup_{n\geqslant 1} E|\varepsilon_n|^p < \infty$, we get

$$\begin{aligned} E|g_n(x) - Eg_n(x)|^p &= E\left|\sum_{i=1}^{n} W_{ni}(x)\varepsilon_{ni}\right|^p = E\left|\sum_{i=1}^{n} W_{ni}(x)\varepsilon_i\right|^p \\ &\leqslant C_5 \sum_{i=1}^{n} |W_{ni}(x)|^p E|\varepsilon_i|^p \\ &\leqslant C_6 \sum_{i=1}^{n} |W_{ni}(x)|^p \to 0, \text{ as } n \to \infty. \end{aligned} \tag{4.5}$$

Besides, noting that the conditions (H_2), (H_3) and (H_5) are also satisfied, we can obtain that $|Eg_n(x) - g(x)| \to 0$, as $n \to \infty$ for any $x \in c(g)$. Therefore, the desired result (3.2) follows from (4.1), (4.2) and (4.5) immediately. The proof is completed. □

Proof of Theorem 3.3. Since g is continuous on the compact set A, we have that g is uniformly continuous on the compact set A. Consequently, similar to the proof of Theorem 3.1, we can get that

$$\lim_{n\to\infty} \sup_{x\in A} E|g_n(x) - Eg_n(x)|^r = 0, \qquad \lim_{n\to\infty} \sup_{x\in A} |Eg_n(x) - g(x)|^r = 0.$$

Therefore,

$$\begin{aligned} \sup_{x\in A} E|g_n(x) - g(x)|^r &\leqslant c_p \left[\sup_{x\in A} E|g_n(x) - Eg_n(x)|^r + \sup_{x\in A} |Eg_n(x) - g(x)|^r\right] \\ &\to 0, \quad \text{as } n \to \infty, \end{aligned}$$

which implies the desired result (3.3) immediately. The proof is completed. □

Proof of Theorem 3.4. Since g is continuous on the compact set A, we can see that g is uniformly continuous on the compact set A. Consequently, similar to the proofs of Theorem 3.1 and Theorem 3.2, we can get that

$$\lim_{n\to\infty} \sup_{x\in A} E|g_n(x) - Eg_n(x)|^p = 0, \qquad \lim_{n\to\infty} \sup_{x\in A} |Eg_n(x) - g(x)|^p = 0.$$

Therefore,

$$\begin{aligned} \sup_{x\in A} E|g_n(x) - g(x)|^p &\leqslant c_p \left[\sup_{x\in A} E|g_n(x) - Eg_n(x)|^p + \sup_{x\in A} |Eg_n(x) - g(x)|^p\right] \\ &\to 0, \quad \text{as } n \to \infty, \end{aligned}$$

which implies the desired result (3.4). This completes the proof of the theorem. □

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