

## A FAMILY OF WINDSCHITL TYPE APPROXIMATIONS FOR GAMMA FUNCTION

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*Abstract.* In this paper, we present a family of high accurate approximation formulas

$$\mathscr{W}_p(x) = \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \exp\left(\frac{1}{1620x^3} \frac{x^2+p}{x^2+p+33/35}\right)$$

for gamma function  $\Gamma(x+1)$  with parameter  $p \geq -33/35$ , and prove the function

$$x \mapsto \ln \Gamma(x+1) - \ln \mathscr{W}_p(x)$$

is strictly increasing and concave on  $(0, \infty)$  if and only if  $p \geq 158/315$ . This yields some new sharp approximations for gamma function.

### 1. Introduction

It is known that the Stirling's formula

$$n! \sim \sqrt{2\pi n} n^n e^{-n} \tag{1.1}$$

for  $n \in \mathbb{N}$  has important applications in probability theory, statistical physics, number theory, combinatorics and other related fields. There are many improvements for the Stirling's formula, see for example, Burnside's [1], Gosper [2], Batir [3], Mortici [4].

Because the gamma function  $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$  for  $x > 0$  is related to the factorial function, many scholars were devoted to seeking various better approximations for the gamma function, for instance, Ramanujan [5, p. 339], Windschitl [6, Eq. (42)], [7], Nemes [8, Corollary 4.1], Mortici [9, 10], Yang and Chu [11, Propositions 4 and 5], Chen [12, 13], Lu et al. [14, 15].

More results involving the approximation formulas for the factorial or gamma function can be found in [16, 17, 18, 19, 20, 21, 22, 23] and the references cited therein. Several nice inequalities between gamma function and the truncations of its asymptotic series can be found in [24], [25].

Now we focus on the Windschitl's approximation formulas (see [6, Eq. (42)], [7]) given by

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} := W_0(x), \tag{1.2}$$

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$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x} + \frac{1}{810x^6}\right)^{x/2} := W_1(x), \tag{1.3}$$

as  $x \rightarrow \infty$ . It was showed in [13] that the rates of Windschitl’s approximation formulas  $W_0(x)$  and  $W_1(x)$  converging to  $\Gamma(x+1)$  is like  $x^{-5}$  and  $x^{-7}$  as  $x \rightarrow \infty$ , respectively. These show that  $W_0(x)$  and  $W_1(x)$  are excellent approximations for gamma function.

Very recently, by using a little known power series Yang and Tian [26] developed Windschitl’s approximation formula  $W_0(x)$  for the gamma function to asymptotic expansion

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \exp\left(\sum_{n=3}^{\infty} \frac{2n(2n-2)! - 2^{2n-1} B_{2n}}{2n(2n)!} \frac{B_{2n}}{x^{2n-1}}\right) \tag{1.4}$$

as  $x \rightarrow \infty$ , where  $B_{2n}$  is the Bernoulli number. While the approximation formula  $W_1(x)$  was developed to another asymptotic expansion by Chen and Paris in [27, Theorem 2]:

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x} + \frac{1}{810x^6} - \frac{163}{170100x^8} + \frac{1019}{680400x^{10}} + \dots\right)^{x/2}$$

as  $x \rightarrow \infty$ . Lu, Song and Ma [28] extended Windschitl’s formula  $W_0(n)$  to another asymptotic expansion in the form of

$$\Gamma(n+1) \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left[n \sinh \left(\frac{1}{n} + \frac{a_7}{n^7} + \frac{a_9}{n^9} + \frac{a_{11}}{n^{11}} + \dots\right)\right]^{n/2}$$

with  $a_7 = 1/810, a_9 = -67/42525, a_{11} = 19/8505, \dots$  without a general formula for the coefficients  $a_k$  for  $k \geq 7$ . This general formula was given by Chen in [12, Theorem 2], and in the same paper, Chen presented a new asymptotic expansion

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2 + \sum_{j=0}^{\infty} r_j x^{-j}}, \quad x \rightarrow \infty, \tag{1.5}$$

which was improved in [26] by Yang and Tian.

On the other hand, Alzer [29] showed that the double inequality

$$\sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \left(1 + \frac{\alpha}{x^5}\right) < \Gamma(x+1) < \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \left(1 + \frac{\beta}{x^5}\right) \tag{1.6}$$

holds for  $x > 0$  with the best possible constants  $\alpha = 0$  and  $\beta = 1/1620$ . Yang and Tian [30] presented a very accurate Windschitl type approximation:

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \exp\left(\frac{7}{324} \frac{1}{x^3(35x^2+33)}\right) = W_2(x), \tag{1.7}$$

as  $x \rightarrow \infty$ , and prove that the function  $x \mapsto \ln \Gamma(x+1) - \ln W_2(x)$  is decreasing and concave on  $(1, \infty)$ . This yields some new inequalities for gamma function in terms of  $W_0(x)$ .

Inspired by the Windschitl type approximation (1.7), we consider a family of Windschitl type approximations for gamma function defined by

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \exp\left(\frac{1}{1620x^5} \frac{x^2+p}{x^2+p+33/35}\right) = \mathscr{W}_p(x) \tag{1.8}$$

as  $x \rightarrow \infty$ , where  $p \geq -33/35$ . It is easy to see that

$$\begin{aligned} \mathcal{W}_\infty(x) &= \lim_{p \rightarrow \infty} \mathcal{W}_p(x) = \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \exp\left(\frac{1}{1620x^5}\right) \\ \mathcal{W}_{-33/35}(x) &= \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \exp\left(\frac{1}{1620x^5} - \frac{11}{18900x^7}\right), \\ \mathcal{W}_0(x) &= \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \exp\left(\frac{1}{1620x^5} \frac{x^2}{x^2 + 33/35}\right) = W_2(x). \end{aligned}$$

The aim of this paper is to determine the parameter  $p \geq -33/35$  such that the function  $x \mapsto \ln \Gamma(x+1) - \ln \mathcal{W}_p(x)$  is increasing and concave on  $(0, \infty)$ . Our main result is the following theorem.

**THEOREM 1.** *Let  $p \geq -33/35$ . Then the function*

$$\begin{aligned} F_p(x) &= \ln \Gamma(x+1) - \ln \sqrt{2\pi} - \left(x + \frac{1}{2}\right) \ln x + x \\ &\quad - \frac{x}{2} \ln \left(x \sinh \frac{1}{x}\right) - \frac{1}{1620x^5} \frac{x^2 + p}{x^2 + 33/35} \end{aligned} \tag{1.9}$$

is strictly increasing and concave on  $(0, \infty)$  if and only if  $p \geq p_0 = 158/315$ .

### 2. Proof of Theorem 1

As is well known, analytic inequality [31, 32, 33] is playing a very important role in different branches of modern mathematics. In order to prove Theorem 1, we need the following inequality.

**LEMMA 1.** *The inequality*

$$\psi' \left(x + \frac{1}{2}\right) < \frac{1}{x} \frac{x^6 + \frac{445}{78}x^4 + \frac{199801}{34320}x^2 + \frac{3072}{5005}}{x^6 + \frac{301}{52}x^4 + \frac{14357}{2288}x^2 + \frac{9069}{9152}} \tag{2.1}$$

holds for  $x > 0$ .

*Proof.* Let

$$g(x) = \psi' \left(x + \frac{1}{2}\right) - \frac{1}{x} \frac{q_6(x)}{p_6(x)},$$

where

$$\begin{aligned} p_6(x) &= x^6 + \frac{301}{52}x^4 + \frac{14357}{2288}x^2 + \frac{9069}{9152}, \\ q_6(x) &= x^6 + \frac{445}{78}x^4 + \frac{199801}{34320}x^2 + \frac{3072}{5005}. \end{aligned}$$

In view of

$$\psi' \left(x + \frac{3}{2}\right) - \psi' \left(x + \frac{1}{2}\right) = -\frac{1}{(x+1/2)^2}$$

(see [34, p. 260, (6.4.6)]), we have

$$\begin{aligned} g(x+1) - g(x) &= \psi' \left( x + \frac{3}{2} \right) - \frac{1}{x+1} \frac{q_6(x+1)}{p_6(x+1)} - \psi' \left( x + \frac{1}{2} \right) + \frac{1}{x} \frac{q_6(x)}{p_6(x)} \\ &= -\frac{1}{(x+1/2)^2} - \frac{1}{x+1} \frac{q_6(x+1)}{p_6(x+1)} + \frac{1}{x} \frac{q_6(x)}{p_6(x)} \\ &= \frac{176400}{20449} \frac{1}{x(x+1)(2x+1)^2 p_6(x)p_6(x+1)} > 0. \end{aligned}$$

This yields

$$g(x) < g(x+1) < \dots < \lim_{n \rightarrow \infty} g(x+n) = 0,$$

which completes the proof.  $\square$

*Proof of Theorem 1.* Using the asymptotic expansion (1.4), we have that as  $x \rightarrow \infty$ ,

$$\begin{aligned} F_p(x) &= \left[ \ln \Gamma(x+1) - \ln \sqrt{2\pi} - \left( x + \frac{1}{2} \right) \ln x + x - \frac{x}{2} \ln \left( x \sinh \frac{1}{x} \right) \right] \\ &\quad - \frac{1}{1620x^5} \frac{x^2 + p}{x^2 + p + 33/35} \\ &\sim \sum_{n=3}^5 \frac{2n(2n-2)! - 2^{2n-1} B_{2n}}{2n(2n)!} \frac{1}{x^{2n-1}} - \frac{1}{1620x^5} \frac{x^2 + p}{x^2 + p + 33/35} \\ &= -\frac{11}{170100} \frac{(315p - 158)x^2 - (455p + 429)}{x^9(35x^2 + 35p + 33)}, \end{aligned}$$

which implies that

$$\lim_{x \rightarrow \infty} \frac{F_p(x)}{x^{-9}} = -\frac{11}{18900} \left( p - \frac{158}{315} \right). \tag{2.2}$$

In view of  $\lim_{x \rightarrow \infty} F_p(x) = 0$ , making use of L'Hospital rule gives

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{F'_p(x)}{x^{-10}} &= \frac{11}{2100} \left( p - \frac{158}{315} \right), \\ \lim_{x \rightarrow \infty} \frac{F''_p(x)}{x^{-11}} &= -\frac{11}{210} \left( p - \frac{158}{315} \right). \end{aligned}$$

(i) We now prove the necessity. If  $F_p$  is increasing and concave on  $(0, \infty)$ , then there must be

$$\lim_{x \rightarrow \infty} \frac{F'_p(x)}{x^{-10}} = \frac{11}{2100} \left( p - \frac{158}{315} \right) \geq 0 \text{ and } \lim_{x \rightarrow \infty} \frac{F''_p(x)}{x^{-11}} = -\frac{11}{210} \left( p - \frac{158}{315} \right) \leq 0,$$

which yields  $p \geq p_0 = 158/315$ .

(ii) We next prove the sufficiency. Differentiation yields

$$\begin{aligned} F'_p(x) &= \psi(x+1) - \frac{1}{2} \ln \left( x \sinh \frac{1}{x} \right) + \frac{1}{2x} \coth \frac{1}{x} - \ln x - \frac{1}{2x} - \frac{1}{2} \\ &\quad + \frac{7}{324} \frac{175x^4 + (350p + 99)x^2 + 5p(35p + 33)}{x^6(35x^2 + 35p + 33)^2}, \end{aligned}$$

$$F_p''(x) = \psi'(x+1) + \frac{1}{2x^3} \frac{1}{\sinh^2(1/x)} - \frac{3}{2x} + \frac{1}{2x^2}$$

$$= \frac{7 \ 6125x^6 + 35(525p + 187)x^4 + 3(35p + 33)(175p + 22)x^2 + 5p(35p + 33)^2}{54 x^7 (35x^2 + 35p + 33)^3}$$

Since

$$\frac{\partial F_p'(x)}{\partial p} = \frac{385}{108} \frac{63x^2 + 35p + 33}{x^6 (35x^2 + 35p + 33)^3} > 0$$

$$\frac{\partial F_p''(x)}{\partial p} = -\frac{385 \ 3675x^4 + 112(35p + 33)x^2 + (35p + 33)^2}{18 x^7 (35x^2 + 35p + 33)^4} < 0$$

for  $p \geq -33/35$  and  $x > 0$ , that are, the functions  $p \mapsto F_p'(x)$  and  $p \mapsto F_p''(x)$  are increasing and decreasing on  $[-33/35, \infty)$ . Thus, if we prove  $F_{p_0}''(x) < 0$ , then  $F_{p_0}'(x) > \lim_{x \rightarrow \infty} F_{p_0}'(x) = 0$  for  $x > 0$ . And, for  $p \geq p_0$ , it is easy to see that  $F_p''(x) \leq F_{p_0}''(x) < 0$  and  $F_p'(x) \geq F_{p_0}'(x) > 0$ , then the sufficiency follows.

Replacing  $x$  by  $(x + 1/2)$  in the inequity of (2.1) leads to

$$\psi'(x+1) < \frac{1}{210} \frac{(60060x^6 + 180180x^5 + 567875x^4 + 835450x^3 + 919933x^2 + 532238x + 146631)}{(2x+1)(143x^6 + 429x^5 + 1364x^4 + 2013x^3 + 2273x^2 + 1338x + 420)},$$

which indicates that

$$F_{p_0}''(x) < \frac{1}{210} \frac{60060x^6 + 180180x^5 + 567875x^4 + 835450x^3 + 919933x^2 + 532238x + 146631}{(2x+1)(143x^6 + 429x^5 + 1364x^4 + 2013x^3 + 2273x^2 + 1338x + 420)} - \frac{3}{2x} + \frac{1}{2x^2} + \frac{1}{2x^3} \frac{1}{\sinh^2(1/x)} - \frac{127575x^6 + 328293x^4 + 346788x^2 + 133510}{9450x^7(9x^2 + 13)^3} := f\left(\frac{1}{x}\right).$$

Simplifying yields

$$f(t) = \frac{t^3}{2 \sinh^2 t} - \frac{t}{9450} \frac{p_{19}(t)}{p_{13}(t)},$$

where

$$p_{19}(t) = 56074200t^{19} + 290784780t^{18} + 806391950t^{17} + 1630996354t^{16} + 2573755772t^{15} - 948850784t^{14} - 5848462963t^{13} - 3534213977t^{12} + 16372329813t^{11} + 57129403014t^{10} + 109206482238t^9 + 152782165188t^8 + 164420940285t^7 + 152143429005t^6 + 108142112610t^5 + 70899015285t^4 + 32357867850t^3 + 14814902025t^2 + 3447969525t + 985134150,$$

$$p_{13}(t) = (t+2)(13t^2+9)^3(420t^6+1338t^5+2273t^4+2013t^3+1364t^2+429t+143).$$

Using the inequality

$$2 \sinh^2 t = \cosh 2t - 1 > \sum_{n=0}^6 \frac{2^{2n}}{(2n)!} t^{2n} - 1 = \sum_{n=1}^6 \frac{2^{2n}}{(2n)!} t^{2n},$$

we have

$$\begin{aligned}
 f(t) &< \frac{t^3}{\sum_{n=1}^6 \frac{2^{2n}}{(2n)!} t^{2n}} - \frac{t}{9450} \frac{p_{19}(t)}{p_{13}(t)} \\
 &= \frac{t}{9450} \frac{9450 p_{13}(t) - \left(\sum_{n=1}^6 \frac{2^{2n}}{(2n)!} t^{2n-2}\right) p_{19}(t)}{\left(\sum_{n=1}^6 \frac{2^{2n}}{(2n)!} t^{2n-2}\right) p_{13}(t)} \\
 &:= -\frac{1}{4725} \frac{t^{13} \times p_{17}(t)}{\left(\sum_{n=1}^6 \frac{2^{2n}}{(2n)!} t^{2n-2}\right) p_{13}(t)} < 0,
 \end{aligned}$$

where

$$\begin{aligned}
 p_{17}(t) &= \frac{213616}{891} t^{17} + \frac{1174888}{945} t^{16} + \frac{30363892}{2673} t^{15} + \frac{22453788188}{467775} t^{14} \\
 &+ \frac{20234223892}{66825} t^{13} + \frac{537563456096}{467775} t^{12} + \frac{360206516968}{66825} t^{11} \\
 &+ \frac{8397752582192}{467775} t^{10} + \frac{1379858652428}{22275} t^9 + \frac{25906996096822}{155925} t^8 \\
 &+ \frac{3144272626043}{7425} t^7 + \frac{8209692140801}{10395} t^6 + \frac{1214784299189}{825} t^5 \\
 &+ \frac{7860969452488}{5775} t^4 + \frac{94725155551}{55} t^3 + \frac{353797595441}{385} t^2 \\
 &+ \frac{3200114871}{5} t + \frac{6400229742}{35} > 0
 \end{aligned}$$

for  $t > 0$ . This implies that  $F''_{p_0}(x) < 0$  for  $x > 0$ , and the proof is complete.  $\square$

### 3. New bounds for gamma function

As a direct consequence of Theorem 1, we immediately get

**COROLLARY 1.** For  $p \geq -33/35$ , let  $\mathscr{W}_p(x)$  be defined by (1.8). Then the double inequality

$$\lambda_p \mathscr{W}_p(x) < \Gamma(x+1) < \mathscr{W}_p(x), \tag{3.1}$$

or equivalently,

$$\begin{aligned}
 &\lambda_p \left(x \sinh \frac{1}{x}\right)^{x/2} \exp\left(\frac{1}{1620x^5} \frac{x^2+p}{x^2+p+33/35}\right) \\
 &\leq \frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x} < \left(x \sinh \frac{1}{x}\right)^{x/2} \exp\left(\frac{1}{1620x^5} \frac{x^2+p}{x^2+p+33/35}\right)
 \end{aligned} \tag{3.2}$$

holds for  $x \geq 1$  if and only if  $p \geq p_0 = 158/315$ , where

$$\lambda_p = \frac{1}{\sqrt{2\pi \sinh 1}} \exp\left(\frac{1}{324} \frac{11333p+22025}{35p+68}\right) \tag{3.3}$$

is the best constant. In particular, we have

$$\lambda_{p_0} \mathscr{W}_{p_0}(x) < \Gamma(x+1) < \mathscr{W}_{p_0}(x),$$

namely,

$$\begin{aligned} & \lambda_{p_0} \left(x \sinh \frac{1}{x}\right)^{x/2} \exp\left(\frac{1}{1620x^5} \frac{x^2 + 158/315}{x^2 + 13/9}\right) \\ & \leq \frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/2)^x} < \left(x \sinh \frac{1}{x}\right)^{x/2} \exp\left(\frac{1}{1620x^5} \frac{x^2 + 158/315}{x^2 + 13/9}\right) \end{aligned} \tag{3.4}$$

holds for  $x \geq 1$  with the best constant

$$\lambda_{p_0} = \frac{1}{\sqrt{2\pi \sinh 1}} \exp\left(\frac{113357}{113400}\right) \approx 0.999963. \tag{3.5}$$

*Proof.* The necessary condition for the inequalities (3.2) to hold for  $x \geq 1$  follows from  $\lim_{x \rightarrow \infty} x^9 F_p(x) \leq 0$ . This together with the limit relation (2.2) yields  $p \geq p_0 = 158/315$ .

The sufficient condition for the inequalities (3.2) to hold for  $x \geq 1$  follows from the increasing property of  $F_p$  on  $(0, \infty)$  if  $p \geq p_0 = 158/315$ .

Putting  $p = p_0 = 158/315$  in inequalities (3.2) gives (3.4), which completes the proof.  $\square$

REMARK 1. We claim that the lower and upper bounds in inequalities (3.1) are decreasing and increasing with respect to the parameter  $p$  on  $[-33/35, \infty)$  for  $x \geq 1$ . In fact, by differentiation we find that

$$\begin{aligned} \frac{\partial}{\partial p} \ln \mathscr{W}_p(x) &= \frac{77}{108x^5(35x^2 + 35p + 33)^2}, \\ \frac{\partial}{\partial p} (\ln(\lambda_p \mathscr{W}_p(x))) &= -\frac{77}{108} \frac{H_p(x)}{x^5(35p + 68)^2(35x^2 + 35p + 33)^2}, \end{aligned}$$

where

$$H_p(x) = 1225(x^5 - 1)p^2 + 70(35x^7 + 33x^5 - 68)p + (1225x^9 + 2310x^7 + 1089x^5 - 4624).$$

Clearly,  $\partial \ln \mathscr{W}_p(x) / \partial p > 0$  for  $p \geq -33/35$  and  $x \geq 1$ ; while  $\partial (\ln(\lambda_p \mathscr{W}_p(x))) / \partial p < 0$  is due to

$$H_p(x) = 1225 \left[ (x^5 - 1) \left(p + \frac{33}{35}\right)^2 + 2(x^7 - 1) \left(p + \frac{33}{35}\right) + (x^9 - 1) \right] > 0$$

for  $p \geq -33/35$  and  $x > 1$ . Thus taking  $p = 158/315, 4/7, \infty$ , we obtain the following corollary.

COROLLARY 2. *The inequalities*

$$\lambda_\infty \mathscr{W}_\infty(x) \leq \lambda_{4/7} \mathscr{W}_{4/7}(x) \leq \lambda_{p_0} \mathscr{W}_{p_0}(x) \leq \Gamma(x+1) < \mathscr{W}_{p_0}(x) < \mathscr{W}_{22/35}(x) < \mathscr{W}_\infty(x),$$

that are,

$$\begin{aligned} \lambda_\infty \exp\left(\frac{1}{1620x^5}\right) &\leq \lambda_{4/7} \exp\left(\frac{1}{324x^5} \frac{7x^2+4}{35x^2+53}\right) \\ &\leq \lambda_{p_0} \exp\left(\frac{1}{1620x^5} \frac{x^2+158/315}{x^2+13/9}\right) \leq \frac{\Gamma(x+1)}{\sqrt{2\pi x} (x/2)^x (x \sinh(1/x))^{x/2}} \\ &< \exp\left(\frac{1}{1620x^5} \frac{x^2+158/315}{x^2+13/9}\right) < \exp\left(\frac{1}{324x^5} \frac{7x^2+4}{35x^2+53}\right) < \exp\left(\frac{1}{1620x^5}\right), \end{aligned}$$

hold for  $x \geq 1$ , where  $\lambda_{p_0} \approx 0.999963$  given in (3.5) and

$$\begin{aligned} \lambda_{4/7} &= \frac{1}{\sqrt{2\pi \sinh 1}} \exp\left(\frac{2591}{2592}\right) \approx 0.999956, \\ \lambda_\infty &= \frac{1}{\sqrt{2\pi \sinh 1}} \exp\left(\frac{1619}{1620}\right) \approx 0.999725 \end{aligned}$$

are the best constants.

It has been shown in [30] that the function  $x \mapsto \ln \Gamma(x+1) - \ln W_2(x)$  is decreasing and concave on  $(1, \infty)$ . Since  $W_2(x) = \mathcal{W}_0(x)$ , by the same technique as the proof of Theorem 1 we can prove

**THEOREM 2.** *Let  $p \geq -33/35$ . Then the function  $F_p$  defined by (1.9) is decreasing and convex on  $[1, \infty)$  if  $-33/35 \leq p \leq 0$ .*

Likewise, let  $p = -33/35, -1/7, 0$  in Theorem 2. Then by using the monotonicity of  $F_p$  in  $x$  on  $(1, \infty)$  and bounds in inequalities (3.2) we can obtain

**COROLLARY 3.** *The inequalities*

$$\mathcal{W}_{-33/35}(x) < \mathcal{W}_{-1/7}(x) < \mathcal{W}_0(x) < \Gamma(x+1) < \lambda_0 \mathcal{W}_0(x) \leq \lambda_{-1/7} \mathcal{W}_{-1/7}(x) \leq \mathcal{W}_{-33/35}(x),$$

namely,

$$\begin{aligned} \exp\left(\frac{1}{1620x^5} - \frac{11}{18900x^7}\right) &< \exp\left(\frac{1}{2268x^5} \frac{7x^2-1}{5x^2+4}\right) < \exp\left(\frac{7}{324x^3(35x^2+33)}\right) \\ &< \frac{\Gamma(x+1)}{\sqrt{2\pi x} (x/2)^x (x \sinh(1/x))^{x/2}} \leq \lambda_0 \exp\left(\frac{7}{324x^3(35x^2+33)}\right) \\ &\leq \lambda_{-1/7} \exp\left(\frac{1}{2268x^5} \frac{7x^2-1}{5x^2+4}\right) \leq \lambda_{-33/35} \exp\left(\frac{1}{1620x^5} - \frac{11}{18900x^7}\right), \end{aligned}$$

hold for  $x \geq 1$  with the best constants

$$\begin{aligned} \lambda_0 &= \frac{1}{\sqrt{2\pi \sinh 1}} \exp\left(\frac{22025}{22032}\right) \approx 1.000024, \\ \lambda_{-1/7} &= \frac{1}{\sqrt{2\pi \sinh 1}} \exp\left(\frac{3401}{3402}\right) \approx 1.000048, \\ \lambda_{-33/35} &= \frac{1}{\sqrt{2\pi \sinh 1}} \exp\left(\frac{28349}{28350}\right) \approx 1.000307. \end{aligned}$$



### 4. Numerical comparisons

As we know, a remarkable approximation for gamma function is fairly accurate but relatively succinct. Such ones can see [5, p. 339], [8, Corollary 4.1], [7], [9], [11, Propositions 4 and 5], [13]. In this section, we list some more accurate but simple approximation formulas for gamma function and compare them with our family of approximation formulas  $\mathscr{W}_p(x)$  defined by (1.8).

It is easy to check that Nemes' approximation formula [7] satisfies

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \exp\left(\frac{210x^2+53}{360x(7x^2+2)}\right) \left(1 + O\left(\frac{1}{x^7}\right)\right) := N_2(x), \quad (4.1)$$

as  $x \rightarrow \infty$ . It was showed in [13] that as  $x \rightarrow \infty$ ,

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^3+24x/7-1/2}\right)^{x^2+53/210} \left(1 + O\left(\frac{1}{x^7}\right)\right) := C(x). \quad (4.2)$$

As shown in Introduction, we have

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x} + \frac{1}{810x^6}\right)^{x/2} \left(1 + O\left(\frac{1}{x^7}\right)\right) = W_1(x) \left(1 + O\left(\frac{1}{x^7}\right)\right).$$

For our family of approximation formulas  $\mathscr{W}_p(x)$  for gamma function, by (2.2) we see that

$$\lim_{x \rightarrow \infty} \frac{\ln \Gamma(x+1) - \ln \mathscr{W}_p(x)}{x^{-9}} = \lim_{x \rightarrow \infty} \frac{F_p(x)}{x^{-9}} = -\frac{11}{18900} \left(p - \frac{158}{315}\right),$$

which implies that

$$\Gamma(x+1) \sim \mathscr{W}_p(x) \left(1 + O\left(\frac{1}{x^9}\right)\right) \text{ if } p \neq p_0 = \frac{158}{315}.$$

And, we easily verify that

$$\lim_{x \rightarrow \infty} \frac{\ln \Gamma(x+1) - \ln \mathscr{W}_{p_0}(x)}{x^{-11}} = -\frac{276277}{392931000},$$

which indicates that

$$\Gamma(x+1) \sim \mathscr{W}_{p_0}(x) \left(1 + O\left(\frac{1}{x^{11}}\right)\right).$$

It thus can be seen that our new Windschitl type approximation formula  $\mathscr{W}_{p_0}(x)$  are the best among ones listed above, which can be also found from the following Table 1.

Table 1: Comparison among  $N_2$  (4.1),  $C$  (4.2),  $W_1$  (1.3),  $\mathscr{W}_0$  and  $\mathscr{W}_{p_0}$

$n$	$\frac{N_2(n)-n!}{n!}$	$\frac{C(n)-n!}{n!}$	$\frac{W_1(n)-n!}{n!}$	$\frac{\mathscr{W}_0(n)-n!}{n!}$	$\frac{\mathscr{W}_{p_0}(n)-n!}{n!}$
1	$1.11 \times 10^{-4}$	$1.340 \times 10^{-4}$	$1.83 \times 10^{-4}$	$2.41 \times 10^{-5}$	$3.74 \times 10^{-5}$
2	$1.90 \times 10^{-6}$	$2.22 \times 10^{-6}$	$2.67 \times 10^{-6}$	$2.31 \times 10^{-7}$	$1.08 \times 10^{-7}$
5	$4.35 \times 10^{-9}$	$4.96 \times 10^{-9}$	$5.74 \times 10^{-9}$	$1.25 \times 10^{-10}$	$1.13 \times 10^{-11}$
10	$3.61 \times 10^{-11}$	$4.09 \times 10^{-11}$	$4.71 \times 10^{-11}$	$2.79 \times 10^{-13}$	$6.59 \times 10^{-15}$
20	$2.86 \times 10^{-13}$	$3.24 \times 10^{-13}$	$3.73 \times 10^{-13}$	$5.63 \times 10^{-16}$	$3.38 \times 10^{-18}$
50	$4.71 \times 10^{-16}$	$5.33 \times 10^{-16}$	$6.13 \times 10^{-16}$	$1.49 \times 10^{-19}$	$1.44 \times 10^{-22}$
100	$3.68 \times 10^{-18}$	$4.17 \times 10^{-18}$	$4.79 \times 10^{-18}$	$2.92 \times 10^{-22}$	$7.01 \times 10^{-26}$

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