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THE SPITZER LAW FOR ψ–MIXING RANDOM VARAIBLES

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Abstract. The Spitzer law is obtained for the maximun partial sums of the identically distributed ψ-mixing random variables without any conditions on mixing rate, and another proof of the classical Kolmogorov strong law of large numbers is also given for them.

1. Introduction and main result

Let $\{X, X_n, n \geqslant 1\}$ be a sequence of independent and identically distributed random variables. The following theorem, related to the classical Kolmogorov strong law of large numbers (see Spitzer, 1965), is well-known.

THEOREM A. *The following statements are equivalent*

$$EX = 0, \tag{1.1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left\{ \left| \sum_{k=1}^{n} X_k \right| > \varepsilon n \right\} < \infty \ \ \forall \varepsilon > 0. \tag{1.2}$$

One can label the formula (1.2) as the Spitzer law. The Spitzer law is a specail case of the complete convergence, which first was established by Hsu and Robbins (1947), and then was extended to the general case by Baum and Katz (1965). In particular, Baum and Katz (1965) pointed out that (1.1), (1.2) and

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left\{ \max_{1 \leqslant m \leqslant n} \left| \sum_{k=1}^{m} X_k \right| > \varepsilon n \right\} < \infty \ \ \forall \varepsilon > 0 \tag{1.3}$$

are equivalent.

In general, it is possible that (1.3) is strictly stronger than (1.2) for the non-independent case. One can see the example that (1.2) does not imply (1.3) in the pairwise independent case in Bai et al. (2014). On the other hand, (1.3) can imply the classical Kolmogorov strong law of large numbers

$$\frac{1}{n} \sum_{k=1}^{n} X_k \to 0 \ \text{ a.s.} \tag{1.4}$$

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by the same argument as Chen et al. (2006). So, it is more interesting to obtain (1.3) than (1.2).

For the sake of clarity, let us recall the concept of the ψ-mixing random variables.

DEFINITION 1.1. Define the ψ-mixing coefficient for $\{X_n, n \geqslant 1\}$, a sequence of random variables, as

$$\psi(n) = \sup_{m\geqslant 1} \sup_{A\in\mathscr{F}_1^m, B\in\mathscr{F}_{m+n}^\infty, P(A)P(B)\neq 0} \left| \frac{P(AB)}{P(A)P(B)} - 1 \right|,$$

where $\mathscr{F}_n^m = \sigma(X_i : n \leqslant i \leqslant m)$. Then $\{X_n, n \geqslant 1\}$ is said to be ψ-mixing or *-mixing, if $\psi(n) \to 0$ as $n \to \infty$.

The concept of ψ-mixing is introduced by Blum et al. (1963). They obtained the classical Kolmogorov strong law of large numbers for identically distributed ψ-mixing random variables without any conditions on mixing rate. Recently, Hu et al. (2017) generalized the result of Blum et al. (1963) to the weighted sums of the identically distributed ψ-mixing random variables without any conditions on mixing rate and gave some applications in EV models.

Due to the works of Blum et al. (1963) and Hu et al. (2017), we will obtain the equivalence of (1.1), (1.3) and (1.4) for the kind of mixing random variables without any conditions on mixing rate.

We now state the main result.

THEOREM 1.1. $\{X, X_n, n \geqslant 1\}$ *be a sequence of identically distributed* ψ*-mixing random variables. Then (1.1), (1.3) and (1.4) are equivalent.*

Throughout this paper, the symbol C denotes a positive constant which is not necessarily the same one in each appearance. The symbol $I(A)$ denotes the indicator function of the event A, and $\lfloor x \rfloor$ denotes the integer part of the real number x.

2. Lemmas and proof of main result

The first lemmas is about the property of the ψ-mixing random variables, and modifies Lemma 3.3.1 in Stout (1974) slightly.

LEMMA 2.1. *Let* $\{Y_n, n \geqslant 1\}$ *be a sequence of* ψ*-mixing random variables with the mixing coefficient* $\psi(\cdot)$. *Suppose* $E|Y_n| < \infty$ *for each* $n \geqslant 1$. *Then*

$$|E(Y_{n+1}|\mathscr{G}) - EY_{n+1}| \leqslant \psi(1)E|Y_{n+1}| \tag{2.1}$$

for each σ *field* $\mathscr{G} \subset \sigma(Y_i : 1 \leqslant i \leqslant n)$, *and each* $n \geqslant 1$.

Proof. It is trivial that (2.1) holds if $\psi(1) = \infty$. Now we assume that $0 \leqslant \psi(1) < \infty$. By the definition of the ψ-mixing coefficient, for any $n \geqslant 1$, $A \in \sigma(Y_i : 1 \leqslant i \leqslant n)$ and $B \in \sigma(Y_i : i \geqslant n+1)$, we have

$$|P(AB) - P(A)P(B)| \leqslant \psi(1)P(A)P(B).$$

Then by the same argument as Lemma 3.3.1 in Stout (1974), (2.1) holds. □

To prove Theorem 1.2, the following two lemmas on φ-mixing random variables are needed. The first one is a Rosenthal type inequality for φ-mixing random variables (see Shao, 1988). The second one shows that the Spitzer law holds for uniformly bounded φ-mixing random variables without any conditions on mixing rate and is interesting in itself. Since ψ-mixing is φ-mixing (see, for example, Lin and Lu, 1997), Lemma 2.3 also holds for ψ-mixing random variables.

LEMMA 2.2. *Let $\{Y_n, n \geqslant 1\}$ be a sequence of φ-mixing random variables with $E|Y_n|^s < \infty$ for all $n \geqslant 1$ and for some $s \geqslant 2$. Then there exists a positive constant C depending only on s and the φ-mixing coefficient $\varphi(\cdot)$ such that for all $n \geqslant 1$,*

$$E \max_{1 \leqslant m \leqslant n} \left| \sum_{k=1}^{m} (Y_k - EY_k) \right|^s \leqslant C \left\{ \left[\exp\left(6 \sum_{i=1}^{\lfloor \log n \rfloor} \varphi^{1/2}(2^i) \right) \cdot n \max_{1 \leqslant k \leqslant n} EY_k^2 \right]^{s/2} + \sum_{k=1}^{n} E|Y_k|^s \right\}.$$

REMARK 2.1. Set $a(x) = \sum_{i=1}^{\lfloor \log x \rfloor} \varphi^{1/2}(2^i)$, $x > 0$. Then by $\varphi(2^n) \to 0$ as $n \to \infty$, $\lim_{x\to\infty} a(x)/\log x = 0$ and hence $\lim_{x\to\infty} x^{-\delta} \exp(sa(x)) = 0$ for any $s > 0$ and $\delta > 0$. Therefore, the series $\sum_{n=1}^{\infty} n^{-\lambda} \exp(sa(n))$ converges for any $s > 0$ and $\lambda > 1$.

LEMMA 2.3. *Let $\{Y_n, n \geqslant 1\}$ be a sequence of φ-mixing random variables with* $\sup_{n\geqslant 1} |Y_n| \leqslant M$ *a.s. for some constant $M > 0$. Then*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left\{ \max_{1 \leqslant m \leqslant n} \left| \sum_{k=1}^{m} (Y_k - EY_k) \right| > \varepsilon n \right\} < \infty \ \forall \varepsilon > 0. \tag{2.2}$$

Proof. By the Markov inequality, Lemma 2.1, and $\sup_{n\geqslant 1} |Y_n| \leqslant M$ a.s., we have for any $s > 2$,

$$\begin{aligned} &P\left\{ \max_{1 \leqslant m \leqslant n} \left| \sum_{k=1}^{m} (Y_k - EY_k) \right| > \varepsilon n \right\} \\ &\leqslant Cn^{-s} E \max_{1 \leqslant m \leqslant n} \left| \sum_{k=1}^{m} (Y_k - EY_k) \right|^s \\ &\leqslant Cn^{-s} \left\{ \left[\exp\left(6 \sum_{i=1}^{\lfloor \log n \rfloor} \varphi^{1/2}(2^i) \right) \cdot n \max_{1 \leqslant k \leqslant n} EY_k^2 \right]^{s/2} + \sum_{k=1}^{n} E|Y_k|^s \right\} \\ &\leqslant Cn^{-s/2} \exp(3sa(n)) + Cn^{1-s}. \end{aligned}$$

By $s > 2$ and Remark 2.1, the series $\sum_{n=1}^{\infty} n^{-s/2} \exp(3sa(n))$ and $\sum_{n=1}^{\infty} n^{1-s}$ converge, and hence (2.2) holds. □

Proof of Theorem 1.1. We first prove that (1.1) implies (1.3). We can assume that $0 \leqslant \psi(1) < \infty$ by the subsequence method. Let us fix $\varepsilon > 0$. By $E|X| < \infty$, there exists a positive integer number $M = M(\varepsilon)$ such that $E|X|I(|X| > M) \leqslant \varepsilon/(8+8\psi(1))$. Note that by $EX = 0$,

$$\left\{\max_{1\leqslant m\leqslant n}\left|\sum_{k=1}^{m} X_k\right| > \varepsilon n\right\} \subset \left\{\max_{1\leqslant m\leqslant n}\left|\sum_{k=1}^{m}(X_kI(|X_k| \leqslant M) - EX_kI(|X_k| \leqslant M))\right| > \varepsilon n/2\right\}$$
$$\cup\left\{\max_{1\leqslant m\leqslant n}\left|\sum_{k=1}^{m}(X_kI(|X_k| > M) - EX_kI(|X_k| > M))\right| > \varepsilon n/2\right\},$$

and by $E|X|I(|X| > M) \leqslant \varepsilon/(8+8\psi(1)) \leqslant \varepsilon/8$,

$$\begin{aligned}
&\left\{\max_{1\leqslant m\leqslant n}\left|\sum_{k=1}^{m}(X_kI(|X_k| > M) - EX_kI(|X_k| > M))\right| > \varepsilon n/2\right\}\\
&\subset \left\{\sum_{k=1}^{n}|X_k|I(|X_k| > M) > (\varepsilon/2 - E|X|I(|X| > M))n\right\}\\
&\subset \left\{\sum_{k=1}^{n}|X_k|I(|X_k| > M) > 3\varepsilon n/8\right\}\\
&\subset \cup_{k=1}^{n}\{|X_k| > n\} \cup \left\{\sum_{k=1}^{n}|X_k|I(M < |X_k| \leqslant n) > 3\varepsilon n/8\right\}
\end{aligned}$$

holds for any $n > M$. Since $|X_nI(|X_n| \leqslant M)| \leqslant M$ for all $n \geqslant 1$, by Lemma 2.2,

$$\sum_{n=1}^{\infty}\frac{1}{n}P\left\{\max_{1\leqslant m\leqslant n}\left|\sum_{k=1}^{m}(X_kI(|X_k| \leqslant M) - EX_kI(|X_k| \leqslant M))\right| > \varepsilon n/2\right\} < \infty,$$

and by $E|X| < \infty$,

$$\sum_{n=1}^{\infty}\frac{1}{n}P\left(\cup_{k=1}^{n}\{|X_k| > n\}\right) \leqslant \sum_{n=1}^{\infty}P\{|X| > n\} \leqslant CE|X| < \infty.$$

Therefore, to prove (1.3), it suffices to prove that

$$\sum_{n=M+1}^{\infty}\frac{1}{n}P\left\{\sum_{k=1}^{n}|X_k|I(M < |X_k| \leqslant n) > 3\varepsilon n/8\right\} < \infty. \tag{2.3}$$

Set $X_{nk} = |X_k|I(M < |X_k| \leqslant n)$, $\mathscr{F}_{n,k} = \sigma(X_{ni} : 1 \leqslant i \leqslant k)$ for $1 \leqslant k \leqslant n$, and $\mathscr{F}_{n,0} =$

$\{\emptyset,\Omega\}$. By Lemma 2.1 and $E|X|I(|X|>M)\leqslant \varepsilon/(8+8\psi(1))$,

$$\begin{aligned}\left|\sum_{k=1}^{n}E(X_{nk}|\mathscr{F}_{n,k-1})\right| &\leqslant \sum_{k=1}^{n}(|E(X_{nk}|\mathscr{F}_{n,k-1})-EX_{nk}|+|EX_{nk}|)\\ &\leqslant (1+\psi(1))\sum_{k=1}^{n}E|X_{nk}|\\ &\leqslant (1+\psi(1))\sum_{k=1}^{n}E|X_k|I(|X_k|>M)\\ &\leqslant \varepsilon n/8.\end{aligned}$$

Thus, to prove (2.3), it suffices to prove that

$$\sum_{n=M+1}^{\infty}\frac{1}{n}P\left\{\left|\sum_{k=1}^{n}(X_{nk}-E(X_{nk}|\mathscr{F}_{n,k-1}))\right|>\varepsilon n/4\right\}<\infty.$$

In fact, note that $\{X_{nk}-E(X_{nk}|\mathscr{F}_{n,k-1}),1\leqslant k\leqslant n\}$ is a sequence of martingale difference for any $n\geqslant M+1$, then by the Markov inequality, the martingale property and using a standard method,

$$\begin{aligned}&\sum_{n=M+1}^{\infty}\frac{1}{n}P\left\{\left|\sum_{k=1}^{n}(X_{nk}-E(X_{nk}|\mathscr{F}_{n,k-1}))\right|>\varepsilon n/4\right\}\\ &\leqslant C\sum_{n=M+1}^{\infty}\frac{1}{n}\cdot\frac{1}{n^2}E\left(\sum_{k=1}^{n}(X_{nk}-E(X_{nk}|\mathscr{F}_{n,k-1}))\right)^2\\ &=C\sum_{n=M+1}^{\infty}\frac{1}{n}\cdot\frac{1}{n^2}\sum_{k=1}^{n}E(X_{nk}-E(X_{nk}|\mathscr{F}_{n,k-1}))^2\\ &\leqslant C\sum_{n=M+1}^{\infty}\frac{1}{n}\cdot\frac{1}{n^2}\sum_{k=1}^{n}EX_{nk}^2\\ &\leqslant C\sum_{n=1}^{\infty}\frac{1}{n^2}EX^2I(|X|\leqslant n)\\ &\leqslant CE|X|<\infty.\end{aligned}$$

Therefore, (1.3) holds.

By the same argument as Chen et al. (2006), (1.3) implies (1.4). And (1.1) and (1.4) are equivalent by Corollary in Blum et al. (1963). So we complete the proof. □

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