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# ON GENERALIZATION OF D'AURIZIO–SÁNDOR INEQUALITIES INVOLVING A PARAMETER

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*Abstract.* In this work, we generalize the D'Aurizio-Sándor inequalities ([2, 4]) using an elementary approach. In particular, our approach provides an alternative proof of the D'Aurizio-Sándor inequalities. Moreover, as an immediate consequence of the generalized D'Aurizio-Sándor inequalities, we establish the D'Aurizio-Sándor-type inequalities for hyperbolic functions.

## 1. Introduction

Based on infinite product expansions and inequalities on series and the Riemann's zeta function, D'Aurizio ([2]) proved the following inequality:

$$\frac{1-\frac{\cos x}{\cos\frac{x}{2}}}{x^2} < \frac{4}{\pi^2}, \tag{1}$$

where $x \in (0, \pi/2)$. Using an elementary approach, Sándor ([4]) offered an alternative proof of (1) by employing trigonometric inequalities and an auxiliary function. In the same paper, Sándor also provided the converse to (1):

$$\frac{1-\frac{\cos x}{\cos\frac{x}{2}}}{x^2} > \frac{3}{8}, \tag{2}$$

where $x \in (0, \pi/2)$. In addition, Sándor found the following analogous inequality (4) holds true for the case of sine functions:

THEOREM 1. (D'Aurizio-Sándor inequalities ([2, 4])) *The two double inequalities*

$$\frac{3}{8} < \frac{1-\frac{\cos x}{\cos\frac{x}{2}}}{x^2} < \frac{4}{\pi^2} \tag{3}$$

*and*

$$\frac{4}{\pi^2}(2-\sqrt{2}) < \frac{2-\frac{\sin x}{\sin\frac{x}{2}}}{x^2} < \frac{1}{4} \tag{4}$$

*hold for any* $x \in (0, \pi/2)$.

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Throughout this paper, we denote $\frac{1-\frac{\cos x}{\cos\frac{x}{p}}}{x^2}$ and $\frac{p-\frac{\sin x}{\sin\frac{x}{p}}}{x^2}$ respectively by

$$f_p^c(x) = \frac{1-\dfrac{\cos x}{\cos\frac{x}{p}}}{x^2}, \tag{5}$$

$$f_p^s(x) = \frac{p-\dfrac{\sin x}{\sin\frac{x}{p}}}{x^2}. \tag{6}$$

Our aim is to generalize the D'Aurizio-Sándor inequalities for the case of $f_p^c(x)$ and $f_p^s(x)$ as follows:

THEOREM 2. (Generalized D'Aurizio-Sándor inequalities) *Let $0<x<\pi/2$. Then the two double inequalities*

$$\frac{4}{\pi^2} < \frac{1-\dfrac{\cos x}{\cos\frac{x}{p}}}{x^2} < \frac{p^2-1}{2p^2} \tag{7}$$

*and*

$$\frac{4}{\pi^2}\left(p-\csc\left(\frac{\pi}{2p}\right)\right) < \frac{p-\dfrac{\sin x}{\sin\frac{x}{p}}}{x^2} < \frac{p^2-1}{6p} \tag{8}$$

*hold for $p=3,4,5,\cdots$. In particular, (8) remains true when $p=2$ while (7) is reversed when $p=2$.*

We remark that the inequality (7) for the cosine function has been established in [5]. In this paper, we prove (7) by using a different approach.

The remainder of this paper is organized as follows. Section 2 is devoted to the proof of Theorem 2 and an alternative proof of Theorem 1. In Section 3, we establish an analogue of Theorem 2 for hyperbolic functions. As an application of Theorem 2, we apply in Section 4 the inequality (8) to the Chebyshev polynomials of the second kind and establish a trigonometric inequality.

## 2. Proof of the main results

At first we will prove the following lemma. The lemma provides some formulas for the higher-order derivative $\frac{d^2}{dx^2}(x^3\frac{d}{dx}f_p^{\triangle}(x))$ involving $f_p^{\triangle}(x)$ $(\triangle = c,s)$, which are the key ingredients of the proof of Theorem 2. It will turn out that the sign of $\frac{d^2}{dx^2}(x^3\frac{d}{dx}f_p^{\triangle}(x))$ plays a crucial role in proving Theorem 2.

LEMMA 1. *Let $0<x<\pi/2$ and $k=1,2,3,\cdots$. Then when $p\in\mathbb{R}$ and $p\neq 0$, we have*

$(i)$

$$\begin{aligned}\frac{d^2}{dx^2}\left(x^3\frac{d}{dx}f_p^c(x)\right) = -\frac{x\csc^4(\frac{x}{p})}{8p^3}\Bigg(&(p+1)^3\sin\left(x-\frac{3x}{p}\right)+(p-1)^3\sin\left(x+\frac{3x}{p}\right)\\ &+\left(3p^3+3p^2-15p-23\right)\sin\left(x-\frac{x}{p}\right)\\ &+\left(3p^3-3p^2-15p+23\right)\sin\left(x+\frac{x}{p}\right)\Bigg);\end{aligned} \tag{9}$$

$(ii)$

$$\begin{aligned}\frac{d^2}{dx^2}\left(x^3\frac{d}{dx}f_p^s(x)\right) = \frac{x\csc^4(\frac{x}{p})}{8p^3}\Bigg(&(p+1)^3\sin\left(x-\frac{3x}{p}\right)-(p-1)^3\sin\left(x+\frac{3x}{p}\right)\\ &+\left(-3p^3-3p^2+15p+23\right)\sin\left(x-\frac{x}{p}\right)\\ &+\left(3p^3-3p^2-15p+23\right)\sin\left(x+\frac{x}{p}\right)\Bigg).\end{aligned} \tag{10}$$

*In particular,*

$(iii)$ *when* $p=2k$,

$$\frac{d^2}{dx^2}\left(x^3\frac{d}{dx}f_p^s(x)\right) = -\frac{x}{4k^3}\sum_{j=0}^{k-1}(2j+1)^3\sin\left(\frac{2j+1}{2k}x\right); \tag{11}$$

$(iv)$ *when* $p=2k+1$,

$$\frac{d^2}{dx^2}\left(x^3\frac{d}{dx}f_p^c(x)\right) = -\frac{16x}{(2k+1)^3}\sum_{j=1}^{k}j^3\sin\left(\frac{2j}{2k+1}x\right)(-1)^{j-1}, \tag{12}$$

$$\frac{d^2}{dx^2}\left(x^3\frac{d}{dx}f_p^s(x)\right) = -\frac{16x}{(2k+1)^3}\sum_{j=1}^{k}j^3\sin\left(\frac{2j}{2k+1}x\right). \tag{13}$$

$(v)$ *For* $p\in\mathbb{R}$ *and* $p\neq 0$,

$$\lim_{x\to 0}\frac{d}{dx}\left(x^3\frac{d}{dx}f_p^{\triangle}(x)\right) = \lim_{x\to 0}x^3\frac{d}{dx}f_p^{\triangle}(x)=0, \tag{14}$$

*where* $\triangle = c,s$.

*Proof.* $(i)$, $(ii)$ and $(v)$ follow directly from calculations using elementary Calculus. In particular, trigonometric addition formulas are used in proving $(i)$ and $(ii)$. To prove (11) in $(iii)$, we claim

$$-\frac{x}{4k^3}\sum_{j=0}^{k-1}(2j+1)^3\sin\left(\frac{2j+1}{2k}x\right) = -x\frac{d^3}{dx^3}\left(\frac{\sin x}{\sin\left(\frac{x}{2k}\right)}\right). \tag{15}$$

We rewrite

$$\frac{1}{4k^3}\sum_{j=0}^{k-1}(2j+1)^3\sin\left(\frac{2j+1}{2k}x\right)=2\frac{d^3}{dx^3}\left(\sum_{j=0}^{k-1}\cos\left(\frac{2j+1}{2k}x\right)\right). \tag{16}$$

On the other hand, making use of Euler's formula $e^{iz}=\cos z+i\sin z$ leads to an alternative expression of the right-hand side of (16):

$$\sum_{j=0}^{k-1}\cos\left(\frac{2j+1}{2k}x\right)=\sum_{j=0}^{k-1}\Re\left\{e^{i(\frac{x}{2k}+\frac{x}{k}j)}\right\}=\Re\left\{e^{i\frac{x}{2k}}\sum_{j=0}^{k-1}\left(e^{i\frac{x}{k}}\right)^j\right\} \tag{17}$$

$$=\Re\left\{e^{i\frac{x}{2k}}\frac{1-e^{ix}}{1-e^{i\frac{x}{k}}}\right\}=\Re\left\{e^{i\frac{x}{2k}}\frac{e^{\frac{ix}{2}}(e^{-\frac{ix}{2}}-e^{\frac{ix}{2}})}{e^{\frac{ix}{2k}}(e^{-\frac{ix}{2k}}-e^{\frac{ix}{2k}})}\right\} \tag{18}$$

$$=\Re\left\{e^{\frac{ix}{2}}\frac{\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2k}\right)}\right\}=\cos\left(\frac{x}{2}\right)\frac{\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2k}\right)}=\frac{\sin x}{2\sin\left(\frac{x}{2k}\right)}, \tag{19}$$

where $\Re\{z\}$ is the real part of $z$ and $i=\sqrt{-1}$. We complete the proof of the claim. Now it suffices to show

$$\frac{d^2}{dx^2}\left(x^3\frac{d}{dx}f_p^s(x)\right)=-x\frac{d^3}{dx^3}\left(\frac{\sin x}{\sin\left(\frac{x}{2k}\right)}\right). \tag{20}$$

Using (10) in $(ii)$, this can be achieved by straightforward calculations of the right-hand side of the above equation. Thus $(iii)$ is proved. The proof of $(iv)$ is similar, and we omit the details. We complete the proof of Lemma 1. □

We provide here an alternative proof of the two double inequalities in Theorem 1.

*Proof of Theorem 1.* To this end, we show that for $x\in(0,\pi/2)$, $f_2^c(x)=\frac{1-\frac{\cos x}{\cos\frac{x}{2}}}{x^2}$ is strictly increasing while $f_2^s(x)=\frac{2-\frac{\sin x}{\sin\frac{x}{2}}}{x^2}$ is strictly decreasing. These lead to the desired inequalities since it is easy to see that

$$\lim_{x\to 0}f_2^c(x)=\frac{3}{8},\quad \lim_{x\to\pi/2}f_2^c(x)=\frac{4}{\pi^2}, \tag{21}$$

$$\lim_{x\to 0}f_2^s(x)=\frac{1}{4},\quad \lim_{x\to\pi/2}f_2^s(x)=\frac{4}{\pi^2}(2-\sqrt{2}). \tag{22}$$

To see if $f_2^c(x)$ is strictly increasing, we employ (9) in Lemma 1 to obtain

$$\frac{d^2}{dx^2}\left(x^3\frac{d}{dx}f_2^c(x)\right)=-\frac{x}{64}\sec^4\left(\frac{x}{2}\right)\left(-44\sin\left(\frac{x}{2}\right)+5\sin\left(\frac{3x}{2}\right)+\sin\left(\frac{5x}{2}\right)\right)$$

$$=-\frac{x}{16}\sec^4\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)(\cos x-2)(\cos x+5)>0. \tag{23}$$

As $\lim_{x\to 0}\frac{d}{dx}\left(x^3\frac{d}{dx}f_2^c(x)\right)=0$, it follows that $\frac{d}{dx}\left(x^3\frac{d}{dx}f_2^c(x)\right)>0$. We are led to $x^3\frac{d}{dx}f_2^c(x)>0$ or $\frac{d}{dx}f_2^c(x)>0$ since $\lim_{x\to 0}\left(x^3\frac{d}{dx}f_2^c(x)\right)=0$. This shows that $f_2^c(x)$ is strictly increasing.

By means of (11) in Lemma 1, we have

$$\frac{d^2}{dx^2}\left(x^3\frac{d}{dx}f_2^s(x)\right)=-\frac{x}{4}\sin\left(\frac{x}{2}\right)<0, \tag{24}$$

from which we infer that $\frac{d}{dx}\left(x^3\frac{d}{dx}f_2^s(x)\right)<0$ because of $\lim_{x\to 0}\frac{d}{dx}\left(x^3\frac{d}{dx}f_2^s(x)\right)=0$. Then

$$\frac{d}{dx}\left(x^3\frac{d}{dx}f_2^s(x)\right)<0, \tag{25}$$

together with the fact that $\lim_{x\to 0}\left(x^3\frac{d}{dx}f_2^s(x)\right)=0$ yields $x^3\frac{d}{dx}f_2^s(x)<0$ or $\frac{d}{dx}f_2^s(x)<0$. Thus we have shown that $f_2^s(x)$ is strictly decreasing. We note that when $p=2$, $(v)$ of Lemma 1

$$\lim_{x\to 0}\frac{d}{dx}\left(x^3\frac{d}{dx}f_2^{\triangle}(x)\right)=\lim_{x\to 0}x^3\frac{d}{dx}f_2^{\triangle}(x)=0, \tag{26}$$

where $\triangle=c,s$ has been employed in the above arguments. This completes the proof of the theorem. □

We are now in the position to give the proof of Theorem 2.

*Proof of Theorem* 2. The proof of the case when $p=2$ has been given in Theorem 2. For $p\geqslant 3$, we prove the desired inequalities by showing that $\frac{d}{dx}f_p^{\triangle}(x)<0$ for $\triangle=c,s$. Due to $(i)$ of Lemma 1, we see that $\frac{d^2}{dx^2}\left(x^3\frac{d}{dx}f_p^c(x)\right)<0$. On the other hand, we use (11) and (13) in Lemma 1 to conclude that $\frac{d^2}{dx^2}\left(x^3\frac{d}{dx}f_p^s(x)\right)<0$. Thus we have shown that for $\triangle=c,s$,

$$\frac{d^2}{dx^2}\left(x^3\frac{d}{dx}f_p^{\triangle}(x)\right)<0. \tag{27}$$

Because of the first vanishing limit in $(v)$ of Lemma 1, it follows that

$$\frac{d}{dx}\left(x^3\frac{d}{dx}f_p^{\triangle}(x)\right)<0, \tag{28}$$

which, together with the fact that the second limit in $(v)$ of Lemma 1 vanishes, implies that $x^3\frac{d}{dx}f_p^{\triangle}(x)<0$ or $\frac{d}{dx}f_p^{\triangle}(x)<0$ for $\triangle=c,s$. It remains to find the following limits:

$$\lim_{x\to 0}f_p^c(x)=\frac{p^2-1}{2p^2},\quad \lim_{x\to\pi/2}f_p^c(x)=\frac{4}{\pi^2}, \tag{29}$$

$$\lim_{x\to 0}f_p^s(x)=\frac{p^2-1}{6p},\quad \lim_{x\to\pi/2}f_p^s(x)=\frac{4}{\pi^2}\left(p-\csc\left(\frac{\pi}{2p}\right)\right). \tag{30}$$

We immediately have

$$\frac{4}{\pi^2} = \lim_{x\to\pi/2} f_p^c(x) < \frac{1-\dfrac{\cos x}{\cos\frac{x}{p}}}{x^2} < \lim_{x\to 0} f_p^c(x) = \frac{p^2-1}{2\,p^2} \tag{31}$$

and

$$\frac{4}{\pi^2}\left(p-\csc\left(\frac{\pi}{2p}\right)\right) = \lim_{x\to\pi/2} f_p^s(x) < \frac{p-\dfrac{\sin x}{\sin\frac{x}{p}}}{x^2} < \lim_{x\to 0} f_p^s(x) = \frac{p^2-1}{6\,p}. \tag{32}$$

The proof is completed. □

## 3. Generalized D'Aurizio-Sándor inequalities for hyperbolic functions

In this section, we show an analogue of Theorem 2 for the case of hyperbolic functions holds true. Let

$$h_p^c(x) = \frac{1-\dfrac{\cosh x}{\cosh\frac{x}{p}}}{x^2}, \tag{33}$$

$$h_p^s(x) = \frac{p-\dfrac{\sinh x}{\sinh\frac{x}{p}}}{x^2}. \tag{34}$$

Following the same arguments for proving Lemma 1, it can be shown that Lemma 1 with $\cos x$, $\sin x$ and $f_p^{\triangle}(x)$ $(\triangle = c,s)$ replaced by $\cosh x$, $\sinh x$ and $h_p^{\triangle}(x)$ $(\triangle = c,s)$ respectively, remains valid. It follows that we can prove $\frac{d}{dx}h_p^{\triangle}(x) < 0$ for $\triangle = c,s$ as in the proof of Theorem 2. It remains to calculate the following limits:

$$\lim_{x\to 0} f_p^c(x) = \frac{1-p^2}{2\,p^2}, \quad \lim_{x\to\pi/2} f_p^c(x) = \frac{4}{\pi^2}\left(1-\cosh\left(\frac{\pi}{2}\right)\operatorname{sech}\left(\frac{\pi}{2p}\right)\right), \tag{35}$$

$$\lim_{x\to 0} f_p^s(x) = \frac{1-p^2}{6\,p}, \quad \lim_{x\to\pi/2} f_p^s(x) = \frac{4}{\pi^2}\left(p-\sinh\left(\frac{\pi}{2}\right)\operatorname{csch}\left(\frac{\pi}{2p}\right)\right). \tag{36}$$

Thus, we have the following analogue of Theorem 2 for $\cosh x$ and $\sinh x$.

THEOREM 3. *Let $0 < x < \pi/2$. Then the two double inequalities*

$$\frac{4}{\pi^2}\left(1-\cosh\left(\frac{\pi}{2}\right)\operatorname{sech}\left(\frac{\pi}{2p}\right)\right) < \frac{1-\dfrac{\cosh x}{\cosh\frac{x}{p}}}{x^2} < \frac{1-p^2}{2\,p^2} \tag{37}$$

*and*

$$\frac{4}{\pi^2}\left(p-\sinh\left(\frac{\pi}{2}\right)\operatorname{csch}\left(\frac{\pi}{2p}\right)\right) < \frac{p-\dfrac{\sinh x}{\sinh\frac{x}{p}}}{x^2} < \frac{1-p^2}{6\,p} \tag{38}$$

*hold for* $p = 3,4,5,\cdots$. *In particular,* (37) *is reversed when* $p = 2$ *while* (38) *remains true when* $p = 2$.

## 4. Application of the generalized D'Aurizio-Sándor inequalities to the Chebyshev polynomials of the second kinds

The first few Chebyshev polynomials of the second kind $U_n(x)$ $(n = 0,1,2,\cdots)$ are ([1, 3])

$$U_0(x) = 1, \tag{39}$$

$$U_1(x) = 2x, \tag{40}$$

$$U_2(x) = 4x^2 - 1, \tag{41}$$

$$U_3(x) = 8x^3 - 4x, \tag{42}$$

$$U_4(x) = 16x^4 - 12x^2 + 1, \tag{43}$$

$$U_5(x) = 32x^5 - 32x^3 + 6x, \tag{44}$$

$$U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1. \tag{45}$$

In this section, we apply Theorem 2 to $U_n(x)$ with $x = \cos\theta$. By means of the formula $U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}$, we obtain the following corollary.

COROLLARY 1. *Let* $y \in (0, \frac{\pi}{2p})$. *The double inequality*

$$\frac{p}{6}\left((1-p^2)y^2+6\right) < U_{p-1}(\cos y) < p - \frac{4}{\pi^2}\left(p - \csc\left(\frac{\pi}{2p}\right)\right)py^2 \tag{46}$$

*holds for* $p = 2,3,4,5,\cdots$.

*Proof.* The double inequality (8) in Theorem 2 can be written as

$$p - \frac{p^2-1}{6p}x^2 < \frac{\sin x}{\sin\frac{x}{p}} < p - \frac{4}{\pi^2}\left(p - \csc\left(\frac{\pi}{2p}\right)\right)x^2, \quad x \in (0, \pi/2). \tag{47}$$

Letting $x/p = y$, we have

$$\frac{p}{6}\left(6-(p^2-1)y^2\right) < \frac{\sin(py)}{\sin y} < p - \frac{4}{\pi^2}\left(p - \csc\left(\frac{\pi}{2p}\right)\right)p^2y^2, \quad y \in (0, \frac{\pi}{2p}). \tag{48}$$

Due to $\frac{\sin(py)}{\sin y} = U_{p-1}(\cos y)$, the proof is completed. □

EXAMPLE 1. Letting $p = 7$ in Corollary 1 results in the following inequality

$$7 - 56y^2 < 64\cos^6 y - 80\cos^4 y + 24\cos^2 y - 1 < 7 - \frac{196\left(7-\csc\left(\frac{\pi}{14}\right)\right)}{\pi^2}y^2, \tag{49}$$

where $y \in (0, \frac{\pi}{14}) \approx (0, 0.2244)$ and $\frac{196\left(7-\csc\left(\frac{\pi}{14}\right)\right)}{\pi^2} \approx 49.7673$.

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