Journal of Mathematical Inequalities
Volume 12, Number 3 (2018), 789–805 doi:10.7153/jmi-2018-12-60

FRACTIONAL INTEGRAL ASSOCIATED WITH SCHRÖDINGER OPERATOR ON VANISHING GENERALIZED MORREY SPACES

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(*Communicated by R. Oinarov*)

Abstract. Let $L=-\triangle+V$ be a Schrödinger operator, where the non-negative potential V belongs to the reverse Hölder class $RH_{n/2}$, let b belong to a new $BMO_\theta(\rho)$ space, and let $\mathscr{I}_\beta^L$ be the fractional integral operator associated with L. In this paper, we study the boundedness of the operator $\mathscr{I}_\beta^L$ and its commutators $[b,\mathscr{I}_\beta^L]$ with $b\in BMO_\theta(\rho)$ on generalized Morrey spaces associated with Schrödinger operator $M_{p,\varphi}^{\alpha,V}$ and vanishing generalized Morrey spaces associated with Schrödinger operator $VM_{p,\varphi}^{\alpha,V}$. We find the sufficient conditions on the pair (φ_1,φ_2) which ensures the boundedness of the operator $\mathscr{I}_\beta^L$ from $M_{p,\varphi_1}^{\alpha,V}$ to $M_{q,\varphi_2}^{\alpha,V}$ and from $VM_{p,\varphi_1}^{\alpha,V}$ to $VM_{q,\varphi_2}^{\alpha,V}$, $1/p-1/q=\beta/n$. When b belongs to $BMO_\theta(\rho)$ and (φ_1,φ_2) satisfies some conditions, we also show that the commutator operator $[b,\mathscr{I}_\beta^L]$ is bounded from $M_{p,\varphi_1}^{\alpha,V}$ to $M_{q,\varphi_2}^{\alpha,V}$ and from $VM_{p,\varphi_1}^{\alpha,V}$ to $VM_{q,\varphi_2}^{\alpha,V}$, $1/p-1/q=\beta/n$.

1. Introduction and results

Let us consider the Schrödinger operator

$$L=-\Delta+V \text{ on } \mathbb{R}^n,\ n\geqslant 3,$$

where V is a non-negative, $V\neq 0$, and belongs to the reverse Hölder class RH_q for some $q\geqslant n/2$, i.e., there exists a constant $C>0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}V^q(y)dy\right)^{1/q}\leqslant\frac{C}{|B(x,r)|}\int_{B(x,r)}V(y)dy$$

holds for every $x\in\mathbb{R}^n$ and $0<r<\infty$, where $B(x,r)$ denotes the ball centered at x with radius r. In particular, if V is a nonnegative polynomial, then $V\in RH_\infty$.

Obviously, $RH_{q_2}\subset RH_{q_1}$, if $q_2>q_1$. The most important property of the class RH_q is its self-improvement, that is, if $V\in RH_q$, then $V\in RH_{q+\varepsilon}$ for some $\varepsilon>0$.

Mathematics subject classification (2010): 42B35, 35J10, 47H50.
Keywords and phrases: Fractional integral associated with Schrödinger operator, commutator, BMO, vanishing generalized Morrey space associated with Schrödinger operator.

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As in [18], for a given potential $V \in RH_q$ with $q \geqslant n/2$, we define the auxiliary function

$$\rho(x) := \frac{1}{m_V(x)} = \sup_{r>0}\left\{r : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy \leqslant 1\right\}.$$

It is well-known that that $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^n$.

According to [4], the new BMO space $BMO_\theta(\rho)$ with $\theta \geqslant 0$ is defined as a set of all locally integrable functions b such that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |b(y) - b_B|dy \leqslant C\left(1 + \frac{r}{\rho(x)}\right)^\theta$$

for all $x \in \mathbb{R}^n$ and $r > 0$, where $b_B = \frac{1}{|B|}\int_B b(y)dy$. A norm for $b \in BMO_\theta(\rho)$, denoted by $[b]_\theta$, is given by the infimum of the constants in the inequalities above. Clearly, $BMO \subset BMO_\theta(\rho)$.

We now present the definition of generalized Morrey spaces (including weak version) related to potential, which introduced by Guliyev in [12].

DEFINITION 1. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $1 \leqslant p < \infty$, $\alpha \geqslant 0$, and $V \in RH_q$, $q \geqslant 1$. We denote by $M^{\alpha,V}_{p,\varphi} = M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space associated with Schrödinger operator, the space of all functions $f \in L^p_{loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M^{\alpha,V}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0}\left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi(x,r)^{-1} r^{-n/p}\|f\|_{L_p(B(x,r))}.$$

Also $WM^{\alpha,V}_{p,\varphi} = WM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space associated with Schrödinger operator,the space of all functions $f \in WL^p_{loc}(\mathbb{R}^n)$ with

$$\|f\|_{WM^{\alpha,V}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0}\left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi(x,r)^{-1} r^{-n/p}\|f\|_{WL_p(B(x,r))} < \infty.$$

REMARK 1. (i) When $\alpha = 0$, and $\varphi(x,r) = r^{(\lambda-n)/p}$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the classical Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ introduced by Morrey in [13];

(ii) When $\varphi(x,r) = r^{(\lambda-n)/p}$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the Morrey space associated with Schrödinger operator $L^{\alpha,V}_{p,\lambda}(\mathbb{R}^n)$ studied by Tang and Dong in [21];

(iii) When $\alpha = 0$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the generalized Morrey space $M_{p,\varphi}(\mathbb{R}^n)$ introduced by Mizuhara and Nakai in [14, 15].

(iv) The generalized Morrey space associated with Schrödinger operator $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ was introduced by Guliyev in [12].

The classical Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ was introduced by Morrey in [13] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [7, 8, 9, 13]. The generalized Morrey spaces are defined with r^{λ} replaced by a general non-negative function $\varphi(x,r)$ satisfying some assumptions (see, for example, [10, 14, 15, 19] and etc).

For brevity, in the sequel we use the notations

$$\mathfrak{A}^{\alpha,V}_{p,\varphi}(f;x,r) := \left(1+\frac{r}{\rho(x)}\right)^{\alpha} r^{-n/p}\,\varphi(x,r)^{-1}\|f\|_{L_p(B(x,r))}$$

and

$$\mathfrak{A}^{W,\alpha,V}_{\Phi,\varphi}(f;x,r) := \left(1+\frac{r}{\rho(x)}\right)^{\alpha} r^{-n/p}\,\varphi(x,r)^{-1}\|f\|_{WL_p(B(x,r))}.$$

DEFINITION 2. The vanishing generalized Morrey space associated with Schrödinger operator $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f\in M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}^{\alpha,V}_{p,\varphi}(f;x,r)=0. \tag{1}$$

The vanishing weak generalized Morrey space associated with Schrödinger operator $VWM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f\in WM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}^{W,\alpha,V}_{p,\varphi}(f;x,r)=0.$$

The vanishing spaces $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ and $VWM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\|f\|_{VM^{\alpha,V}_{p,\varphi}} \equiv \|f\|_{M^{\alpha,V}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n,r>0}\mathfrak{A}^{\alpha,V}_{p,\varphi}(f;x,r),$$
$$\|f\|_{VWM^{\alpha,V}_{p,\varphi}} \equiv \|f\|_{WM^{\alpha,V}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n,r>0}\mathfrak{A}^{\alpha,V}_{W,p,\varphi}(f;x,r),$$

respectively.

In the case $\alpha=0$, and $\varphi(x,r)=r^{(\lambda-n)/p}$ $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the vanishing Morrey space $VM_{p,\lambda}$ introduced in [22], where applications to PDE were considered.

We refer to [1, 6, 16, 17] for some properties of vanishing generalized Morrey spaces.

DEFINITION 3. Let $L=-\triangle+V$ with $V\in RH_{n/2}$. The fractional integral associated with L is defined by

$$\mathscr{I}^L_{\beta}f(x)=L^{-\beta/2}f(x)=\int_0^{\infty}e^{-tL}(f)(x)\,t^{\beta/2-1}dt$$

for $0<\beta<n$. The commutator of $\mathscr{I}^L_{\beta}$ is defined by

$$[b,\mathscr{I}^L_{\beta}]f(x)=b(x)\mathscr{I}^L_{\beta}f(x)-\mathscr{I}^L_{\beta}(bf)(x).$$

In this paper, we consider the boundedness of the fractional integral operator $\mathscr{I}_\beta^L$ on the generalized Morrey spaces $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ and the vanishing generalized Morrey spaces $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$. When b belongs to the new BMO space $BMO_\theta(\rho)$, we also show that $[b,\mathscr{I}_\beta^L]$ is bounded from $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ to $M_{q,\varphi}^{\alpha,V}(\mathbb{R}^n)$ and from $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ to $VM_{q,\varphi}^{\alpha,V}(\mathbb{R}^n)$.

Our main results are as follows.

THEOREM 1. *Let $V \in RH_{n/2}$, $\alpha \geqslant 0$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ and $\varphi_1 \in \Omega_p^{\alpha,V}$, $\varphi_2 \in \Omega_q^{\alpha,V}$ satisfies the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t} \leqslant c_0 \varphi_2(x,r), \tag{2}$$

where c_0 does not depend on x and r. Then the operator $\mathscr{I}_\beta^L$ is bounded on $M_{p,\varphi_1}^{\alpha,V}$ to $M_{q,\varphi_2}^{\alpha,V}$ for $p>1$ and from $M_{1,\varphi_1}^{\alpha,V}$ to $WM_{\frac{n}{n-\beta},\varphi_2}^{\alpha,V}$. Moreover, for $p>1$

$$\|\mathscr{I}_\beta^L f\|_{M_{q,\varphi_2}^{\alpha,V}} \leqslant C\|f\|_{M_{p,\varphi_1}^{\alpha,V}},$$

and for $p=1$

$$\|\mathscr{I}_\beta^L f\|_{WM_{\frac{n}{n-\beta},\varphi_2}^{\alpha,V}} \leqslant C\|f\|_{M_{1,\varphi_1}^{\alpha,V}},$$

where C does not depend on f.

THEOREM 2. *Let $V \in RH_{n/2}$, $\alpha \geqslant 0$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ and $\varphi_1 \in \Omega_p^{\alpha,V}$, $\varphi_2 \in \Omega_q^{\alpha,V}$ satisfies the condition*

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t} \leqslant c_0 \varphi_2(x,r), \tag{3}$$

where c_0 does not depend on x and r. If $b \in BMO_\theta(\rho)$, then the operator $[b,\mathscr{I}_\beta^L]$ is bounded from $M_{p,\varphi_1}^{\alpha,V}$ to $M_{q,\varphi_2}^{\alpha,V}$ and

$$\|[b,\mathscr{I}_\beta^L] f\|_{M_{q,\varphi_2}^{\alpha,V}} \leqslant C[b]_\theta \|f\|_{M_{p,\varphi_1}^{\alpha,V}},$$

where C does not depend on f.

THEOREM 3. *Let* $V \in RH_{n/2}$, $\alpha \geqslant 0$, $1 \leqslant p < n/\beta$, $1/q = 1/p - \beta/n$ *and* $\varphi_1 \in \Omega^{\alpha,V}_{p,1}$, $\varphi_2 \in \Omega^{\alpha,V}_{q,1}$ *satisfies the conditions*

$$c_\delta := \int_\delta^\infty \sup_{x\in\mathbb{R}^n} \varphi_1(x,t)\frac{dt}{t} < \infty$$

for every $\delta > 0$, *and*

$$\int_r^\infty \varphi_1(x,t)\frac{dt}{t^{1-\beta}} \leqslant C_0\varphi_2(x,r), \tag{4}$$

where C_0 *does not depend on* $x \in \mathbb{R}^n$ *and* $r > 0$. *Then the operator* $\mathscr{I}^L_\beta$ *is bounded from* $VM^{\alpha,V}_{p,\varphi_1}$ *to* $VM^{\alpha,V}_{q,\varphi_2}$ *for* $p > 1$ *and from* $VM^{\alpha,V}_{1,\varphi_1}$ *to* $VWM^{\alpha,V}_{\frac{n}{n-\beta},\varphi_2}$.

THEOREM 4. *Let* $V \in RH_{n/2}$, $b \in BMO_\theta(\rho)$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$, *and* $\varphi_1 \in \Omega^{\alpha,V}_{p,1}$, $\varphi_2 \in \Omega^{\alpha,V}_{q,1}$ *satisfies the conditions*

$$\int_r^\infty \Big(1+\ln\frac{t}{r}\Big)\varphi_1(x,t)\frac{dt}{t^{1-\beta}} \leqslant c_0\varphi_2(x,r), \tag{5}$$

where c_0 *does not depend on* x *and* r,

$$\lim_{r\to 0} \frac{\ln\frac{1}{r}}{\inf limits_{x\in\mathbb{R}^n}\varphi_2(x,r)} = 0 \tag{6}$$

and

$$c_\delta := \int_\delta^\infty \Big(1+|\ln t|\Big) \sup_{x\in\mathbb{R}^n} \varphi_1(x,t)\frac{dt}{t^{1-\beta}} < \infty \tag{7}$$

for every $\delta > 0$. *Then the operator* $[b,\mathscr{I}^L_\beta]$ *is bounded from* $VM^{\alpha,V}_{p,\varphi_1}$ *to* $VM^{\alpha,V}_{q,\varphi_2}$.

REMARK 2. Note that, Theorems 1 and 2 in the case of $V \equiv 0$ was proved in [11, Corollary 5.5 and 7.5] and in the case of $\varphi(x,r) = r^{(\lambda-n)/p}$ in [21, Theorems 1.3 and 1.4].

REMARK 3. Note that, in [2] the Nikolskii-Morrey type spaces were introduced and the authors studied some embedding theorems. In the next paper, we shall introduce the generalized Nikolskii-Morrey spaces associated with Schrödinger operator and will study some embedding theorems. We will also investigate the boundedness of fractional integral associated with Schrödinger operator on the generalized Nikolskii-Morrey spaces associated with Schrödinger operator.

In this paper, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \leqslant CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2. Some preliminaries

We would like to recall the important properties concerning the critical function.

LEMMA 1. [18] *Let $V \in RH_{n/2}$. For the associated function ρ there exist C and $k_0 \geqslant 1$ such that*

$$C^{-1}\rho(x)\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{-k_0} \leqslant \rho(y) \leqslant C\rho(x)\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{\frac{k_0}{1+k_0}} \tag{8}$$

for all $x,y \in \mathbb{R}^n$.

LEMMA 2. [3] *Suppose $x \in B(x_0,r)$. Then for $k \in N$ we have*

$$\frac{1}{\Big(1+\frac{2^k r}{\rho(x)}\Big)^N} \lesssim \frac{1}{\Big(1+\frac{2^k r}{\rho(x_0)}\Big)^{N/(k_0+1)}}.$$

We give some inequalities about the new BMO space $BMO_\theta(\rho)$.

LEMMA 3. [4] *Let $1 \leqslant s < \infty$. If $b \in BMO_\theta(\rho)$, then*

$$\Big(\frac{1}{|B|}\int_B |b(y)-b_B|^s dy\Big)^{1/s} \leqslant [b]_\theta \Big(1+\frac{r}{\rho(x)}\Big)^{\theta'}$$

for all $B = B(x,r)$, with $x \in \mathbb{R}^n$ and $r>0$, where $\theta' = (k_0+1)\theta$ and k_0 is the constant appearing in (8).

LEMMA 4. [4] *Let $1 \leqslant s < \infty$, $b \in BMO_\theta(\rho)$, and $B = B(x,r)$. Then*

$$\Big(\frac{1}{|2^k B|}\int_{2^k B} |b(y)-b_B|^s dy\Big)^{1/s} \leqslant [b]_\theta k\Big(1+\frac{2^k r}{\rho(x)}\Big)^{\theta'}$$

for all $k \in \mathbb{N}$, with θ' as in Lemma 3.

Let K_β be the kernel of $\mathscr{I}_\beta^L$. The following result give the estimate on the kernel $K_\beta(x,y)$.

LEMMA 5. [5] *If $V \in RH_{n/2}$, then for every N, there exists a constant C such that*

$$|K_\beta(x,y)| \leqslant \frac{C}{\Big(1+\frac{|x-y|}{\rho(x)}\Big)^N}\frac{1}{|x-y|^{n-\beta}}. \tag{9}$$

Finally, we recall a relationship between essential supremum and essential infimum.

LEMMA 6. [23] *Let f be a real-valued nonnegative function and measurable on E. Then*

$$\left(\operatorname*{ess\,inf}_{x\in E} f(x)\right)^{-1} = \operatorname*{ess\,sup}_{x\in E} \frac{1}{f(x)}.$$

LEMMA 7. [3] *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $1 \leqslant p < \infty$, $\alpha \geqslant 0$, and $V \in RH_q$, $q \geqslant 1$.*

(i) If $\sup\limits_{t<r<\infty} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \frac{r^{-\frac{n}{p}}}{\varphi(x,r)} = \infty$ *for some $t>0$ and for all $x\in\mathbb{R}^n$, then* $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n) = \Theta$.

(ii) If $\sup\limits_{0<r<\tau} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi(x,r)^{-1} = \infty$ *for some $\tau>0$ and for all $x\in\mathbb{R}^n$, then* $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n) = \Theta$.

REMARK 4. We denote by $\Omega^{\alpha,V}_{p}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that for all $t>0$,

$$\sup_{x\in\mathbb{R}^n} \left\| \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \frac{r^{-\frac{n}{p}}}{\varphi(x,r)} \right\|_{L_\infty(t,\infty)} < \infty, \quad \text{and} \quad \sup_{x\in\mathbb{R}^n} \left\| \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi(x,r)^{-1} \right\|_{L_\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 7, we always assume that $\varphi \in \Omega^{\alpha,V}_{p}$.

REMARK 5. We denote by $\Omega^{\alpha,V}_{p,1}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that

$$\inf_{x\in\mathbb{R}^n} \inf_{r>\delta} \left(1+\frac{r}{\rho(x)}\right)^{-\alpha} \varphi(x,r) > 0, \text{ for some } \delta>0, \tag{10}$$

and

$$\lim_{r\to 0} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \frac{r^{n/p}}{\varphi(x,r)} = 0.$$

For the non-triviality of the space $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ we always assume that $\varphi \in \Omega^{\alpha,V}_{p,1}$.

3. Proof of Theorem 1

We first prove the following conclusions

THEOREM 5. *Let $V \in RH_{n/2}$. If $1<p<n/\beta$, $1/q = 1/p - \beta/n$ then the inequality*

$$\|\mathscr{I}^{L}_{\beta}(f)\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}$$

holds for any $f \in L^p_{loc}(\mathbb{R}^n)$. *Moreover, for* $p=1$ *the inequality*

$$\|\mathscr{I}^L_\beta(f)\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))} \lesssim r^{n-\beta} \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t}$$

holds for any $f \in L^1_{loc}(\mathbb{R}^n)$.

Proof. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ and $\lambda B = B(x_0,\lambda r)$ for any $\lambda > 0$. We write f as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{B(x_0,2r)}(y)$, and $\chi_{B(x_0,2r)}$ denotes the characteristic function of $B(x_0,2r)$. Then

$$\|\mathscr{I}^L_\beta(f)\|_{L_q(B(x_0,r))} \leqslant \|\mathscr{I}^L_\beta(f_1)\|_{L_q(B(x_0,r))} + \|\mathscr{I}^L_\beta(f_2)\|_{L_q(B(x_0,r))}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$ and from the boundedness of $\mathscr{I}^L_\beta$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ (see [20]) it follows that

$$\begin{aligned}
\|\mathscr{I}^L_\beta(f_1)\|_{L_q(B(x_0,r))} &\lesssim \|f\|_{L_p(B(x_0,2r))} \\
&\lesssim r^{\frac{n}{q}} \|f\|_{L_p(B(x_0,2r))} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{q}+1}} \\
&\lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}.
\end{aligned} \tag{11}$$

To estimate $\|\mathscr{I}^L_\beta(f_2)\|_{L_p(B(x_0,r))}$, obverse that $x \in B$, $y \in (2B)^c$ implies $|x-y| \approx |x_0 - y|$. Then by (9) we have

$$\begin{aligned}
\sup_{x\in B} |\mathscr{I}^L_\beta(f_2)(x)| &\leqslant \sup_{x\in B} \int_{(2B)^c} |K_\beta(x,y)f(y)|dy \\
&\lesssim \int_{(2B)^c} \frac{|f(y)|}{|x_0-y|^{n-\beta}} dy \\
&\lesssim \sum_{k=1}^{\infty} (2^{k+1}r)^{-n+\beta} \int_{2^{k+1}B} |f(y)|dy.
\end{aligned}$$

By Hölder's inequality we get

$$\begin{aligned}
\sup_{x\in B} |\mathscr{I}^L_\beta(f_2)(x)| &\lesssim \sum_{k=1}^{\infty} \|f\|_{L_p(2^{k+1}B)} (2^{k+1}r)^{-1-\frac{n}{p}+\beta} \int_{2^k r}^{2^{k+1}r} dt \\
&\lesssim \sum_{k=1}^{\infty} \int_{2^k r}^{2^{k+1}r} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \\
&\lesssim \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}.
\end{aligned} \tag{12}$$

Then

$$\|\mathscr{I}_\beta^L(f_2)\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \tag{13}$$

holds for $1 \leqslant p < n/\beta$. Therefore, by (11) and (13) we get

$$\|\mathscr{I}_\beta^L(f)\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \tag{14}$$

holds for $1 \leqslant p < n/\beta$.

When $p=1$, by the boundedness of $\mathscr{I}_\beta^L$ from $L_1(\mathbb{R}^n)$ to $WL_{\frac{n}{n-\beta}}(\mathbb{R}^n)$, we get

$$\|\mathscr{I}_\beta^L(f_1)\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))} \lesssim \|f\|_{L_1(B(x_0,2r))} \lesssim r^{n-\beta} \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t}.$$

By (13) we have

$$\|\mathscr{I}_\beta^L(f_2)\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))} \leqslant \|\mathscr{I}_\beta^L(f_2)\|_{L_{\frac{n}{n-\beta}}(B(x_0,2r))} \lesssim r^{n-\beta} \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t}.$$

Then

$$\|\mathscr{I}_\beta^L(f)\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))} \lesssim r^{n-\beta} \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t}. \qquad \square$$

Proof of Theorem 1. From Lemma 6, we have

$$\frac{1}{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}} = \operatorname*{ess\,sup}_{t<s<\infty} \frac{1}{\varphi_1(x,s)s^{\frac{n}{p}}}.$$

Note the fact that $\|f\|_{L_p(B(x_0,t))}$ is a nondecresing function of t, and $f \in M_{p,\varphi_1}^{\alpha,V}$, then

$$\begin{aligned}
\frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_p(B(x_0,t))}}{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}} &\lesssim \operatorname*{ess\,sup}_{t<s<\infty} \frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_p(B(x_0,t))}}{\varphi_1(x_0,s)s^{\frac{n}{p}}} \\
&\lesssim \sup_{0<s<\infty} \frac{\left(1+\frac{s}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_p(B(x_0,s))}}{\varphi_1(x_0,s)s^{\frac{n}{p}}} \\
&\lesssim \|f\|_{M_{p,\varphi_1}^{\alpha,V}}.
\end{aligned}$$

Since $\alpha \geqslant 0$, and (φ_1,φ_2) satisfies the condition (2), then

$$\int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} = \int_{2r}^{\infty} \frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_p(B(x_0,t))}}{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha} t^{\frac{n}{q}}} \frac{dt}{t}$$

$$\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \int_{2r}^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha} t^{\frac{n}{q}}} \frac{dt}{t}$$

$$\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \int_{r}^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t}$$

$$\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \varphi_2(x_0,r). \tag{15}$$

Then by Theorem 5 we get

$$\begin{aligned}\|\mathscr{I}^L_\beta(f)\|_{M^{\alpha,V}_{q,\varphi_2}} &\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} r^{-n/q} \|\mathscr{I}^L_\beta(f)\|_{L_p(B(x_0,r))} \\ &\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \\ &\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}}.\end{aligned}$$

Let $q = \frac{n}{n-\beta}$, similar to the estimates of (15) we have

$$\int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t} \lesssim \|f\|_{M^{\alpha,V}_{1,\varphi_1}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \varphi_2(x_0,r).$$

Thus by Theorem 5 we get

$$\begin{aligned}\|\mathscr{I}^L_\beta(f)\|_{WM^{\alpha,V}_{\frac{n}{n-\beta},\varphi_2}} &\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} r^{\beta-n} \|\mathscr{I}^L_\beta(f)\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))} \\ &\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t} \\ &\lesssim \|f\|_{M^{\alpha,V}_{1,\varphi_1}}. \quad \square\end{aligned}$$

4. Proof of Theorem 2

As the proof of Theorem 1, it suffices to prove the following result.

THEOREM 6. *Let $V \in RH_{n/2}$, $b \in BMO_\theta(\rho)$. If $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ then the inequality*

$$\|[b,\mathscr{I}^L_\beta(f)]\|_{L_q(B(x_0,r))} \lesssim [b]_\theta r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \tag{16}$$

holds for any $f \in L^p_{loc}(\mathbb{R}^n)$.

Proof. We write f as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{B(x_0,2r)}(y)$. Then

$$\|[b,\mathscr{I}_\beta^L](f)\|_{L_q(B(x_0,r))} \leqslant \|[b,\mathscr{I}_\beta^L](f_1)\|_{L_q(B(x_0,r))} + \|[b,\mathscr{I}_\beta^L](f_2)\|_{L_q(B(x_0,r))}.$$

By the boundedness of $[b,\mathscr{I}_\beta^L]$ on $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ (see [21]) and (11) we get

$$\begin{aligned}\|[b,\mathscr{I}_\beta^L](f_1)\|_{L_q(B(x_0,r))} &\lesssim [b]_\theta \|f\|_{L_p(B(x_0,2r))}\\ &\lesssim [b]_\theta r^{\frac{n}{q}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}\\ &\lesssim [b]_\theta r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}. \qquad (17)\end{aligned}$$

We now turn to deal with the term $\|[b,\mathscr{I}_\beta^L](f_2)\|_{L_q(B(x_0,r))}$. For any given $x \in B(x_0,r)$ we have

$$|[b,\mathscr{I}_\beta^L]f_2(x)| \leqslant |b(x)-b_{2B}|\,|\mathscr{I}_\beta^L(f_2)(x)| + |\mathscr{I}_\beta^L((b-b_{2B})f_2)(x)|.$$

Then by (12), Lemma 3, and taking $N \geqslant (k_0+1)\theta$ we get

$$\begin{aligned}\|(b(x)-b_{2B})\mathscr{I}_\beta^L(f_2)\|_{L_q(B(x_0,r))} &\lesssim [b]_\theta r^{\frac{n}{q}} \left(1+\frac{2r}{\rho(x_0)}\right)^{\theta-N/(k_0+1)} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}\\ &\lesssim [b]_\theta r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}. \qquad (18)\end{aligned}$$

Finally, let us estimate $\|\mathscr{I}_\beta^L((b-b_{2B})f_2)\|_{L_q(B(x_0,r))}$. By (9), Lemma 2 and (12) we have

$$\begin{aligned}\sup_{x\in B}|\mathscr{I}_\beta^L((b-b_{2B})f_2)(x)| &\lesssim \sup_{x\in B}\int_{(2B)^c} \frac{1}{\left(1+\frac{|x-y|}{\rho(x)}\right)^N}\frac{|b(y)-b_{2B}||f(y)|}{|x_0-y|^{n-\beta}}dy\\ &\lesssim \sup_{x\in B}\sum_{k=1}^{\infty}\frac{1}{(2^k r)^{n-\beta}\left(1+\frac{2^k r}{\rho(x)}\right)^N}\int_{2^{k+1}B}|b(y)-b_{2B}||f(y)|dy\\ &\lesssim \sum_{k=1}^{\infty}\frac{1}{(2^k r)^{n-\beta}\left(1+\frac{2^k r}{\rho(x_0)}\right)^{N/(k_0+1)}}\int_{2^{k+1}B}|b(y)-b_{2B}||f(y)|dy.\end{aligned}$$

Note that

$$\int_{2^{k+1}B} |b(y)-b_{2B}||f(y)|dy \lesssim \Big(\int_{2^{k+1}B} |b(y)-b_{2B}|^{p'}\Big)^{1/p'} \|f\|_{L_p(B(x_0,2^{k+1}r))}$$
$$\lesssim [b]_\theta k\Big(1+\frac{2^k r}{\rho(x_0)}\Big)^{\theta'} (2^k r)^{\frac{n}{p'}} \|f\|_{L_p(B(x_0,2^{k+1}r))}.$$

Then

$$\sup_{x\in B} |\mathscr{I}_\beta^L((b-b_{2B})f_2)(x)| \lesssim [b]_\theta \sum_{k=1}^{\infty} \frac{k(2^k r)^{-\frac{n}{p}+\beta}}{\Big(1+\frac{2^k r}{\rho(x_0)}\Big)^{N/(k_0+1)-\theta'}} \|f\|_{L_p(B(x_0,2^{k+1}r))}$$
$$\lesssim [b]_\theta \sum_{k=1}^{\infty} k(2^k r)^{-\frac{n}{q}} \|f\|_{L_p(B(x_0,2^{k+1}r))}$$
$$\lesssim [b]_\theta \sum_{k=1}^{\infty} k \int_{2^k r}^{2^{k+1}r} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}.$$

Since $2^k r \leqslant t \leqslant 2^{k+1} r$, then $k \approx \ln \frac{t}{r}$. Thus

$$\sup_{x\in B} |\mathscr{I}_\beta^L((b-b_{2B})f_2)(x)| \lesssim [b]_\theta \sum_{k=1}^{\infty} k \int_{2^k r}^{2^{k+1}r} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}$$
$$\lesssim [b]_\theta \sum_{k=1}^{\infty} \int_{2^k r}^{2^{k+1}r} \ln\frac{t}{r} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}$$
$$\lesssim [b]_\theta \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}.$$

Then

$$\|\mathscr{I}_\beta^L((b-b_{2B})f_2)\|_{L_q(B(x_0,r))} \lesssim [b]_\theta r^{\frac{n}{q}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}. \tag{19}$$

Combining (17), (18) and (19), the proof of Theorem 6 is completed. □

Proof of Theorem 2. Since $f \in M_{p,\varphi_1}^{\alpha,V}$ and (φ_1,φ_2) satisfies the condition (3), by (15) we have

$$\int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}$$
$$= \int_{2r}^{\infty} \frac{\Big(1+\frac{t}{\rho(x_0)}\Big)^{\alpha} \|f\|_{L_p(B(x_0,t))}}{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s) s^{\frac{n}{p}}} \Big(1+\ln\frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s) s^{\frac{n}{p}}}{\Big(1+\frac{t}{\rho(x_0)}\Big)^{\alpha} t^{\frac{n}{q}}} \frac{dt}{t}$$
$$\lesssim \|f\|_{M_{p,\varphi_1}^{\alpha,V}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s) s^{\frac{n}{p}}}{\Big(1+\frac{t}{\rho(x_0)}\Big)^{\alpha} t^{\frac{n}{q}}} \frac{dt}{t}$$

$$\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \int_r^\infty \left(1+\ln\frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t}$$
$$\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \varphi_2(x_0,r). \tag{20}$$

Then from Theorem 6 and by (20) we get

$$\|[b,\mathscr{I}^L_\beta](f)\|_{M^{\alpha,V}_{q,\varphi_2}}$$
$$\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} r^{-n/q} \|[b,\mathscr{I}^L_\beta](f)\|_{L_q(B(x_0,r))}$$
$$\lesssim [b]_\theta \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} \int_{2r}^\infty \left(1+\ln\frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}$$
$$\lesssim [b]_\theta \|f\|_{M^{\alpha,V}_{p,\varphi_1}}. \quad \square$$

5. Proof of Theorem 3

The statement is derived from the estimate (14). The estimation of the norm of the operator, that is, the boundedness in the non-vanishing space, immediately follows from by Theorem 1. So we only have to prove that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{p,\varphi_1}(f;x,r) = 0 \;\Rightarrow\; \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{q,\varphi_2}(\mathscr{I}^L_\beta(f);x,r) = 0 \tag{21}$$

and

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{1,\varphi_1}(f;x,r) = 0 \;\Rightarrow\; \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{W,\alpha,V}_{n/(n-\beta),\varphi_2}(\mathscr{I}^L_\beta(f);x,r) = 0. \tag{22}$$

To show that $\sup_{x\in\mathbb{R}^n} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi_2(x,r)^{-1} r^{-n/p} \|\mathscr{I}^L_\beta(f)\|_{L_q(B(x,r))} < \varepsilon$ for small r, we split the right-hand side of (14):

$$\left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi_2(x,r)^{-1} r^{-n/p} \|\mathscr{I}^L_\beta(f)\|_{L_q(B(x,r))} \leqslant C[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)], \tag{23}$$

where $\delta_0 > 0$ (we may take $\delta_0 > 1$), and

$$I_{\delta_0}(x,r) := \frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)} \int_r^{\delta_0} t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x,t))} dt$$

and

$$J_{\delta_0}(x,r) := \frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)} \int_{\delta_0}^\infty t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x,t))} dt$$

and it is supposed that $r < \delta_0$. We use the fact that $f \in VM^{\alpha,V}_{p,\varphi_1}(\mathbb{R}^n)$ and choose any fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n}\left(1+\frac{t}{\rho(x)}\right)^{\alpha}\varphi_1(x,t)^{-1}t^{-n/p}\|f\|_{L_p(B(x,t))} < \frac{\varepsilon}{2CC_0},$$

where C and C_0 are constants from (4) and (23). This allows to estimate the first term uniformly in $r \in (0,\delta_0)$:

$$\sup_{x\in\mathbb{R}^n} CI_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \quad 0 < r < \delta_0.$$

The estimation of the second term now my be made already by the choice of r sufficiently small. Indeed, thanks to the condition (10) we have

$$J_{\delta_0}(x,r) \leqslant c_{\sigma_0}\frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_1(x,r)}\|f\|_{VM^{\alpha,V}_{p,\varphi_1}},$$

where c_{σ_0} is the constant from (1). Then, by (10) it suffices to choose r small enough such that

$$\sup_{x\in\mathbb{R}^n}\frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)} \leqslant \frac{\varepsilon}{2c_{\sigma_0}\|f\|_{VM^{\alpha,V}_{p,\varphi_1}}},$$

which completes the proof of (21).

The proof of (22) is similar to the proof of (21).

6. Proof of Theorem 4

The norm inequality having already been provided by Theorem 2, we only have to prove the implication

$$\begin{aligned}
&\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi_1(x,r)^{-1}r^{-n/p}\|f\|_{L_p(B(x,r))} = 0\\
&\Longrightarrow \lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi_2(x,r)^{-1}r^{-n/p}\|[b,\mathscr{I}^L_\beta(f)]\|_{L_q(B(x,r))} = 0.
\end{aligned}$$

To check that

$$\sup_{x\in\mathbb{R}^n}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi_2(x,r)^{-1}r^{-n/p}\|[b,\mathscr{I}^L_\beta(f)]\|_{L_q(B(x,r))} < \varepsilon \quad \text{for small } r,$$

we use the estimate (16):

$$\varphi_2(x,r)^{-1} r^{-n/p} \|[b, \mathscr{I}_\beta^L(f)]\|_{L_q(B(x,r))} \lesssim \frac{[b]_\theta}{\varphi_2(x,r)} \int_r^\infty \left(1+\ln\frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}.$$

We take $r<\delta_0$, where δ_0 will be chosen small enough and split the integration:

$$\left(1+\frac{r}{\rho(x)}\right)^\alpha \varphi_2(x,r)^{-1} r^{-n/p} \|[b, \mathscr{I}_\beta^L(f)]\|_{L_q(B(x,r))} \leqslant C[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)], \qquad (24)$$

where

$$I_{\delta_0}(x,r) := \frac{\left(1+\frac{r}{\rho(x)}\right)^\alpha}{\varphi_2(x,r)} \int_r^{\delta_0} \left(1+\ln\frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}$$

and

$$J_{\delta_0}(x,r) := \frac{\left(1+\frac{r}{\rho(x)}\right)^\alpha}{\varphi_2(x,r)} \int_{\delta_0}^\infty \left(1+\ln\frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}.$$

We choose a fixed $\delta_0>0$ such that

$$\sup_{x\in\mathbb{R}^n} \left(1+\frac{r}{\rho(x)}\right)^\alpha \varphi_1(x,r)^{-1} r^{-n/p} \|f\|_{L_p(B(x,r))} < \frac{\varepsilon}{2CC_0}, \quad r\leqslant\delta_0,$$

where C and C_0 are constants from (24) and (5), which yields the estimate of the first term uniform in $r\in(0,\delta_0)$: $\sup\limits_{x\in\mathbb{R}^n} CI_{\delta_0}(x,r) < \frac{\varepsilon}{2}$, $0<r<\delta_0$.

For the second term, writing $1+\ln\frac{t}{r}\leqslant 1+|\ln t|+\ln\frac{1}{r}$, we obtain

$$J_{\delta_0}(x,r) \leqslant \frac{c_{\delta_0}+\widetilde{c_{\delta_0}}\ln\frac{1}{r}}{\varphi_2(x,r)} \|f\|_{M^{\alpha,V}_{p,\varphi_1}},$$

where c_{δ_0} is the constant from (7) with $\delta=\delta_0$ and $\widetilde{c_{\delta_0}}$ is a similar constant with omitted logarithmic factor in the integrand. Then, by (6) we can choose small r such that $\sup_{x\in\mathbb{R}^n} J_{\delta_0}(x,r)<\frac{\varepsilon}{2}$, which completes the proof.

7. Conclusions

In this paper, we study the boundedness of the of the fractional integral operator $\mathscr{I}_\beta^L$ associated with Schrödinger operator and its commutators $[b,\mathscr{I}_\beta^L]$ with $b\in BMO_\theta(\rho)$ on generalized Morrey spaces $M^{\alpha,V}_{p,\varphi}$ associated with Schrödinger operator and vanishing generalized Morrey spaces $VM^{\alpha,V}_{p,\varphi}$ associated with Schrödinger operator. We find the sufficient conditions on the pair (φ_1,φ_2) which ensures the boundedness of the operator $\mathscr{I}_\beta^L$ from $M^{\alpha,V}_{p,\varphi_1}$ to $M^{\alpha,V}_{q,\varphi_2}$ and from $VM^{\alpha,V}_{p,\varphi_1}$ to $VM^{\alpha,V}_{q,\varphi_2}$, $1/p-1/q=\beta/n$. When b belongs to $BMO_\theta(\rho)$ and (φ_1,φ_2) satisfies some conditions, we also show that the commutator operator $[b,\mathscr{I}_\beta^L]$ is bounded from $M^{\alpha,V}_{p,\varphi_1}$ to $M^{\alpha,V}_{q,\varphi_2}$ and from $VM^{\alpha,V}_{p,\varphi_1}$ to $VM^{\alpha,V}_{q,\varphi_2}$, $1/p-1/q=\beta/n$.

Acknowledgements. We thank the referee(s) for careful reading the paper and useful comments. The research of A. Akbulut was partially supported by the grant of Ahi Evran University Scientific Research Project (FEF.A3.16.023). The research of M. Omarova was partially supported by the grant of Presidium of Azerbaijan National Academy of Science 2015.

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(Received September 15, 2017)

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