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COEFFICIENT PROBLEMS FOR UNIFIED STARLIKE AND CONVEX CLASSES OF m–FOLD SYMMETRIC BI–UNIVALENT FUNCTIONS

ZHENHAN TU AND LIANGPENG XIONG

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Abstract. Let $\mathscr{T}_m$ denote the class of m-fold symmetric bi-univalent functions in the open unit disk. We obtain the coefficient bounds of $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in a new general subclass $\mathscr{C}_m^{h,p}(\alpha)$ of $\mathscr{T}_m$, where h and p are in Carathéodory class of functions. We investigate the initial Taylor-Maclaurin coefficients estimate problems associated with $\mathscr{C}_m^{h,p}(\alpha)$ also. Our conclusion improves some earlier related results.

1. Introduction

Let $\mathscr{A}$ be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are normalized analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. We denote by $\mathscr{S}$ the class of all functions $f(z) \in \mathscr{A}$ which are univalent in $\mathbb{U}$.

Let $\mathscr{P}$ be the class of all analytic functions $p : \mathbb{U} \to \mathbb{C}$ satisfying $p(0) = 1$ and the real part $\Re p(z) > 0$ on $\mathbb{U}$.

The Koebe one-quarter theorem ensures that the image of $\mathbb{U}$ under every $f \in \mathscr{S}$ contains a disk of radius $\frac{1}{4}$ (see, Duren [11]). Thus, every function $f(z) \in \mathscr{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z, \quad z \in \mathbb{U}$$

and

$$f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f), \quad r_0(f) \geqslant \frac{1}{4}\right).$$

A function $f \in \mathscr{S}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let $\mathscr{T}$ denote the class of bi-univalent functions.

In 1967, Lewin [20] investigated the class $\mathscr{T}$ and showed that, for every function $f \in \mathscr{S}$ of the form (1), the second coefficient of f satisfies the estimate $|a_2| < 1.51$.

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Also, Brannan-Clunie [7] conjectured that $|a_2| \leqslant \sqrt{2}$ for $f \in \mathscr{T}$. Furthermore, Netanyahu [22] proved that $\max\{|a_2| : f \in \mathscr{T}\} = \frac{4}{3}$. In 1985, Kedzierawski [19] proved the Brannan-Clunie conjecture for bi-starlike functions and Tan [35] obtained the bound with $|a_2| < 1.485$, which is the best known estimate for functions in the class $\mathscr{T}$. In addition, Brannan-Taha [8] obtained estimates on the initial coefficients $|a_2|$ and $|a_3|$ for functions in the classes of bi-starlike functions of order β $(0 \leqslant \beta < 1)$ and bi-convex functions of order β $(0 \leqslant \beta < 1)$.

The study of bi-univalent functions was revived in recent years by Srivastava–Mishra–Gochhayat [24], and a considerably large number of sequels to Srivastava–Mishra–Gochhayat [24] have appeared in the literature since then (see, e.g., [3, 12, 15, 23, 25, 33, 36, 37, 38]). Recently, Çağlar-Deniz-Srivastava [10] studied the second Hankel determinant for certain subclasses of bi-univalent functions, Deniz [13] and Srivastava-Bansal [27] both extended and improved the results of Brannan–Taha [8] by the principle of subordination between analytic functions, and Srivastava-Gaboury-Ghanim [30] obtained the coefficient estimates for some general subclasses of analytic and bi-univalent functions.

Faber polynomials plays a considerable act in geometric function theory (see, e.g., [4, 6, 17]), which was introduced by Faber [16]. In particular, Srivastava-Eker-Ali [28] and Sakar-Güney [34] used the Faber polynomial expansion techniques to derive bounds for the general Taylor-Maclaurin coefficients $|a_n|$ of the functions in different subclasses of $\mathscr{T}$, and Srivastava-Eker-Hamidi-Jahangiri [31] studied the Faber polynomial coefficients for bi-univalent functions defined by the Tremblay fractional derivative operator.

Now, using the Faber polynomial expansion of functions $f \in \mathscr{A}$ of the form (1), the coefficients of its inverse map $g = f^{-1}$ can be expressed as (see, Airault-Bouali [4]):

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots) w^n, \tag{2}$$

where

$$\begin{aligned} K_{n-1}^{-n} = & \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{[2(-n-1)]!(n-3)!} a_2^{n-3} a_3 \\ & + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 + \frac{(-n)!}{[2(-n+2)]!(n-5)!} a_2^{n-5} [a_5 + (-n+2) a_3^2] \\ & + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} [a_6 + (-2n+5) a_3 a_4] + \sum_{j \geqslant 7} a_2^{n-1} V_j, \end{aligned}$$

in which V_j $(7 \leqslant j \leqslant n)$ is a homogeneous polynomial in the variables $a_2, a_3, \ldots, a_n$ (see, Airault-Ren [5]). In particular, the first three terms of K_{n-1}^{-n} are

$$\frac{1}{2} K_1^{-2} = -a_2, \quad \frac{1}{3} K_2^{-3} = 2a_2^2 - a_3, \quad \frac{1}{4} K_3^{-4} = -(5a_2^3 - 5a_2 a_3 + a_4). \tag{3}$$

Thus, the inverse function f^{-1} may analytically continued to $\mathbb{U}$ as follows:

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots. \tag{4}$$

For each $f \in \mathscr{S}$, the function

$$h(z) = \sqrt[m]{f(z^m)}, \;\; z \in \mathbb{U}, \;\; m \in \mathbb{N},$$

is univalent and maps the unit disk $\mathbb{U}$ into a region with m-fold symmetry. A function is said to be m-fold symmetric (see, e.g., [26, 29]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \;\; z \in \mathbb{U}, \;\; m \in \mathbb{N}. \tag{5}$$

We denote by $\mathscr{S}_m$ the class of m-fold symmetric univalent functions in $\mathbb{U}$. The functions in the class $\mathscr{S}$ are said to be one-fold symmetric.

Each bi-univalent function generates an m-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. The normalized form of f is given as in (5) and the series expansion for f^{-1}, which has been recently proven by Srivastava-Sivasubramanian-Sivakumar [26], is given as follows:

$$g(w) = w - a_{m+1} w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}] w^{2m+1} + \cdots, \tag{6}$$

where $f^{-1} = g$. We denote by $\mathscr{T}_m$ the class of m-fold symmetric bi-univalent functions in $\mathbb{U}$. Thus, when $m = 1$, the formula (6) coincides with the formula (4).

Here are some examples of m-fold symmetric bi-univalent functions (see, e.g., [26, 29])

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \quad \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)\right]^{\frac{1}{m}}, \quad [-\log(1-z^m)]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1-w^m}\right)^{\frac{1}{m}}, \quad \left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{\frac{1}{m}}, \quad \left(\frac{e^{w^m}-1}{e^{w^n}}\right)^{\frac{1}{m}}.$$

Srivastava-Gaboury-Ghanim [29] and Sivasubramanian-Sivakumar [32] deal with the coefficients problems for $f \in \mathscr{T}_m$. Bounds for the initial coefficients of different classes of m-fold symmetric bi-univalent functions were also investigated by the other authors (see, e.g., [14, 17, 26]).

DEFINITION 1. Let the function $h, \; p : \mathbb{U} \to \mathbb{C}$ be constrained that $h(0) = p(0) = 1$ and

$$\min\{\Re(h(z)), \; \Re((p(z))\} > 0 \quad (z \in \mathbb{U}).$$

For a function $f \in \mathscr{T}_m$, we say $f \in \mathscr{C}_m^{h,p}(\alpha)$ if the following conditions are satisfied:

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \in h(\mathbb{U}) \quad (z \in \mathbb{U}, \;\; 0 \leqslant \alpha \leqslant 1)$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\alpha}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\alpha} \in p(\mathbb{U}) \quad (w \in \mathbb{U}, \;\; 0 \leqslant \alpha \leqslant 1),$$

where $g(w) = f^{-1}(w)$.

REMARK 1. Obviously, $\mathscr{C}_m^{h,p}(\alpha)$ generalizes the class of m-fold symmetric bi-starlike and bi-convex functions. Specially, $\mathscr{C}_1^{h,p}(\alpha)$ was introduced and studied by Xiong-Liu [36] with $\mathscr{C}^{h,p}(\alpha)$. Some closely-related classes were investigated by Bulut [9] and Xu-Xiao-Srivastava [37] also.

If we let

$$h(z)=\sqrt[m]{\frac{1+(1-2\beta)z^m}{1-z^m}},\quad p(z)=\sqrt[m]{\frac{1-(1-2\beta)z^m}{1+z^m}},\quad (0\leqslant\beta<1,\ z\in\mathbb{U})$$

and

$$h(z)=\sqrt[m]{\left(\frac{1+z^m}{1-z^m}\right)^{\beta}},\quad p(z)=\sqrt[m]{\left(\frac{1-z^m}{1+z^m}\right)^{\beta}},\quad (0<\beta\leqslant 1,\ z\in\mathbb{U})$$

in Definition 1 respectively, then we have the definition 2 and definition 3 as follows.

DEFINITION 2. For a function $f\in\mathscr{T}_m$, we say $f\in\mathscr{C}_m^{\beta}(\alpha)$ if the following conditions are satisfied:

$$\Re\left\{\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}\right\}>\beta\quad(z\in\mathbb{U})$$

and

$$\Re\left\{\left(\frac{wg'(w)}{g(w)}\right)^{\alpha}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\alpha}\right\}>\beta\quad(w\in\mathbb{U}),$$

where $g(w)=f^{-1}(w)$, $0\leqslant\beta<1$, $0\leqslant\alpha\leqslant 1$.

REMARK 2. (i) If $m=1$ in Definition 2, then the class $\mathscr{C}_1^{\beta}(\alpha)$ was introduced and studied by Ali-Lee-Ravichandran-Supramaniama [3] with $\mathscr{C}^{\beta}(\alpha)$. Also the classes $\mathscr{C}^{\beta}(1)\equiv\mathscr{S}^{\beta}$ and $\mathscr{C}^{\beta}(0)\equiv\mathscr{K}^{\beta}$ were introduced by Brannan-Taha [8].

(ii) If $\alpha=0$ in Definition 2, then the class $\mathscr{C}_m^{\beta}(0)$ was introduced and studied by Sivasubramanian-Sivakumar [32] with $\mathscr{K}_m^{\beta}$.

(iii) If $\alpha=1$ in Definition 2, then the class $\mathscr{C}_m^{\beta}(1)$ was introduced and studied by Hamidi-Jahangiri [17] with $\mathscr{S}_m^{\beta}$.

DEFINITION 3. For a function $f\in\mathscr{T}_m$, we say $f\in\mathscr{C}_m^{*\beta}(\alpha)$ if the following conditions are satisfied:

$$\left|\arg\left[\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}\right]\right|<\frac{\beta\pi}{2},\quad z\in\mathbb{U}$$

and

$$\left|\arg\left[\left(\frac{wg'(w)}{g(w)}\right)^{\alpha}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\alpha}\right]\right|<\frac{\beta\pi}{2},\quad w\in\mathbb{U},$$

where $g(w)=f^{-1}(w)$, $0\leqslant\alpha\leqslant 1$, $0<\beta\leqslant 1$.

REMARK 3. (i) If $m=1$ in Definition 3, then the class was introduced and studied by Ali-Lee-Ravichandran-Supramaniama [3] with $\mathscr{C}^{*\beta}(\alpha)$. Also the classes $\mathscr{C}^{*\beta}(1)\equiv\mathscr{S}^{*}_{\beta}$ and $\mathscr{C}^{*\beta}(0)\equiv\mathscr{K}^{*}_{\beta}$ were introduced by Brannan-Taha [8].

(ii) If $\alpha=0$ or $\alpha=1$ in Definition 3, then the classes were introduced and studied by Sivasubramanian-Sivakumar [32] with $\mathscr{K}^{*\beta}_{m}$ or $\mathscr{S}^{*\beta}_{m}$, respectively.

Motivated and stimulated especially by the works of Srivastava–Mishra–Gochhayat [24], Xiong-Liu [36], Xu-Xiao-Srivastava [37] and Xu-Gui-Srivastava [38], we give the estimates on the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for the subclass $\mathscr{C}^{h,p}_{m}(\alpha)$ of m-fold symmetric bi-univalent functions in this paper. The corresponding results about the classes $\mathscr{C}^{\beta}_{m}(\alpha)$ and $\mathscr{C}^{*\beta}_{m}(\alpha)$ were given also. Our results generalize and improve some earlier related works.

2. Main results

We begin by finding the estimates on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in the class $\mathscr{C}^{h,p}_{m}(\alpha)$.

THEOREM 1. *Let the function $f(z)$ given by (5) be in the class $\mathscr{C}^{h,p}_{m}(\alpha)$. Then*

$$|a_{m+1}|\leqslant\min\left\{\sqrt{\frac{|h^{(2m)}(0)|+|p^{(2m)}(0)|}{(2m)!|L_m|}},\sqrt{\frac{|h^{(m)}(0)|^2+|p^{(m)}(0)|^2}{2(m!)^2[(1-\alpha)m^2+1]^2}}\right\} \tag{7}$$

and

$$|a_{2m+1}|\leqslant\min\left\{\left|\frac{A+L_m}{L_mB}\right|\frac{|h^{(2m)}(0)|}{(2m)!}+\left|\frac{A-L_m}{L_mB}\right|\frac{|p^{(2m)}(0)|}{(2m)!},\mathfrak{B}\right\}, \tag{8}$$

where $A=m(m+1)[2(1-\alpha)(2m+1)+2\alpha]$, $B=4(1-\alpha)m(2m+1)+4m\alpha$, $L_m=(m+1)[2(1-\alpha)m(2m+1)+2m\alpha]+\alpha(\alpha-1)m^2+2\alpha(1-\alpha)m^2(m+1)-2m\alpha-\alpha(1-\alpha)m^2(m+1)^2-2m(1-\alpha)(m+1)^2$ *and*

$$\mathfrak{B}=\frac{A}{B}\frac{|h^{(m)}(0)|^2+|p^{(m)}(0)|^2}{2(m!)^2[(1-\alpha)m^2+1]^2}+\frac{|h^{(2m)}(0)|+|p^{(2m)}(0)|}{(2m)!B}.$$

Proof. For the function $f\in\mathscr{C}^{h,p}_{m}(\alpha)$ and for the inverse map $g=f^{-1}$, we obtain

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}=h(z)\quad(z\in\mathbb{U}) \tag{9}$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\alpha}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\alpha}=p(w)\quad(w\in\mathbb{U}), \tag{10}$$

where h and p satisfy the hypotheses in Definition 1. Now suppose that the functions $h(z)$ and $p(w)$ have the following series expansions:

$$h(z)=1+h_mz^m+h_{2m}z^{2m}+\cdots \tag{11}$$

and

$$p(w) = 1 + p_m w + p_{2m} w^{2m} + \cdots, \tag{12}$$

respectively.

Following (5), we write:

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha} = 1 + T_m z^m + T_{2m} z^{2m} + \cdots,$$

where

$$T_m = [(1-\alpha)m^2 + 1]a_{m+1} \tag{13}$$

and

$$T_{2m} = [2(1-\alpha)m(2m+1) + 2m\alpha]a_{2m+1} + \Big[\frac{\alpha(\alpha-1)}{2}m^2 + \alpha(1-\alpha)m^2(1+m) \\ -m\alpha - \frac{\alpha(1-\alpha)}{2}m^2(m+1)^2 - m(1-\alpha)(m+1)^2\Big]a_{m+1}^2.$$

Also from (5) and (6), we get

$$\left(\frac{wg'(w)}{g(w)}\right)^{\alpha}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\alpha} = 1 + G_m w^m + G_{2m} w^{2m} + \cdots, \tag{14}$$

where

$$G_m = -[(1-\alpha)m^2 + 1]a_{m+1}$$

and

$$G_{2m} = \Big[m(m+1)[2(1-\alpha)(2m+1) + 2\alpha] + \frac{\alpha(\alpha-1)}{2}m^2 + \alpha(1-\alpha)m^2(m+1) \\ -m\alpha - \frac{\alpha(1-\alpha)}{2}m^2(m+1)^2 - m(1-\alpha)(m+1)^2\Big]a_{m+1}^2 \\ -[2(1-\alpha)m(2m+1) + 2m\alpha]a_{2m+1}.$$

Now, combining (9)-(14), we have

$$T_m = h_m, \tag{15}$$

$$T_{2m} = h_{2m}, \tag{16}$$

$$G_m = p_m, \tag{17}$$

$$G_{2m} = p_{2m}. \tag{18}$$

From (15) and (17), it follows

$$h_m = -p_m \tag{19}$$

and

$$2[(1-\alpha)m^2 + 1]^2 a_{m+1}^2 = h_m^2 + p_m^2. \tag{20}$$

Also from (16) and (18), we get

$$L_m a_{m+1}^2 = h_{2m} + p_{2m}, \tag{21}$$

where

$$\begin{aligned} L_m = &(m+1)[2(1-\alpha)m(2m+1)+2m\alpha]+\alpha(\alpha-1)m^2 \\ &+2\alpha(1-\alpha)m^2(m+1)-2m\alpha-\alpha(1-\alpha)m^2(m+1)^2-2m(1-\alpha)(m+1)^2. \end{aligned}$$

Therefore, from (20) and (21), we have

$$a_{m+1}^2 = \frac{h_m^2+p_m^2}{2[(1-\alpha)m^2+1]^2}, \tag{22}$$

and

$$a_{m+1}^2 = \frac{h_{2m}+p_{2m}}{L_m}, \tag{23}$$

which give the desired estimate on $|a_{m+1}|$ as asserted in (7).

Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (18) from (16), we get

$$4[(1-\alpha)m(2m+1)+m\alpha]a_{2m+1} - m(m+1)[2(1-\alpha)(2m+1)+2\alpha]a_{m+1}^2 = h_{2m}-p_{2m}. \tag{24}$$

By (22) and (24), it follows

$$\begin{aligned} a_{2m+1} = &\frac{m(m+1)[2(1-\alpha)(2m+1)+2\alpha]}{4[(1-\alpha)m(2m+1)+m\alpha]}\,\frac{h_m^2+p_m^2}{2[(1-\alpha)m^2+1]^2} \\ &+\frac{h_{2m}-p_{2m}}{4[(1-\alpha)m(2m+1)+m\alpha]}. \end{aligned} \tag{25}$$

On the other hand, from (23) and (24), it follows

$$L_m B a_{2m+1} = A(h_{2m}+p_{2m}) + L_m(h_{2m}-p_{2m}),$$

where

$$A = m(m+1)[2(1-\alpha)(2m+1)+2\alpha],$$

$$B = 4(1-\alpha)m(2m+1)+4m\alpha.$$

Thus we obtain

$$a_{2m+1} = \frac{A+L_m}{L_m B}h_{2m} + \frac{A-L_m}{L_m B}p_{2m},$$

which yields the desired estimate on $|a_{2m+1}|$ as asserted in (8). □

THEOREM 2. *Let the function $f(z)$ given by (5) be in the class $\mathscr{C}_m^{*\beta}(\alpha)$. Then*

$$|a_{m+1}| \leqslant \min\left\{\frac{2\beta}{m}\sqrt{\frac{1}{(2m)!|L_m|}}, \frac{2\beta}{m(m!)[(1-\alpha)m^2+1]}\right\}$$

and

$$|a_{2m+1}| \leqslant \min\left\{ \left(\left|\frac{A+L_m}{L_m B}\right| + \left|\frac{A-L_m}{L_m B}\right|\right) \frac{2\beta^2}{m^2(2m)!}, \mathfrak{B}_2 \right\},$$

where $A = m(m+1)[2(1-\alpha)(2m+1)+2\alpha]$, $B = 4(1-\alpha)m(2m+1)+4m\alpha$, $L_m = (m+1)[2(1-\alpha)m(2m+1)+2m\alpha]+\alpha(\alpha-1)m^2+2\alpha(1-\alpha)m^2(m+1) - 2m\alpha - \alpha(1-\alpha)m^2(m+1)^2 - 2m(1-\alpha)(m+1)^2$ *and*

$$\mathfrak{B}_2 = \frac{A}{B}\frac{4\beta^2}{m^2(m!)^2[(1-\alpha)m^2+1]^2} + \frac{4\beta^2}{m^2(2m)!B}.$$

Proof. Let

$$h(z) = \sqrt[m]{\left(\frac{1+z^m}{1-z^m}\right)^{\beta}} = 1 + 2\frac{\beta}{m}z^m + 2\frac{\beta^2}{m^2}z^{2m} + \cdots, \quad z \in \mathbb{U}$$

and

$$p(z) = \sqrt[m]{\left(\frac{1-z^m}{1+z^m}\right)^{\beta}} = 1 - 2\frac{\beta}{m}z^m + 2\frac{\beta^2}{m^2}z^{2m} + \cdots, \quad z \in \mathbb{U}$$

in Theorem 1. Then we have Theorem 2. □

THEOREM 3. *Let the function* $f(z)$ *given by* (5) *be in the class* $\mathscr{C}_m^{\beta}(\alpha)$. *Then*

$$|a_{m+1}| \leqslant \min\left\{ \frac{2}{m}\sqrt{\frac{(1-\beta)[m+(1-m)(1-\beta)]}{(2m)!|L_m|}}, \frac{2(1-\beta)}{m(m!)[(1-\alpha)m^2+1]} \right\}$$

and

$$|a_{2m+1}| \leqslant \min\left\{ \left(\left|\frac{A+L_m}{L_m B}\right| + \left|\frac{A-L_m}{L_m B}\right|\right) \frac{|2(1-\beta)[m+(1-m)(1-\beta)]|}{m^2(2m)!}, \mathfrak{B}_3 \right\},$$

where $A = m(m+1)[2(1-\alpha)(2m+1)+2\alpha]$, $B = 4(1-\alpha)m(2m+1)+4m\alpha$, $L_m = (m+1)[2(1-\alpha)m(2m+1)+2m\alpha]+\alpha(\alpha-1)m^2+2\alpha(1-\alpha)m^2(m+1) - 2m\alpha - \alpha(1-\alpha)m^2(m+1)^2 - 2m(1-\alpha)(m+1)^2$ *and*

$$\mathfrak{B}_3 = \frac{A}{B}\frac{4(1-\beta)^2}{m^2(m!)^2[(1-\alpha)m^2+1]^2} + \frac{4(1-\beta)|m+(1-m)(1-\beta)|}{m^2(2m)!B}.$$

Proof. Let

$$\begin{aligned} h(z) &= \sqrt[m]{\frac{1+(1-2\beta)z^m}{1-z^m}} \\ &= 1 + \frac{2}{m}(1-\beta)z^m + \left[\frac{2}{m}(1-\beta) + \frac{1-m}{2m^2}(2-2\beta)^2\right]z^{2m} + \cdots, \quad z \in \mathbb{U} \end{aligned}$$

and

$$p(z) = \sqrt[m]{\frac{1-(1-2\beta)z^m}{1+z^m}}$$
$$= 1 - \frac{2}{m}(1-\beta)z^m + \left[\frac{2}{m}(1-\beta) + \frac{1-m}{2m^2}(2-2\beta)^2\right]z^{2m} + \cdots, \quad z \in \mathbb{U}$$

in Theorem 1. Then we have Theorem 3. □

3. Corollaries and consequences

In this section, we give some corollaries by using the above theorems.

COROLLARY 1. *Let the function* $f(z)$ *given by* (5) *be in the class* $\mathscr{C}^{h,p}(\alpha)$*, then*

$$|a_2| \leqslant \min\left\{\sqrt{\frac{|h''(0)|+|p''(0)|}{2|\alpha^2-3\alpha+4|}}, \sqrt{\frac{|h'(0)|^2+|p'(0)|^2}{2(2-\alpha)^2}}\right\}$$

and

$$|a_3| \leqslant \min\left\{\frac{|\alpha^2-11\alpha+16||h''(0)|}{8(3-2\alpha)(\alpha^2-3\alpha+4)} + \frac{|\alpha^2+5\alpha-8||p''(0)|}{8(3-2\alpha)(\alpha^2-3\alpha+4))}, \mathfrak{B}_1\right\},$$

where

$$\mathfrak{B}_1 = \frac{|h'(0)|^2+|p'(0)|^2}{2(2-\alpha)^2} + \frac{|h''(0)|+|p''(0)|}{8(3-2\alpha)}.$$

Proof. By taking $m=1$ in Theorem 1, we get Corollary 1, which is an improvement of the estimates given by Xiong-Liu [36]. □

COROLLARY 2. *Let the function* $f(z)$ *given by* (5) *be in the class* $\mathscr{K}_m^{*\beta}$*. Then*

$$|a_{m+1}| \leqslant \min\left\{\frac{2\beta}{m}\sqrt{\frac{1}{2(2m)!m^2(m+1)}}, \frac{2\beta}{m(m!)(m^2+1)}\right\}$$

and

$$|a_{2m+1}| \leqslant \min\left\{\frac{\beta^2}{m^4(2m)!}, \frac{2(m+1)\beta^2}{m^2(m!)^2(m^2+1)^2} + \frac{\beta^2}{m^3(2m+1)!}\right\}.$$

Proof. By letting $\alpha=0$ in Theorem 2, we have Corollary 2. □

COROLLARY 3. *Let the function* $f(z)$ *given by* (5) *be in the class* $\mathscr{S}_m^{*\beta}$*. Then*

$$|a_{m+1}| \leqslant \min\left\{\frac{2\beta}{m}\sqrt{\frac{1}{(2m)!2m^2}}, \frac{2\beta}{m(m!)}\right\}$$

and

$$|a_{2m+1}| \leqslant \min\left\{\frac{(m+1)\beta^2}{m^4(2m)!}, \frac{2(m+1)\beta^2}{m^2(m!)^2} + \frac{\beta^2}{m^3(2m)!}\right\}.$$

Proof. Let $\alpha = 1$ in Theorem 2. Then we have Corollary 3. $\square$

COROLLARY 4. *Let the function $f(z)$ given by (5) be in the class $\mathscr{K}_m^{\beta}$. Then*

$$|a_{m+1}| \leqslant \min\left\{\frac{2}{m}\sqrt{\frac{(1-\beta)[m+(1-m)(1-\beta)]}{2(2m)!m^2(m+1)}}, \frac{2(1-\beta)}{m(m!)(m^2+1)}\right\}$$

and

$$|a_{2m+1}| \leqslant \min\left\{\frac{|(1-\beta)[m+(1-m)(1-\beta)]|}{m^4(2m)!}, \mathfrak{B}_4\right\},$$

where

$$\mathfrak{B}_4 = \frac{2(m+1)(1-\beta)^2}{m^2(m!)^2(m^2+1)^2} + \frac{(1-\beta)|m+(1-m)(1-\beta)|}{m^3(2m+1)!}.$$

Proof. Let $\alpha = 0$ in Theorem 3. Then we have Corollary 4. $\square$

COROLLARY 5. *Let the function $f(z)$ given by (5) be in the class $\mathscr{S}_m^{\beta}$. Then*

$$|a_{m+1}| \leqslant \min\left\{\frac{2}{m}\sqrt{\frac{(1-\beta)[m+(1-m)(1-\beta)]}{2(2m)!m^2}}, \frac{2(1-\beta)}{m(m!)}\right\}$$

and

$$|a_{2m+1}| \leqslant \min\left\{\frac{(m+1)|(1-\beta)[m+(1-m)(1-\beta)]|}{m^4(2m)!}, \mathfrak{B}_5\right\},$$

where

$$\mathfrak{B}_5 = \frac{2(m+1)(1-\beta)^2}{m^2(m!)^2} + \frac{(1-\beta)|m+(1-m)(1-\beta)|}{m^3(2m)!}.$$

Proof. Let $\alpha = 1$ in Theorem 3. Then we have Corollary 5. $\square$

REMARK 4. In the case of one fold symmetric functions, Corollary 1 to Corollary 5 improve the estimates obtained by Brannan-Taha [8]. Sharp estimates for the coefficients $|a_{m+1}|$, $|a_{2m+1}|$ and other coefficients of functions belonging to the classes investigated in this paper are yet open problems.

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REFERENCES

[1] M. ABRAMOWITZ, I. A. STEGUN (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 9th printing, Washington, 1970.

[2] W. BURNSIDE, *A rapidly convergent series for* $\log(N!)$, Messenger Math. **46** (1917), 157–159.

[3] R. M. ALI, S. K. LEE, V. RAVICHANDRAN, S. SUPRAMANIAMA, *Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions*, Appl. Math. Lett. **25** (2012), 344–351.

[4] H. AIRAULT, A. BOUALI, *Differential calculus on the Faber polynomials*, Bull. Sci. Math. **130** (2006), 179–222.

[5] H. AIRAULT, J. REN, *An algebra of differential operators and generating functions on the set of univalent functions*, Bull. Sci. Math. **126** (2002), 343–367.

[6] Ş. ALTINKAYA, S. YALÇIN, *Faber polynomial coefficient bounds for a subclass of bi-univalent functions*, Stud. Univ. Babeş-Bolyai Math. **61** (2016), 37–44.

[7] D. A. BRANNAN, J. G. CLUNIE, *Aspects of contemporary complex analysis*, in: Proc. NATO Advanced Study Institute, University of Durham, Durham, July 1–20, 1979.

[8] D. A. BRANNAN, T. S. TAHA, *On some classes of bi-univalent functions*, In: Mazhar SM, Hamoui A, Faour NS, editors, KFAS Proceedings Series, Vol. 3, Oxford, UK: Pergamon Press, 1988, pp. 53–60.

[9] S. BULUT, *Coefficient estimates for a new subclass of analytic and bi-univalent functions defined by convolution*, Math. Notes. Miskolc **17** (2016), 101–110.

[10] M. ÇAĞLAR, E. DENIZ, H. M. SRIVASTAVA, *Second Hankel determinant for certain subclasses of bi-univalent functions*, Turk. J. Math. **41** (2017), 694–706.

[11] P. L. DUREN, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Vol. 259, Berlin, Germany: Springer, 1983.

[12] E. DENIZ, M. ÇAĞLAR, H. ORHAN, *Second Hankel determinant for bi-starlike and bi-convex functions of order* β, Appl. Math. Comput. **271** (2015), 301–307.

[13] E. DENIZ, *Certain subclasses of bi-univalent functions satisfying subordinate conditions*, J. Class. Anal. **2** (2013), 49–60.

[14] S. S. EKER, *Coefficient bounds for subclasses of m-fold symmetric bi-univalent functions*, Turk. J. Math. **40** (2016), 641–646.

[15] B. A. FRASIN, M. K. AOUF, *New subclasses of bi-univalent functions*, Appl. Math. Lett. **24** (2011), 1569–1573.

[16] G. FABER, *Uber polynomische Entwickelungen*, Math. Ann. **57** (1903), 389–408.

[17] S. G. HAMIDI, J. M. JAHANGIRI, *Unpredictability of the coefficients of m-fold symmetric bi-starlike functions*, Int. J. Math. **25** (2014), 1–8.

[18] M. JAHANGIRI, *On the coefficients of powers of a class of Bazilevic functions*, Indian J. Pure Appl. Math. **17** (1986), 1140–1144.

[19] A. W. KEDZIERAWSKI, *Some remarks on bi-univalent functions*, Ann. Univ. Mariae Curie-Sklodowska (Sect A) **39** (1985), 77–81.

[20] M. LEWIN, *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc. **18** (1967), 63–68.

[21] G. MURUGUSUNDARAMOORTHY, T. JANANI, N. E. CHO, *Coefficient estimates of Mocanu-type meromorphic bi-univalent functions of complex order*, Proc. Jangjeon Math. Soc. **19** (2016), 691–700.

[22] E. NETANYAHU, *The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in* $|z| < 1$, Arch. Rational Mech. Anal. **32** (1969), 100–112.

[23] H. ORHAN, N. MAGESH, V. K. BALAJI, *Initial coefficient bounds for a general class of bi-univalent functions*, Filomat, **29** (2015), 1259–1267.

[24] H. M. SRIVASTAVA, A. K. MISHRA, P. GOCHHAYAT, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett. **23** (2010), 1188–1192.

[25] H. M. SRIVASTAVA, G. MURUGUSUNDARAMOORTHY, N. MAGESH, *Certain subclasses of bi-univalent functions associated with the Hohlov operator*, Global J. Math. Anal. **1** (2013), 67–73.

[26] H.M. SRIVASTAVA, S. SIVASUBRAMANIAN, R. SIVAKUMAR, *Initial coefficient bounds for a subclass of* m*-fold symmetric bi-univalent functions*, Tbilisi Math. J., **7** (2014), 1–10.

[27] H.M. SRIVASTAVA, D. BANSAL, *Coefficient estimates for a subclass of analytic and bi-univalent functions*, J. Egyptian Math. Soc., **23** (2015), 242–246.

[28] H.M. SRIVASTAVA, S.S. EKER, R.M. ALI, *Coefficient Bounds for a Certain Class of Analytic and Bi-Univalent Functions*, Filomat **29** (2015), 1839–1845.

[29] H.M. SRIVASTAVA, S. GABOURY, F. GHANIM, *Initial coefficient estimates for some subclasses of m-Fold symmetric bi-univalent functions*, Acta. Math. Sci., **36** (2016), 863–871.

[30] H.M. SRIVASTAVA, S. GABOURY, F. GHANIM, *Coefficient estimates for some general subclasses of analytic and bi-univalent functions*, Afrika Mat. **28** (2017), 693–706.

[31] H.M. SRIVASTAVA, S.S. EKER, S.G. HAMIDI, J.M. JAHANGIRI, *Faber polynomial coefficient estimates for bi-Univalent Functions defined by the tremblay fractional derivative operator*, Bull. Iran. Math. Soc. **44** (2018), 149–157.

[32] S. SIVASUBRAMANIAN, R. SIVAKUMAR, *Initial coefficient bound for m-fold symmetric bi-λ-convex functions*, J. Math. Inequal., **10** (2016), 783–791.

[33] S. SIVASUBRAMANIAN, R. SIVAKUMAR, T. BULBOACĂ, T.N. SHANMUGAM, *On the class of bi-univalent functions*, C. R. Acad. Sci. Paris., **352** (2014), 890–895.

[34] F.M. SAKAR, H.O. GÜNEY, *Coefficient bounds for a new subclass of analytic bi-close-to-convex functions by making use of Faber polynomial expansion*, Turk. J. Math., **41** (2017), 888–895.

[35] D.L. TAN, *Coefficient estimates for bi-univalent functions*, Chin. Ann. Math. (Ser A), **5** (1984), 559–568.

[36] L.P. XIONG, X.L. LIU, *Some extensions of coefficient problems for bi-univalent ma-minda starlike and convex functions*, Filomat, **29** (2015), 1645–1650.

[37] Q.H. XU, H.G. XIAO, H.M. SRIVASTAVA, *A certain general subclass of analytic and bi- univalent functions and associated coefficient estimate problems*, Appl. Math. Comput., **218** (2012), 11461–11465.

[38] Q.H. XU, Y.C. GUI, H.M. SRIVASTAVA, *Coefficient estimates for a certain subclass of analytic and bi-univalent functions*, Appl. Math. Lett., **25** (2012), 990–994.

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Zhenhan Tu
School of Mathematics and Statistics
Wuhan University
Wuhan, Hubei 430072, People's Republic of China
e-mail: zhhtu.math@whu.edu.cn

Corresponding author:
Liangpeng Xiong
School of Mathematics and Statistics
Wuhan University
Wuhan, Hubei 430072, People's Republic of China
e-mail: lpxiong2016@whu.edu.cn