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# FRACTIONAL INTEGRAL ASSOCIATED WITH SCHRÖDINGER OPERATOR ON VANISHING GENERALIZED MORREY SPACES

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*Abstract.* Let $L=-\triangle+V$ be a Schrödinger operator, where the non-negative potential $V$ belongs to the reverse Hölder class $RH_{n/2}$, let $b$ belong to a new $BMO_\theta(\rho)$ space, and let $\mathscr{I}_\beta^L$ be the fractional integral operator associated with $L$. In this paper, we study the boundedness of the operator $\mathscr{I}_\beta^L$ and its commutators $[b,\mathscr{I}_\beta^L]$ with $b\in BMO_\theta(\rho)$ on generalized Morrey spaces associated with Schrödinger operator $M_{p,\varphi}^{\alpha,V}$ and vanishing generalized Morrey spaces associated with Schrödinger operator $VM_{p,\varphi}^{\alpha,V}$. We find the sufficient conditions on the pair $(\varphi_1,\varphi_2)$ which ensures the boundedness of the operator $\mathscr{I}_\beta^L$ from $M_{p,\varphi_1}^{\alpha,V}$ to $M_{q,\varphi_2}^{\alpha,V}$ and from $VM_{p,\varphi_1}^{\alpha,V}$ to $VM_{q,\varphi_2}^{\alpha,V}$, $1/p-1/q=\beta/n$. When $b$ belongs to $BMO_\theta(\rho)$ and $(\varphi_1,\varphi_2)$ satisfies some conditions, we also show that the commutator operator $[b,\mathscr{I}_\beta^L]$ is bounded from $M_{p,\varphi_1}^{\alpha,V}$ to $M_{q,\varphi_2}^{\alpha,V}$ and from $VM_{p,\varphi_1}^{\alpha,V}$ to $VM_{q,\varphi_2}^{\alpha,V}$, $1/p-1/q=\beta/n$.

## 1. Introduction and results

Let us consider the Schrödinger operator

$$L=-\Delta+V \text{ on } \mathbb{R}^n,\ n\geqslant 3,$$

where $V$ is a non-negative, $V\neq 0$, and belongs to the reverse Hölder class $RH_q$ for some $q\geqslant n/2$, i.e., there exists a constant $C>0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}V^q(y)dy\right)^{1/q}\leqslant \frac{C}{|B(x,r)|}\int_{B(x,r)}V(y)dy$$

holds for every $x\in\mathbb{R}^n$ and $0<r<\infty$, where $B(x,r)$ denotes the ball centered at $x$ with radius $r$. In particular, if $V$ is a nonnegative polynomial, then $V\in RH_\infty$.

Obviously, $RH_{q_2}\subset RH_{q_1}$, if $q_2>q_1$. The most important property of the class $RH_q$ is its self-improvement, that is, if $V\in RH_q$, then $V\in RH_{q+\varepsilon}$ for some $\varepsilon>0$.

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As in [18], for a given potential $V \in RH_q$ with $q \geqslant n/2$, we define the auxiliary function

$$\rho(x) := \frac{1}{m_V(x)} = \sup_{r>0}\left\{r : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy \leqslant 1\right\}.$$

It is well-known that that $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^n$.

According to [4], the new BMO space $BMO_\theta(\rho)$ with $\theta \geqslant 0$ is defined as a set of all locally integrable functions $b$ such that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |b(y) - b_B|dy \leqslant C\left(1 + \frac{r}{\rho(x)}\right)^\theta$$

for all $x \in \mathbb{R}^n$ and $r > 0$, where $b_B = \frac{1}{|B|}\int_B b(y)dy$. A norm for $b \in BMO_\theta(\rho)$, denoted by $[b]_\theta$, is given by the infimum of the constants in the inequalities above. Clearly, $BMO \subset BMO_\theta(\rho)$.

We now present the definition of generalized Morrey spaces (including weak version) related to potential, which introduced by Guliyev in [12].

DEFINITION 1. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $1 \leqslant p < \infty$, $\alpha \geqslant 0$, and $V \in RH_q$, $q \geqslant 1$. We denote by $M^{\alpha,V}_{p,\varphi} = M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space associated with Schrödinger operator, the space of all functions $f \in L^p_{loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M^{\alpha,V}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0}\left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi(x,r)^{-1} r^{-n/p}\|f\|_{L_p(B(x,r))}.$$

Also $WM^{\alpha,V}_{p,\varphi} = WM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space associated with Schrödinger operator,the space of all functions $f \in WL^p_{loc}(\mathbb{R}^n)$ with

$$\|f\|_{WM^{\alpha,V}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0}\left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi(x,r)^{-1} r^{-n/p}\|f\|_{WL_p(B(x,r))} < \infty.$$

REMARK 1. $(i)$ When $\alpha = 0$, and $\varphi(x,r) = r^{(\lambda-n)/p}$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the classical Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ introduced by Morrey in [13];

$(ii)$ When $\varphi(x,r) = r^{(\lambda-n)/p}$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the Morrey space associated with Schrödinger operator $L^{\alpha,V}_{p,\lambda}(\mathbb{R}^n)$ studied by Tang and Dong in [21];

$(iii)$ When $\alpha = 0$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the generalized Morrey space $M_{p,\varphi}(\mathbb{R}^n)$ introduced by Mizuhara and Nakai in [14, 15].

$(iv)$ The generalized Morrey space associated with Schrödinger operator $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ was introduced by Guliyev in [12].

The classical Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ was introduced by Morrey in [13] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [7, 8, 9, 13]. The generalized Morrey spaces are defined with $r^\lambda$ replaced by a general non-negative function $\varphi(x,r)$ satisfying some assumptions (see, for example, [10, 14, 15, 19] and etc).

For brevity, in the sequel we use the notations

$$\mathfrak{A}^{\alpha,V}_{p,\varphi}(f;x,r) := \left(1+\frac{r}{\rho(x)}\right)^{\alpha} r^{-n/p}\,\varphi(x,r)^{-1}\|f\|_{L_p(B(x,r))}$$

and

$$\mathfrak{A}^{W,\alpha,V}_{\Phi,\varphi}(f;x,r) := \left(1+\frac{r}{\rho(x)}\right)^{\alpha} r^{-n/p}\,\varphi(x,r)^{-1}\|f\|_{WL_p(B(x,r))}.$$

DEFINITION 2. The vanishing generalized Morrey space associated with Schrödinger operator $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f\in M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}^{\alpha,V}_{p,\varphi}(f;x,r)=0. \tag{1}$$

The vanishing weak generalized Morrey space associated with Schrödinger operator $VWM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f\in WM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}^{W,\alpha,V}_{p,\varphi}(f;x,r)=0.$$

The vanishing spaces $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ and $VWM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\|f\|_{VM^{\alpha,V}_{p,\varphi}}\equiv\|f\|_{M^{\alpha,V}_{p,\varphi}}=\sup_{x\in\mathbb{R}^n,r>0}\mathfrak{A}^{\alpha,V}_{p,\varphi}(f;x,r),$$
$$\|f\|_{VWM^{\alpha,V}_{p,\varphi}}\equiv\|f\|_{WM^{\alpha,V}_{p,\varphi}}=\sup_{x\in\mathbb{R}^n,r>0}\mathfrak{A}^{\alpha,V}_{W,p,\varphi}(f;x,r),$$

respectively.

In the case $\alpha=0$, and $\varphi(x,r)=r^{(\lambda-n)/p}$ $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the vanishing Morrey space $VM_{p,\lambda}$ introduced in [22], where applications to PDE were considered.

We refer to [1, 6, 16, 17] for some properties of vanishing generalized Morrey spaces.

DEFINITION 3. Let $L=-\triangle+V$ with $V\in RH_{n/2}$. The fractional integral associated with $L$ is defined by

$$\mathscr{I}^L_\beta f(x)=L^{-\beta/2}f(x)=\int_0^\infty e^{-tL}(f)(x)\,t^{\beta/2-1}dt$$

for $0<\beta<n$. The commutator of $\mathscr{I}^L_\beta$ is defined by

$$[b,\mathscr{I}^L_\beta]f(x)=b(x)\mathscr{I}^L_\beta f(x)-\mathscr{I}^L_\beta(bf)(x).$$

In this paper, we consider the boundedness of the fractional integral operator $\mathscr{I}_\beta^L$ on the generalized Morrey spaces $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ and the vanishing generalized Morrey spaces $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$. When $b$ belongs to the new $BMO$ space $BMO_\theta(\rho)$, we also show that $[b,\mathscr{I}_\beta^L]$ is bounded from $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ to $M_{q,\varphi}^{\alpha,V}(\mathbb{R}^n)$ and from $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ to $VM_{q,\varphi}^{\alpha,V}(\mathbb{R}^n)$.

Our main results are as follows.

THEOREM 1. *Let $V \in RH_{n/2}$, $\alpha \geqslant 0$, $1<p<n/\beta$, $1/q=1/p-\beta/n$ and $\varphi_1 \in \Omega_p^{\alpha,V}$, $\varphi_2 \in \Omega_q^{\alpha,V}$ satisfies the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{q}}}\frac{dt}{t} \leqslant c_0\varphi_2(x,r), \tag{2}$$

*where $c_0$ does not depend on $x$ and $r$. Then the operator $\mathscr{I}_\beta^L$ is bounded on $M_{p,\varphi_1}^{\alpha,V}$ to $M_{q,\varphi_2}^{\alpha,V}$ for $p>1$ and from $M_{1,\varphi_1}^{\alpha,V}$ to $WM_{\frac{n}{n-\beta},\varphi_2}^{\alpha,V}$. Moreover, for $p>1$*

$$\|\mathscr{I}_\beta^L f\|_{M_{q,\varphi_2}^{\alpha,V}} \leqslant C\|f\|_{M_{p,\varphi_1}^{\alpha,V}},$$

*and for $p=1$*

$$\|\mathscr{I}_\beta^L f\|_{WM_{\frac{n}{n-\beta},\varphi_2}^{\alpha,V}} \leqslant C\|f\|_{M_{1,\varphi_1}^{\alpha,V}},$$

*where $C$ does not depend on $f$.*

THEOREM 2. *Let $V \in RH_{n/2}$, $\alpha \geqslant 0$, $1<p<n/\beta$, $1/q=1/p-\beta/n$ and $\varphi_1 \in \Omega_p^{\alpha,V}$, $\varphi_2 \in \Omega_q^{\alpha,V}$ satisfies the condition*

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right)\frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{q}}}\frac{dt}{t} \leqslant c_0\varphi_2(x,r), \tag{3}$$

*where $c_0$ does not depend on $x$ and $r$. If $b \in BMO_\theta(\rho)$, then the operator $[b,\mathscr{I}_\beta^L]$ is bounded from $M_{p,\varphi_1}^{\alpha,V}$ to $M_{q,\varphi_2}^{\alpha,V}$ and*

$$\|[b,\mathscr{I}_\beta^L]f\|_{M_{q,\varphi_2}^{\alpha,V}} \leqslant C[b]_\theta\|f\|_{M_{p,\varphi_1}^{\alpha,V}},$$

*where $C$ does not depend on $f$.*

THEOREM 3. *Let* $V \in RH_{n/2}$, $\alpha \geqslant 0$, $1 \leqslant p < n/\beta$, $1/q = 1/p - \beta/n$ *and* $\varphi_1 \in \Omega^{\alpha,V}_{p,1}$, $\varphi_2 \in \Omega^{\alpha,V}_{q,1}$ *satisfies the conditions*

$$c_\delta := \int_\delta^\infty \sup_{x\in\mathbb{R}^n} \varphi_1(x,t)\frac{dt}{t} < \infty$$

*for every* $\delta > 0$, *and*

$$\int_r^\infty \varphi_1(x,t)\frac{dt}{t^{1-\beta}} \leqslant C_0\varphi_2(x,r), \tag{4}$$

*where* $C_0$ *does not depend on* $x \in \mathbb{R}^n$ *and* $r > 0$. *Then the operator* $\mathscr{I}^L_\beta$ *is bounded from* $VM^{\alpha,V}_{p,\varphi_1}$ *to* $VM^{\alpha,V}_{q,\varphi_2}$ *for* $p > 1$ *and from* $VM^{\alpha,V}_{1,\varphi_1}$ *to* $VWM^{\alpha,V}_{\frac{n}{n-\beta},\varphi_2}$.

THEOREM 4. *Let* $V \in RH_{n/2}$, $b \in BMO_\theta(\rho)$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$, *and* $\varphi_1 \in \Omega^{\alpha,V}_{p,1}$, $\varphi_2 \in \Omega^{\alpha,V}_{q,1}$ *satisfies the conditions*

$$\int_r^\infty \Big(1+\ln\frac{t}{r}\Big)\varphi_1(x,t)\frac{dt}{t^{1-\beta}} \leqslant c_0\varphi_2(x,r), \tag{5}$$

*where* $c_0$ *does not depend on* $x$ *and* $r$,

$$\lim_{r\to 0}\frac{\ln\frac{1}{r}}{\inf limits_{x\in\mathbb{R}^n}\varphi_2(x,r)} = 0 \tag{6}$$

*and*

$$c_\delta := \int_\delta^\infty \Big(1+|\ln t|\Big)\sup_{x\in\mathbb{R}^n}\varphi_1(x,t)\frac{dt}{t^{1-\beta}} < \infty \tag{7}$$

*for every* $\delta > 0$. *Then the operator* $[b,\mathscr{I}^L_\beta]$ *is bounded from* $VM^{\alpha,V}_{p,\varphi_1}$ *to* $VM^{\alpha,V}_{q,\varphi_2}$.

REMARK 2. Note that, Theorems 1 and 2 in the case of $V \equiv 0$ was proved in [11, Corollary 5.5 and 7.5] and in the case of $\varphi(x,r) = r^{(\lambda-n)/p}$ in [21, Theorems 1.3 and 1.4].

REMARK 3. Note that, in [2] the Nikolskii-Morrey type spaces were introduced and the authors studied some embedding theorems. In the next paper, we shall introduce the generalized Nikolskii-Morrey spaces associated with Schrödinger operator and will study some embedding theorems. We will also investigate the boundedness of fractional integral associated with Schrödinger operator on the generalized Nikolskii-Morrey spaces associated with Schrödinger operator.

In this paper, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant $C$, independent of all important parameters, such that $A \leqslant CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

## 2. Some preliminaries

We would like to recall the important properties concerning the critical function.

LEMMA 1. [18] *Let $V \in RH_{n/2}$. For the associated function $\rho$ there exist $C$ and $k_0 \geqslant 1$ such that*

$$C^{-1}\rho(x)\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{-k_0} \leqslant \rho(y) \leqslant C\rho(x)\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{\frac{k_0}{1+k_0}} \tag{8}$$

*for all $x,y \in \mathbb{R}^n$.*

LEMMA 2. [3] *Suppose $x \in B(x_0,r)$. Then for $k \in N$ we have*

$$\frac{1}{\Big(1+\frac{2^k r}{\rho(x)}\Big)^{N}} \lesssim \frac{1}{\Big(1+\frac{2^k r}{\rho(x_0)}\Big)^{N/(k_0+1)}}.$$

We give some inequalities about the new BMO space $BMO_\theta(\rho)$.

LEMMA 3. [4] *Let $1 \leqslant s < \infty$. If $b \in BMO_\theta(\rho)$, then*

$$\Big(\frac{1}{|B|}\int_B |b(y)-b_B|^s dy\Big)^{1/s} \leqslant [b]_\theta\Big(1+\frac{r}{\rho(x)}\Big)^{\theta'}$$

*for all $B = B(x,r)$, with $x \in \mathbb{R}^n$ and $r > 0$, where $\theta' = (k_0+1)\theta$ and $k_0$ is the constant appearing in* (8).

LEMMA 4. [4] *Let $1 \leqslant s < \infty$, $b \in BMO_\theta(\rho)$, and $B = B(x,r)$. Then*

$$\Big(\frac{1}{|2^k B|}\int_{2^k B} |b(y)-b_B|^s dy\Big)^{1/s} \leqslant [b]_\theta k\Big(1+\frac{2^k r}{\rho(x)}\Big)^{\theta'}$$

*for all $k \in \mathbb{N}$, with $\theta'$ as in Lemma 3.*

Let $K_\beta$ be the kernel of $\mathscr{I}_\beta^L$. The following result give the estimate on the kernel $K_\beta(x,y)$.

LEMMA 5. [5] *If $V \in RH_{n/2}$, then for every $N$, there exists a constant $C$ such that*

$$|K_\beta(x,y)| \leqslant \frac{C}{\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{N}}\frac{1}{|x-y|^{n-\beta}}. \tag{9}$$

Finally, we recall a relationship between essential supremum and essential infimum.

LEMMA 6. [23] *Let $f$ be a real-valued nonnegative function and measurable on $E$. Then*

$$\left(\operatorname*{ess\,inf}_{x\in E} f(x)\right)^{-1} = \operatorname*{ess\,sup}_{x\in E} \frac{1}{f(x)}.$$

LEMMA 7. [3] *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $1 \leqslant p < \infty$, $\alpha \geqslant 0$, and $V \in RH_q$, $q \geqslant 1$.*

*(i) If $\sup\limits_{t<r<\infty} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \frac{r^{-\frac{n}{p}}}{\varphi(x,r)} = \infty$ for some $t > 0$ and for all $x \in \mathbb{R}^n$, then $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n) = \Theta$.*

*(ii) If $\sup\limits_{0<r<\tau} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi(x,r)^{-1} = \infty$ for some $\tau > 0$ and for all $x \in \mathbb{R}^n$, then $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n) = \Theta$.*

REMARK 4. We denote by $\Omega_p^{\alpha,V}$ the sets of all positive measurable functions $\varphi$ on $\mathbb{R}^n \times (0,\infty)$ such that for all $t > 0$,

$$\sup_{x\in\mathbb{R}^n} \left\| \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \frac{r^{-\frac{n}{p}}}{\varphi(x,r)} \right\|_{L_\infty(t,\infty)} < \infty, \quad \text{and} \quad \sup_{x\in\mathbb{R}^n} \left\| \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi(x,r)^{-1} \right\|_{L_\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 7, we always assume that $\varphi \in \Omega_p^{\alpha,V}$.

REMARK 5. We denote by $\Omega_{p,1}^{\alpha,V}$ the sets of all positive measurable functions $\varphi$ on $\mathbb{R}^n \times (0,\infty)$ such that

$$\inf_{x\in\mathbb{R}^n} \inf_{r>\delta} \left(1+\frac{r}{\rho(x)}\right)^{-\alpha} \varphi(x,r) > 0, \text{ for some } \delta > 0, \tag{10}$$

and

$$\lim_{r\to 0} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \frac{r^{n/p}}{\varphi(x,r)} = 0.$$

For the non-triviality of the space $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ we always assume that $\varphi \in \Omega_{p,1}^{\alpha,V}$.

## 3. Proof of Theorem 1

We first prove the following conclusions

THEOREM 5. *Let $V \in RH_{n/2}$. If $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ then the inequality*

$$\|\mathscr{I}_\beta^L(f)\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}$$

*holds for any* $f \in L^p_{loc}(\mathbb{R}^n)$. *Moreover, for* $p=1$ *the inequality*

$$\|\mathscr{I}^L_\beta(f)\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))} \lesssim r^{n-\beta} \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t}$$

*holds for any* $f \in L^1_{loc}(\mathbb{R}^n)$.

*Proof.* For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ and $\lambda B = B(x_0,\lambda r)$ for any $\lambda > 0$. We write $f$ as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{B(x_0,2r)}(y)$, and $\chi_{B(x_0,2r)}$ denotes the characteristic function of $B(x_0,2r)$. Then

$$\|\mathscr{I}^L_\beta(f)\|_{L_q(B(x_0,r))} \leqslant \|\mathscr{I}^L_\beta(f_1)\|_{L_q(B(x_0,r))} + \|\mathscr{I}^L_\beta(f_2)\|_{L_q(B(x_0,r))}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$ and from the boundedness of $\mathscr{I}^L_\beta$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ (see [20]) it follows that

$$\begin{aligned}
\|\mathscr{I}^L_\beta(f_1)\|_{L_q(B(x_0,r))} &\lesssim \|f\|_{L_p(B(x_0,2r))} \\
&\lesssim r^{\frac{n}{q}} \|f\|_{L_p(B(x_0,2r))} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{q}+1}} \\
&\lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}.
\end{aligned} \tag{11}$$

To estimate $\|\mathscr{I}^L_\beta(f_2)\|_{L_p(B(x_0,r))}$, obverse that $x \in B$, $y \in (2B)^c$ implies $|x-y| \approx |x_0 - y|$. Then by (9) we have

$$\begin{aligned}
\sup_{x\in B} |\mathscr{I}^L_\beta(f_2)(x)| &\leqslant \sup_{x\in B} \int_{(2B)^c} |K_\beta(x,y)f(y)|dy \\
&\lesssim \int_{(2B)^c} \frac{|f(y)|}{|x_0-y|^{n-\beta}} dy \\
&\lesssim \sum_{k=1}^{\infty} (2^{k+1}r)^{-n+\beta} \int_{2^{k+1}B} |f(y)|dy.
\end{aligned}$$

By Hölder's inequality we get

$$\begin{aligned}
\sup_{x\in B} |\mathscr{I}^L_\beta(f_2)(x)| &\lesssim \sum_{k=1}^{\infty} \|f\|_{L_p(2^{k+1}B)} (2^{k+1}r)^{-1-\frac{n}{p}+\beta} \int_{2^k r}^{2^{k+1}r} dt \\
&\lesssim \sum_{k=1}^{\infty} \int_{2^k r}^{2^{k+1}r} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \\
&\lesssim \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}.
\end{aligned} \tag{12}$$

Then

$$\|\mathscr{I}_\beta^L(f_2)\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \tag{13}$$

holds for $1 \leqslant p < n/\beta$. Therefore, by (11) and (13) we get

$$\|\mathscr{I}_\beta^L(f)\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \tag{14}$$

holds for $1 \leqslant p < n/\beta$.

When $p=1$, by the boundedness of $\mathscr{I}_\beta^L$ from $L_1(\mathbb{R}^n)$ to $WL_{\frac{n}{n-\beta}}(\mathbb{R}^n)$, we get

$$\|\mathscr{I}_\beta^L(f_1)\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))} \lesssim \|f\|_{L_1(B(x_0,2r))} \lesssim r^{n-\beta} \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t}.$$

By (13) we have

$$\|\mathscr{I}_\beta^L(f_2)\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))} \leqslant \|\mathscr{I}_\beta^L(f_2)\|_{L_{\frac{n}{n-\beta}}(B(x_0,2r))} \lesssim r^{n-\beta} \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t}.$$

Then

$$\|\mathscr{I}_\beta^L(f)\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))} \lesssim r^{n-\beta} \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t}. \qquad \square$$

*Proof of Theorem 1.* From Lemma 6, we have

$$\frac{1}{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}} = \operatorname*{ess\,sup}_{t<s<\infty} \frac{1}{\varphi_1(x,s)s^{\frac{n}{p}}}.$$

Note the fact that $\|f\|_{L_p(B(x_0,t))}$ is a nondecresing function of $t$, and $f \in M_{p,\varphi_1}^{\alpha,V}$, then

$$\begin{aligned}
\frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_p(B(x_0,t))}}{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}} &\lesssim \operatorname*{ess\,sup}_{t<s<\infty} \frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_p(B(x_0,t))}}{\varphi_1(x_0,s)s^{\frac{n}{p}}} \\
&\lesssim \sup_{0<s<\infty} \frac{\left(1+\frac{s}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_p(B(x_0,s))}}{\varphi_1(x_0,s)s^{\frac{n}{p}}} \\
&\lesssim \|f\|_{M_{p,\varphi_1}^{\alpha,V}}.
\end{aligned}$$

Since $\alpha \geqslant 0$, and $(\varphi_1, \varphi_2)$ satisfies the condition (2), then

$$\int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} = \int_{2r}^{\infty} \frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_p(B(x_0,t))}}{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha} t^{\frac{n}{q}}} \frac{dt}{t}$$

$$
\begin{aligned}
&\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \int_{2r}^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha} t^{\frac{n}{q}}} \frac{dt}{t} \\
&\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \int_{r}^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t} \\
&\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \varphi_2(x_0,r). \qquad (15)
\end{aligned}
$$

Then by Theorem 5 we get

$$
\begin{aligned}
\|\mathscr{I}^L_\beta(f)\|_{M^{\alpha,V}_{q,\varphi_2}} &\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} r^{-n/q} \|\mathscr{I}^L_\beta(f)\|_{L_p(B(x_0,r))} \\
&\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \\
&\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}}.
\end{aligned}
$$

Let $q = \frac{n}{n-\beta}$, similar to the estimates of (15) we have

$$
\int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t} \lesssim \|f\|_{M^{\alpha,V}_{1,\varphi_1}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \varphi_2(x_0,r).
$$

Thus by Theorem 5 we get

$$
\begin{aligned}
\|\mathscr{I}^L_\beta(f)\|_{WM^{\alpha,V}_{\frac{n}{n-\beta},\varphi_2}} &\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} r^{\beta-n} \|\mathscr{I}^L_\beta(f)\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))} \\
&\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t} \\
&\lesssim \|f\|_{M^{\alpha,V}_{1,\varphi_1}}. \quad \square
\end{aligned}
$$

## 4. Proof of Theorem 2

As the proof of Theorem 1, it suffices to prove the following result.

THEOREM 6. *Let $V \in RH_{n/2}$, $b \in BMO_\theta(\rho)$. If $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ then the inequality*

$$
\|[b, \mathscr{I}^L_\beta(f)]\|_{L_q(B(x_0,r))} \lesssim [b]_\theta r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \qquad (16)
$$

*holds for any $f \in L^p_{loc}(\mathbb{R}^n)$.*

*Proof.* We write $f$ as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{B(x_0,2r)}(y)$. Then

$$\|[b,\mathscr{I}_\beta^L](f)\|_{L_q(B(x_0,r))} \leqslant \|[b,\mathscr{I}_\beta^L](f_1)\|_{L_q(B(x_0,r))} + \|[b,\mathscr{I}_\beta^L](f_2)\|_{L_q(B(x_0,r))}.$$

By the boundedness of $[b,\mathscr{I}_\beta^L]$ on $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ (see [21]) and (11) we get

$$\begin{aligned}
\|[b,\mathscr{I}_\beta^L](f_1)\|_{L_q(B(x_0,r))} &\lesssim [b]_\theta \|f\|_{L_p(B(x_0,2r))} \\
&\lesssim [b]_\theta r^{\frac{n}{q}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \\
&\lesssim [b]_\theta r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}.
\end{aligned} \tag{17}$$

We now turn to deal with the term $\|[b,\mathscr{I}_\beta^L](f_2)\|_{L_q(B(x_0,r))}$. For any given $x \in B(x_0,r)$ we have

$$|[b,\mathscr{I}_\beta^L]f_2(x)| \leqslant |b(x)-b_{2B}|\,|\mathscr{I}_\beta^L(f_2)(x)| + |\mathscr{I}_\beta^L((b-b_{2B})f_2)(x)|.$$

Then by (12), Lemma 3, and taking $N \geqslant (k_0+1)\theta$ we get

$$\begin{aligned}
\|(b(x)-b_{2B})\mathscr{I}_\beta^L(f_2)\|_{L_q(B(x_0,r))} &\lesssim [b]_\theta r^{\frac{n}{q}} \left(1+\frac{2r}{\rho(x_0)}\right)^{\theta-N/(k_0+1)} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \\
&\lesssim [b]_\theta r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}.
\end{aligned} \tag{18}$$

Finally, let us estimate $\|\mathscr{I}_\beta^L((b-b_{2B})f_2)\|_{L_q(B(x_0,r))}$. By (9), Lemma 2 and (12) we have

$$\begin{aligned}
\sup_{x\in B} |\mathscr{I}_\beta^L((b-b_{2B})f_2)(x)| &\lesssim \sup_{x\in B} \int_{(2B)^c} \frac{1}{\left(1+\frac{|x-y|}{\rho(x)}\right)^N} \frac{|b(y)-b_{2B}||f(y)|}{|x_0-y|^{n-\beta}} dy \\
&\lesssim \sup_{x\in B} \sum_{k=1}^{\infty} \frac{1}{(2^k r)^{n-\beta}\left(1+\frac{2^k r}{\rho(x)}\right)^N} \int_{2^{k+1}B} |b(y)-b_{2B}||f(y)|dy \\
&\lesssim \sum_{k=1}^{\infty} \frac{1}{(2^k r)^{n-\beta}\left(1+\frac{2^k r}{\rho(x_0)}\right)^{N/(k_0+1)}} \int_{2^{k+1}B} |b(y)-b_{2B}||f(y)|dy.
\end{aligned}$$

Note that

$$\int_{2^{k+1}B} |b(y)-b_{2B}||f(y)|dy \lesssim \Big(\int_{2^{k+1}B} |b(y)-b_{2B}|^{p'}\Big)^{1/p'} \|f\|_{L_p(B(x_0,2^{k+1}r))}$$
$$\lesssim [b]_\theta k\Big(1+\frac{2^k r}{\rho(x_0)}\Big)^{\theta'} (2^k r)^{\frac{n}{p'}} \|f\|_{L_p(B(x_0,2^{k+1}r))}.$$

Then

$$\sup_{x\in B} |\mathscr{I}_\beta^L((b-b_{2B})f_2)(x)| \lesssim [b]_\theta \sum_{k=1}^\infty \frac{k(2^k r)^{-\frac{n}{p}+\beta}}{\Big(1+\frac{2^k r}{\rho(x_0)}\Big)^{N/(k_0+1)-\theta'}} \|f\|_{L_p(B(x_0,2^{k+1}r))}$$
$$\lesssim [b]_\theta \sum_{k=1}^\infty k(2^k r)^{-\frac{n}{q}} \|f\|_{L_p(B(x_0,2^{k+1}r))}$$
$$\lesssim [b]_\theta \sum_{k=1}^\infty k \int_{2^k r}^{2^{k+1}r} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}.$$

Since $2^k r \leqslant t \leqslant 2^{k+1} r$, then $k \approx \ln \frac{t}{r}$. Thus

$$\sup_{x\in B} |\mathscr{I}_\beta^L((b-b_{2B})f_2)(x)| \lesssim [b]_\theta \sum_{k=1}^\infty k \int_{2^k r}^{2^{k+1}r} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}$$
$$\lesssim [b]_\theta \sum_{k=1}^\infty \int_{2^k r}^{2^{k+1}r} \ln\frac{t}{r} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}$$
$$\lesssim [b]_\theta \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}.$$

Then

$$\|\mathscr{I}_\beta^L((b-b_{2B})f_2)\|_{L_q(B(x_0,r))} \lesssim [b]_\theta r^{\frac{n}{q}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}. \tag{19}$$

Combining (17), (18) and (19), the proof of Theorem 6 is completed. □

*Proof of Theorem 2.* Since $f \in M^{\alpha,V}_{p,\varphi_1}$ and $(\varphi_1,\varphi_2)$ satisfies the condition (3), by (15) we have

$$\int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}$$
$$= \int_{2r}^{\infty} \frac{\Big(1+\frac{t}{\rho(x_0)}\Big)^{\alpha} \|f\|_{L_p(B(x_0,t))}}{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s) s^{\frac{n}{p}}} \Big(1+\ln\frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s) s^{\frac{n}{p}}}{\Big(1+\frac{t}{\rho(x_0)}\Big)^{\alpha} t^{\frac{n}{q}}} \frac{dt}{t}$$
$$\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s) s^{\frac{n}{p}}}{\Big(1+\frac{t}{\rho(x_0)}\Big)^{\alpha} t^{\frac{n}{q}}} \frac{dt}{t}$$

$$\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \int_r^\infty \left(1+\ln\frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t}$$

$$\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \varphi_2(x_0,r). \tag{20}$$

Then from Theorem 6 and by (20) we get

$$\begin{aligned}
&\|[b,\mathscr{I}^L_\beta](f)\|_{M^{\alpha,V}_{q,\varphi_2}} \\
&\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} r^{-n/q} \|[b,\mathscr{I}^L_\beta](f)\|_{L_q(B(x_0,r))} \\
&\lesssim [b]_\theta \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} \int_{2r}^\infty \left(1+\ln\frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \\
&\lesssim [b]_\theta \|f\|_{M^{\alpha,V}_{p,\varphi_1}}. \quad \square
\end{aligned}$$

## 5. Proof of Theorem 3

The statement is derived from the estimate (14). The estimation of the norm of the operator, that is, the boundedness in the non-vanishing space, immediately follows from by Theorem 1. So we only have to prove that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{p,\varphi_1}(f;x,r) = 0 \;\Rightarrow\; \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{q,\varphi_2}(\mathscr{I}^L_\beta(f);x,r) = 0 \tag{21}$$

and

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{1,\varphi_1}(f;x,r) = 0 \;\Rightarrow\; \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{W,\alpha,V}_{n/(n-\beta),\varphi_2}(\mathscr{I}^L_\beta(f);x,r) = 0. \tag{22}$$

To show that $\sup_{x\in\mathbb{R}^n} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi_2(x,r)^{-1} r^{-n/p} \|\mathscr{I}^L_\beta(f)\|_{L_q(B(x,r))} < \varepsilon$ for small $r$, we split the right-hand side of (14):

$$\left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi_2(x,r)^{-1} r^{-n/p} \|\mathscr{I}^L_\beta(f)\|_{L_q(B(x,r))} \leqslant C[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)], \tag{23}$$

where $\delta_0>0$ (we may take $\delta_0>1$), and

$$I_{\delta_0}(x,r) := \frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)} \int_r^{\delta_0} t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x,t))} dt$$

and

$$J_{\delta_0}(x,r) := \frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)} \int_{\delta_0}^\infty t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x,t))} dt$$

and it is supposed that $r < \delta_0$. We use the fact that $f \in VM^{\alpha,V}_{p,\varphi_1}(\mathbb{R}^n)$ and choose any fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n}\left(1+\frac{t}{\rho(x)}\right)^{\alpha}\varphi_1(x,t)^{-1}t^{-n/p}\|f\|_{L_p(B(x,t))} < \frac{\varepsilon}{2CC_0},$$

where $C$ and $C_0$ are constants from (4) and (23). This allows to estimate the first term uniformly in $r \in (0,\delta_0)$ :

$$\sup_{x\in\mathbb{R}^n} CI_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \quad 0 < r < \delta_0.$$

The estimation of the second term now my be made already by the choice of $r$ sufficiently small. Indeed, thanks to the condition (10) we have

$$J_{\delta_0}(x,r) \leqslant c_{\sigma_0}\frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_1(x,r)}\|f\|_{VM^{\alpha,V}_{p,\varphi_1}},$$

where $c_{\sigma_0}$ is the constant from (1). Then, by (10) it suffices to choose $r$ small enough such that

$$\sup_{x\in\mathbb{R}^n}\frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)} \leqslant \frac{\varepsilon}{2c_{\sigma_0}\|f\|_{VM^{\alpha,V}_{p,\varphi_1}}},$$

which completes the proof of (21).

The proof of (22) is similar to the proof of (21).

## 6. Proof of Theorem 4

The norm inequality having already been provided by Theorem 2, we only have to prove the implication

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi_1(x,r)^{-1}r^{-n/p}\|f\|_{L_p(B(x,r))} = 0$$
$$\Longrightarrow \lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi_2(x,r)^{-1}r^{-n/p}\|[b,\mathcal{I}^L_\beta(f)]\|_{L_q(B(x,r))} = 0.$$

To check that

$$\sup_{x\in\mathbb{R}^n}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi_2(x,r)^{-1}r^{-n/p}\|[b,\mathcal{I}^L_\beta(f)]\|_{L_q(B(x,r))} < \varepsilon \quad \text{for small } r,$$

we use the estimate (16):

$$\varphi_2(x,r)^{-1}r^{-n/p}\|[b,\mathscr{I}_\beta^L(f)]\|_{L_q(B(x,r))} \lesssim \frac{[b]_\theta}{\varphi_2(x,r)}\int_r^\infty\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}.$$

We take $r<\delta_0$, where $\delta_0$ will be chosen small enough and split the integration:

$$\left(1+\frac{r}{\rho(x)}\right)^\alpha\varphi_2(x,r)^{-1}r^{-n/p}\|[b,\mathscr{I}_\beta^L(f)]\|_{L_q(B(x,r))} \leqslant C[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)], \tag{24}$$

where

$$I_{\delta_0}(x,r) := \frac{\left(1+\frac{r}{\rho(x)}\right)^\alpha}{\varphi_2(x,r)}\int_r^{\delta_0}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}$$

and

$$J_{\delta_0}(x,r) := \frac{\left(1+\frac{r}{\rho(x)}\right)^\alpha}{\varphi_2(x,r)}\int_{\delta_0}^\infty\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}.$$

We choose a fixed $\delta_0>0$ such that

$$\sup_{x\in\mathbb{R}^n}\left(1+\frac{r}{\rho(x)}\right)^\alpha\varphi_1(x,r)^{-1}r^{-n/p}\|f\|_{L_p(B(x,r))} < \frac{\varepsilon}{2CC_0}, \quad r\leqslant\delta_0,$$

where $C$ and $C_0$ are constants from (24) and (5), which yields the estimate of the first term uniform in $r\in(0,\delta_0)$ : $\sup\limits_{x\in\mathbb{R}^n} CI_{\delta_0}(x,r) < \frac{\varepsilon}{2}$, $0<r<\delta_0$.

For the second term, writing $1+\ln\frac{t}{r}\leqslant 1+|\ln t|+\ln\frac{1}{r}$, we obtain

$$J_{\delta_0}(x,r) \leqslant \frac{c_{\delta_0}+\widetilde{c_{\delta_0}}\ln\frac{1}{r}}{\varphi_2(x,r)}\|f\|_{M^{\alpha,V}_{p,\varphi_1}},$$

where $c_{\delta_0}$ is the constant from (7) with $\delta=\delta_0$ and $\widetilde{c_{\delta_0}}$ is a similar constant with omitted logarithmic factor in the integrand. Then, by (6) we can choose small $r$ such that $\sup_{x\in\mathbb{R}^n}J_{\delta_0}(x,r)<\frac{\varepsilon}{2}$, which completes the proof.

## 7. Conclusions

In this paper, we study the boundedness of the of the fractional integral operator $\mathscr{I}_\beta^L$ associated with Schrödinger operator and its commutators $[b,\mathscr{I}_\beta^L]$ with $b\in BMO_\theta(\rho)$ on generalized Morrey spaces $M^{\alpha,V}_{p,\varphi}$ associated with Schrödinger operator and vanishing generalized Morrey spaces $VM^{\alpha,V}_{p,\varphi}$ associated with Schrödinger operator. We find the sufficient conditions on the pair $(\varphi_1,\varphi_2)$ which ensures the boundedness of the operator $\mathscr{I}_\beta^L$ from $M^{\alpha,V}_{p,\varphi_1}$ to $M^{\alpha,V}_{q,\varphi_2}$ and from $VM^{\alpha,V}_{p,\varphi_1}$ to $VM^{\alpha,V}_{q,\varphi_2}$, $1/p-1/q=\beta/n$. When $b$ belongs to $BMO_\theta(\rho)$ and $(\varphi_1,\varphi_2)$ satisfies some conditions, we also show that the commutator operator $[b, \mathscr{I}_\beta^L]$ is bounded from $M_{p,\varphi_1}^{\alpha,V}$ to $M_{q,\varphi_2}^{\alpha,V}$ and from $VM_{p,\varphi_1}^{\alpha,V}$ to $VM_{q,\varphi_2}^{\alpha,V}$, $1/p-1/q=\beta/n$.

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## REFERENCES

[1] A. AKBULUT, O. KUZU, *Marcinkiewicz integrals with rough kernel associated with Schrödinger operator on vanishing generalized Morrey spaces*, Azerb. J. Math. **4** (1) (2014), 40–54.

[2] A. AKBULUT, A. EROGLU, A. M. NAJAFOV, *Some embedding theorems on the Nikolskii-Morrey type spaces*, Advances in Analysis, 2016, **1** (1), 18–26.

[3] A. AKBULUT, V. S. GULIYEV, M. N. OMAROVA, *Marcinkiewicz integrals associated with Schrödinger operators and their commutators on vanishing generalized Morrey spaces*, Bound. Value Probl. (2017) 2017:121.

[4] B. BONGIOANNI, E. HARBOURE, O. SALINAS, *Commutators of Riesz transforms related to Schödinger operators*, J. Fourier Anal. Appl. **17** (1) (2011), 115–134.

[5] T. BUI, *Weighted estimates for commutators of some singular integrals related to Schrödinger operators*, Bull. Sci. Math. **138** (2) (2014), 270–292.

[6] X. CAO, D. CHEN, *The boundedness of Toeplitz-type operators on vanishing-Morrey spaces*, Anal. Theory Appl. **27** (4) (2011), 309–319.

[7] F. CHIARENZA, M. FRASCA, *Morrey spaces and Hardy-Littlewood maximal function*, Rend Mat. **7** (1987), 273–279.

[8] G. DI FAZIO, M. A. RAGUSA, *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients*, J. Funct. Anal. **112** (1993) 241–256.

[9] D. FAN, S. LU, D. YANG, *Boundedness of operators in Morrey spaces on homogeneous spaces and its applications*, Acta Math. Sinica (N. S.) **14** (1998), 625–634.

[10] V. S. GULIYEV, *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. 2009, Art. ID 503948, 20 pp.

[11] V. S. GULIYEV, S. S. ALIYEV, T. KARAMAN, P. SHUKUROV, *Boundedness of sublinear operators and commutators on generalized Morrey spaces*, Integral Equations and Operator Theory **71** (3) 2011, 327–355.

[12] V. S. GULIYEV, *Function spaces and integral operators associated with Schrödinger operators: an overview*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **40** (2014), 178–202.

[13] C. MORREY, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126–166.

[14] T. MIZUHARA, *Boundedness of some classical operators on generalized Morrey spaces, Harmonic Analysis* (S. Igari, Ed.), ICM 90 Satellite Proceedings, Springer-Verlag, Tokyo (1991), 183–189.

[15] E. NAKAI, *Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces*, Math. Nachr. **166** (1994), 95–103.

[16] M. A. RAGUSA, *Commutators of fractional integral operators on vanishing-Morrey spaces*, J. Global Optim. **40** (1–3) (2008), 361–368.

[17] N. SAMKO, *Maximal, potential and singular operators in vanishing generalized Morrey spaces*, J. Global Optim. **57** (4) (2013), 1385–1399.

[18] Z. SHEN, *$L_p$ estimates for Schrödinger operators with certain potentials*, Ann. Inst. Fourier (Grenoble) **45** (2) (1995), 513–546.

[19] L. SOFTOVA, *Singular integrals and commutators in generalized Morrey spaces*, Acta Math. Sin. (Engl. Ser.) **22** (3) (2006), 757–766.

[20] E. M. STEIN, *Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton, NJ, 1993.

[21] L. TANG, J. DONG, *Boundedness for some Schrödinger type operator on Morrey spaces related to certain nonnegative potentials*, J. Math. Anal. Appl. **355** (2009), 101–109.
[22] C. VITANZA, *Functions with vanishing Morrey norm and elliptic partial differential equations*, In: Proceedings of methods of real analysis and partial differential equations, Capri, pp. 147–150, Springer, 1990.
[23] R. WHEEDEN, A. ZYGMUND, *Measure and integral, An introduction to real analysis*, Pure and Applied Mathematics, 43, Marcel Dekker, Inc., New York-Basel, 1977.

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