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SOME NEW INEQUALITIES FOR K–FRAMES

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Abstract. In this paper, we establish some inequalities for dual K-frames from the point of view of operator theory. We also present a new inequality for Parseval K-frames associated with a scalar $\lambda \in [0,1]$, which is more general and covers one existing corresponding result recently obtained by F. Arabyani Neyshaburi et al.

1. Introduction

Frames (classical frames), appeared first in the early 1950's, offer us an important tool in dozens of fields because of their flexibility and redundancy, see [4, 5, 6, 7, 8, 14, 15] for more information on frame theory and its applications.

The concept of K-frames was introduced by L. Găvruţa in [10] to investigate the atomic systems associated with a linear and bounded operator K. A K-frame is a generalization of a frame, which allows an atomic decomposition of elements from the range of K and, in general, the range may not be closed. When K is an orthogonal projection, a K-frame turns to be an atom system for subspace which was proposed by H. G. Feichtinger and T. Werther in [9]. It should be remarked that in many ways K-frames behave completely differently from frames as shown in [1, 12, 16, 17], though the definition of a K-frame is similar to a frame in form.

We need to recall some notations and basic definitions.

Throughout this paper, we use $\mathscr{H}$, $\mathbb{R}$, and $\mathbb{J}$ respectively to denote a separable Hilbert space, the field of real numbers, and a countable index set. The notation $B(\mathscr{H})$ is reserved for the set of all linear bounded operators on $\mathscr{H}$.

DEFINITION 1.1. Suppose $K \in B(\mathscr{H})$. A sequence of vectors $\{f_j\}_{j\in\mathbb{J}}$ is said to be a K-frame for $\mathscr{H}$, if there exist two constants $0 < C \leqslant D < \infty$ such that

$$C\|K^*f\|^2 \leqslant \sum_{j\in\mathbb{J}} |\langle f, f_j\rangle|^2 \leqslant D\|f\|^2, \quad \forall f \in \mathscr{H}. \tag{1.1}$$

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If only the right-hand inequality of (1.1) is satisfied, then we call $\{f_j\}_{j\in\mathbb{J}}$ a Bessel sequence for $\mathscr{H}$. A K-frame $\{f_j\}_{j\in\mathbb{J}}$ for $\mathscr{H}$ is said to be Parseval, if

$$\|K^*f\|^2 = \sum_{j\in\mathbb{J}} |\langle f, f_j\rangle|^2, \quad \forall f \in \mathscr{H}. \tag{1.2}$$

Let $\{f_j\}_{j\in\mathbb{J}}$ be a Parseval K-frame for $\mathscr{H}$. Then it is easy to check that

$$KK^*f = \sum_{j\in\mathbb{J}} \langle f, f_j\rangle f_j, \quad \forall f \in \mathscr{H}. \tag{1.3}$$

DEFINITION 1.2. Suppose $K \in B(\mathscr{H})$ and $\{f_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathscr{H}$. A Bessel sequence $\{g_j\}_{j\in\mathbb{J}}$ for $\mathscr{H}$ is called a dual K-frame of $\{f_j\}_{j\in\mathbb{J}}$ if

$$Kf = \sum_{j\in\mathbb{J}} \langle f, g_j\rangle f_j, \quad \forall f \in \mathscr{H}. \tag{1.4}$$

Let $\{f_j\}_{j\in\mathbb{J}}$ be a Bessel sequence for $\mathscr{H}$. For any $\mathbb{I} \subset \mathbb{J}$, we let $\mathbb{I}^c = \mathbb{J}\backslash\mathbb{I}$ and define the operator $S_{\mathbb{I}} : \mathscr{H} \to \mathscr{H}$ by

$$S_{\mathbb{I}}f = \sum_{j\in\mathbb{I}} \langle f, f_j\rangle f_j, \tag{1.5}$$

which is positive, linear bounded and self-adjoint.

R. Balan et al. [3] found a surprising identity for Parseval frames when they devoted to the study of efficient algorithms for signal reconstruction. Moreover, in [3] the following inequality was given:

THEOREM 1.3. *If $\{f_j\}_{j\in\mathbb{J}}$ is a Parseval frame for $\mathscr{H}$, then for every $\mathbb{I} \subset \mathbb{J}$ and every $f \in \mathscr{H}$, we have*

$$\sum_{j\in\mathbb{I}} |\langle f, f_j\rangle|^2 + \Big\|\sum_{j\in\mathbb{I}^c} \langle f, f_j\rangle f_j\Big\|^2 = \sum_{j\in\mathbb{I}^c} |\langle f, f_j\rangle|^2 + \Big\|\sum_{j\in\mathbb{I}} \langle f, f_j\rangle f_j\Big\|^2 \geqslant \frac{3}{4}\|f\|^2. \tag{1.6}$$

Later on, P. Găvruţa in [11] extended inequality (1.6) to general frames and dual frames. Recently, F. Arabyani Neyshaburi et al. [2] obtained the following inequalities for Parseval K-frames on the basis of the work in [3, 11].

THEOREM 1.4. *Suppose $K \in B(\mathscr{H})$ and $\{f_j\}_{j\in\mathbb{J}}$ is a Parseval K-frame for $\mathscr{H}$. Then for every $\mathbb{I} \subset \mathbb{J}$ and every $f \in \mathscr{H}$,*

$$\begin{aligned}
&\mathrm{Re}\Big(\sum_{j\in\mathbb{I}^c} \langle f, f_j\rangle \overline{\langle KK^*f, f_j\rangle}\Big) + \Big\|\sum_{j\in\mathbb{I}} \langle f, f_j\rangle f_j\Big\|^2 \\
&\quad = \mathrm{Re}\Big(\sum_{j\in\mathbb{I}} \langle f, f_j\rangle \overline{\langle KK^*f, f_j\rangle}\Big) + \Big\|\sum_{j\in\mathbb{I}^c} \langle f, f_j\rangle f_j\Big\|^2 \geqslant \frac{3}{4}\|KK^*f\|^2.
\end{aligned} \tag{1.7}$$

THEOREM 1.5. *Suppose $K \in B(\mathscr{H})$ and $\{f_j\}_{j\in\mathbb{J}}$ is a Parseval K-frame for $\mathscr{H}$. Then for every $\mathbb{I} \subset \mathbb{J}$ and every $f \in \mathscr{H}$, we obtain*

$$\begin{aligned} \frac{1}{2}\|KK^*f\|^2 &\leqslant \Big\|\sum_{j\in\mathbb{I}}\langle f,f_j\rangle f_j\Big\|^2 + \Big\|\sum_{j\in\mathbb{I}^c}\langle f,f_j\rangle f_j\Big\|^2 \\ &\leqslant 2\|K\|^2\|K^*f\|^2 - \frac{1}{2}\|KK^*f\|^2. \end{aligned} \tag{1.8}$$

In this paper, we establish some inequalities for dual K-frames from the point of view of operator theory. We also present a new inequality for Parseval K-frames associated with a scalar $\lambda \in [0,1]$ and show that Theorem 1.4 is a particular case of our result when taking $\lambda = \frac{1}{2}$. Finally, we point out that the upper bound condition of the middle term in inequality (1.8) can be replaced with a better one under the condition that $S_{\mathbb{I}}$ commutes with $S_{\mathbb{I}^c}$.

2. Main results and their proofs

To prove the main results, we need the following lemmas.

LEMMA 2.1. (see [6]) *Suppose that $\mathscr{T} \in B(\mathscr{H})$ has closed range, then there exists a unique operator $\mathscr{T}^\dagger \in B(\mathscr{H})$, called the pseudo-inverse of $\mathscr{T}$, satisfying*

$$\mathscr{T}\mathscr{T}^\dagger\mathscr{T} = \mathscr{T}, \quad \mathscr{T}^\dagger\mathscr{T}\mathscr{T}^\dagger = \mathscr{T}^\dagger, \quad (\mathscr{T}^\dagger)^* = (\mathscr{T}^*)^\dagger.$$

LEMMA 2.2. (see [13]) *Suppose that $U,V,\mathscr{T} \in B(\mathscr{H})$, that $U+V=\mathscr{T}$, and that $\mathscr{T}$ has closed range. Then we have*

$$\mathscr{T}^*\mathscr{T}^\dagger U + V^*\mathscr{T}^\dagger V = V^*\mathscr{T}^\dagger\mathscr{T} + U^*\mathscr{T}^\dagger U.$$

LEMMA 2.3. *If $U,V,K \in B(\mathscr{H})$ satisfy $U+V=K$, then*

$$U^*U + \frac{1}{2}(V^*K + K^*V) = V^*V + \frac{1}{2}(U^*K + K^*U) \geqslant \frac{3}{4}K^*K.$$

Proof. We have

$$\begin{aligned} U^*U + \frac{1}{2}(V^*K + K^*V) &= U^*U + \frac{1}{2}\big((K^* - U^*)K + K^*(K-U)\big) \\ &= U^*U - \frac{1}{2}(U^*K + K^*U) + K^*K \end{aligned}$$

and

$$\begin{aligned} V^*V + \frac{1}{2}(U^*K + K^*U) &= (K^* - U^*)(K-U) + \frac{1}{2}(U^*K + K^*U) \\ &= U^*U - (K^*U + U^*K) + K^*K + \frac{1}{2}(U^*K + K^*U) \end{aligned}$$

$$= U^*U - \frac{1}{2}(U^*K + K^*U) + K^*K$$
$$= \left(U - \frac{1}{2}K\right)^* \left(U - \frac{1}{2}K\right) + \frac{3}{4}K^*K$$
$$\geqslant \frac{3}{4}K^*K. \quad \square$$

LEMMA 2.4. *If $U,V,K \in B(\mathscr{H})$ satisfy $U+V = KK^*$, then for any $\lambda \in [0,1]$ we have*

$$U^*U + \lambda(V^*KK^* + KK^*V) = V^*V + (1-\lambda)(U^*KK^* + KK^*U) + (2\lambda-1)(KK^*)^2$$
$$\geqslant (2\lambda - \lambda^2)(KK^*)^2. \tag{2.1}$$

Proof. For any $\lambda \in [0,1]$ we obtain

$$\begin{aligned} U^*U + \lambda(V^*KK^* + KK^*V) &= U^*U + \lambda\big((KK^* - U^*)KK^* + KK^*(KK^* - U)\big) \\ &= U^*U + \lambda\big((KK^*)^2 - U^*KK^* + (KK^*)^2 - KK^*U\big) \\ &= U^*U - \lambda(U^*KK^* + KK^*U) + 2\lambda(KK^*)^2. \end{aligned} \tag{2.2}$$

We also have

$$\begin{aligned} &V^*V + (1-\lambda)(U^*KK^* + KK^*U) + (2\lambda-1)(KK^*)^2 \\ &\quad = (KK^* - U^*)(KK^* - U) + (1-\lambda)(U^*KK^* + K^*KU) + (2\lambda-1)(KK^*)^2 \\ &\quad = U^*U - (U^*KK^* + KK^*U) + (1-\lambda)(U^*KK^* + KK^*U) + 2\lambda(KK^*)^2 \\ &\quad = U^*U - \lambda(U^*KK^* + KK^*U) + 2\lambda(KK^*)^2 \\ &\quad = (U - \lambda KK^*)^*(U - \lambda KK^*) + (2\lambda - \lambda^2)(KK^*)^2 \\ &\quad \geqslant (2\lambda - \lambda^2)(KK^*)^2. \end{aligned} \tag{2.3}$$

Combination of (2.2) and (2.3) yields (2.1). $\square$

We first establish some inequalities for dual K-frames.

THEOREM 2.5. *Suppose $K \in B(\mathscr{H})$. Let $\{f_j\}_{j\in\mathbb{J}}$ be a K-frame for $\mathscr{H}$ and $\{g_j\}_{j\in\mathbb{J}}$ be a dual K-frame of $\{f_j\}_{j\in\mathbb{J}}$. Then for every $\mathbb{I} \subset \mathbb{J}$ and every $f \in \mathscr{H}$, we have*

$$\begin{aligned} \frac{3}{4}\|Kf\|^2 &\leqslant \operatorname{Re}\sum_{j\in\mathbb{I}} \langle f, g_j\rangle \overline{\langle Kf, f_j\rangle} + \Big\|\sum_{j\in\mathbb{I}^c} \langle f, g_j\rangle f_j\Big\|^2 \\ &= \operatorname{Re}\sum_{j\in\mathbb{I}^c} \langle f, g_j\rangle \overline{\langle Kf, f_j\rangle} + \Big\|\sum_{j\in\mathbb{I}} \langle f, g_j\rangle f_j\Big\|^2 \\ &\leqslant \frac{3\|K\|^2 + \|F_{\mathbb{I}} - F_{\mathbb{I}^c}\|^2}{4}\|f\|^2, \end{aligned} \tag{2.4}$$

where the operator $F_{\mathbb{I}} : \mathscr{H} \to \mathscr{H}$ is defined by $F_{\mathbb{I}} = \sum_{j\in\mathbb{I}} \langle f, g_j\rangle f_j$.

Proof. For any $\mathbb{I} \subset \mathbb{J}$ we have $F_{\mathbb{I}} + F_{\mathbb{I}^c} = K$. Noting that

$$\langle K^* F_{\mathbb{I}} f, f\rangle = \langle F_{\mathbb{I}} f, Kf\rangle = \sum_{j\in\mathbb{I}} \langle f, g_j\rangle\langle f_j, Kf\rangle = \sum_{j\in\mathbb{I}} \langle f, g_j\rangle\overline{\langle Kf, f_j\rangle}$$

for each $f \in \mathscr{H}$, by Lemma 2.3 we obtain

$$\begin{aligned}
&\mathrm{Re}\sum_{j\in\mathbb{I}} \langle f, g_j\rangle\overline{\langle Kf, f_j\rangle} + \Big\|\sum_{j\in\mathbb{I}^c} \langle f, g_j\rangle f_j\Big\|^2 \\
&= \frac{1}{2}\big(\langle F_{\mathbb{I}}^* Kf, f\rangle + \langle K^* F_{\mathbb{I}} f, f\rangle\big) + \|F_{\mathbb{I}^c} f\|^2 \\
&= \frac{1}{2}\big(\langle F_{\mathbb{I}^c}^* Kf, f\rangle + \langle K^* F_{\mathbb{I}^c} f, f\rangle\big) + \|F_{\mathbb{I}} f\|^2 \\
&= \mathrm{Re}\sum_{j\in\mathbb{I}^c} \langle f, g_j\rangle\overline{\langle Kf, f_j\rangle} + \Big\|\sum_{j\in\mathbb{I}} \langle f, g_j\rangle f_j\Big\|^2 \\
&\geqslant \frac{3}{4}\|Kf\|^2.
\end{aligned}$$

We now prove the right-hand inequality of (2.4). For any $f \in \mathscr{H}$ we get

$$\begin{aligned}
&\mathrm{Re}\sum_{j\in\mathbb{I}^c} \langle f, g_j\rangle\overline{\langle Kf, f_j\rangle} + \Big\|\sum_{j\in\mathbb{I}} \langle f, g_j\rangle f_j\Big\|^2 \\
&= \mathrm{Re}\langle F_{\mathbb{I}^c}^* Kf, f\rangle + \langle F_{\mathbb{I}} f, F_{\mathbb{I}} f\rangle \\
&= \mathrm{Re}\langle Kf, (K - F_{\mathbb{I}})f\rangle + \langle F_{\mathbb{I}} f, F_{\mathbb{I}} f\rangle \\
&= \mathrm{Re}\big(\langle Kf, Kf\rangle - \langle Kf, F_{\mathbb{I}} f\rangle\big) + \langle F_{\mathbb{I}} f, F_{\mathbb{I}} f\rangle \\
&= \langle Kf, Kf\rangle - \mathrm{Re}\langle Kf, F_{\mathbb{I}} f\rangle + \langle F_{\mathbb{I}} f, F_{\mathbb{I}} f\rangle \\
&= \langle Kf, Kf\rangle - \mathrm{Re}\big(\langle (K - F_{\mathbb{I}})f, F_{\mathbb{I}} f\rangle\big) \\
&= \langle Kf, Kf\rangle - \mathrm{Re}\langle F_{\mathbb{I}^c} f, F_{\mathbb{I}} f\rangle \\
&= \langle Kf, Kf\rangle - \frac{1}{2}\langle F_{\mathbb{I}} f, F_{\mathbb{I}^c} f\rangle - \frac{1}{2}\langle F_{\mathbb{I}^c} f, F_{\mathbb{I}} f\rangle \\
&= \frac{3}{4}\|Kf\|^2 + \frac{1}{4}\langle F_{\mathbb{I}} f + F_{\mathbb{I}^c} f, F_{\mathbb{I}} f + F_{\mathbb{I}^c} f\rangle - \frac{1}{2}\langle F_{\mathbb{I}} f, F_{\mathbb{I}^c} f\rangle - \frac{1}{2}\langle F_{\mathbb{I}^c} f, F_{\mathbb{I}} f\rangle \\
&= \frac{3}{4}\|Kf\|^2 + \frac{1}{4}\langle (F_{\mathbb{I}} - F_{\mathbb{I}^c})f, (F_{\mathbb{I}} - F_{\mathbb{I}^c})f\rangle \\
&\leqslant \frac{3}{4}\|K\|^2\|f\|^2 + \frac{1}{4}\|F_{\mathbb{I}} - F_{\mathbb{I}^c}\|^2\|f\|^2 = \frac{3\|K\|^2 + \|F_{\mathbb{I}} - F_{\mathbb{I}^c}\|^2}{4}\|f\|^2.
\end{aligned}$$

This completes the proof. □

Let $\{\upsilon_j\}_{j\in\mathbb{J}}$ be a bounded sequence of complex numbers. If in Lemma 2.3 we take

$$Uf = \sum_{j\in\mathbb{J}} \upsilon_j\langle f, g_j\rangle f_j, \quad Vf = \sum_{j\in\mathbb{J}} (1 - \upsilon_j)\langle f, g_j\rangle f_j,$$

then we have

THEOREM 2.6. *Suppose $K \in B(\mathscr{H})$. Let $\{f_j\}_{j\in\mathbb{J}}$ be a K-frame for $\mathscr{H}$ and $\{g_j\}_{j\in\mathbb{J}}$ be a dual K-frame of $\{f_j\}_{j\in\mathbb{J}}$. Then for all bounded sequence $\{\upsilon_j\}_{j\in\mathbb{J}}$ and all $f \in \mathscr{H}$, we have*

$$\begin{aligned} &\mathrm{Re}\sum_{j\in\mathbb{J}} \upsilon_j \langle f, g_j\rangle \overline{\langle Kf, f_j\rangle} + \Big\|\sum_{j\in\mathbb{J}}(1-\upsilon_j)\langle f, g_j\rangle f_j\Big\|^2 \\ &\quad = \mathrm{Re}\sum_{j\in\mathbb{J}}(1-\upsilon_j)\langle f, g_j\rangle \overline{\langle Kf, f_j\rangle} + \Big\|\sum_{j\in\mathbb{J}}\upsilon_j\langle f, g_j\rangle f_j\Big\|^2 \geqslant \frac{3}{4}\|Kf\|^2. \end{aligned}$$

Proof. The result follows immediately from the left-hand inequality of (2.4) if we take $\mathbb{I} \subset \mathbb{J}$ and

$$\upsilon_j = \begin{cases} 1, & j \in \mathbb{I}, \\ 0, & j \in \mathbb{I}^c. \end{cases} \quad \square$$

THEOREM 2.7. *Suppose that $K \in B(\mathscr{H})$ is positive and that it has closed range. Let $\{f_j\}_{j\in\mathbb{J}}$ be a K-frame for $\mathscr{H}$ and $\{g_j\}_{j\in\mathbb{J}}$ be a dual K-frame of $\{f_j\}_{j\in\mathbb{J}}$. Then for every $\mathbb{I} \subset \mathbb{J}$ and every $f \in \mathscr{H}$, we have*

$$\begin{aligned} &\mathrm{Re}\sum_{j\in\mathbb{I}}\langle f, g_j\rangle\langle f_j, K^\dagger K f\rangle + \Big\langle \sum_{j\in\mathbb{I}^c}\langle f, g_j\rangle K^\dagger f_j, \sum_{j\in\mathbb{I}^c}\langle f, g_j\rangle f_j \Big\rangle \\ &\quad = \mathrm{Re}\sum_{j\in\mathbb{I}^c}\langle K^\dagger K f, f_j\rangle\langle g_j, f\rangle + \Big\langle \sum_{j\in\mathbb{I}}\langle f, g_j\rangle K^\dagger f_j, \sum_{j\in\mathbb{I}^c}\langle f, g_j\rangle f_j \Big\rangle \\ &\quad \geqslant \frac{3}{4}\|K^{\frac{1}{2}} f\|^2, \end{aligned} \tag{2.5}$$

where $K^\dagger$ denotes the pseudo-inverse of K.

Proof. Since K is positive, it is self-adjoint and thus by Lemma 2.1, $(K^\dagger)^* = (K^*)^\dagger = K^\dagger$. Hence, $\langle K^\dagger F_{\mathbb{I}} f, F_{\mathbb{I}} f\rangle, \langle K^\dagger F_{\mathbb{I}^c} f, F_{\mathbb{I}^c} f\rangle \in \mathbb{R}$ for each $f \in \mathscr{H}$. By Lemma 2.2 we get

$$\begin{aligned} &\mathrm{Re}\sum_{j\in\mathbb{I}}\langle f, g_j\rangle\langle f_j, K^\dagger K f\rangle + \Big\langle \sum_{j\in\mathbb{I}^c}\langle f, g_j\rangle K^\dagger f_j, \sum_{j\in\mathbb{I}^c}\langle f, g_j\rangle f_j \Big\rangle \\ &\quad = \mathrm{Re}\langle F_{\mathbb{I}} f, K^\dagger K f\rangle + \langle K^\dagger F_{\mathbb{I}^c} f, F_{\mathbb{I}^c} f\rangle \\ &\quad = \mathrm{Re}\langle K K^\dagger F_{\mathbb{I}} f, f\rangle + \langle F_{\mathbb{I}^c}^* K^\dagger F_{\mathbb{I}^c} f, f\rangle \\ &\quad = \mathrm{Re}\langle (K K^\dagger F_{\mathbb{I}} + F_{\mathbb{I}^c}^* K^\dagger F_{\mathbb{I}^c}) f, f\rangle \\ &\quad = \mathrm{Re}\langle (F_{\mathbb{I}^c}^* K^\dagger K + F_{\mathbb{I}}^* K^\dagger F_{\mathbb{I}}) f, f\rangle \\ &\quad = \mathrm{Re}\big(\langle F_{\mathbb{I}^c}^* K^\dagger K f, f\rangle + \langle F_{\mathbb{I}}^* K^\dagger F_{\mathbb{I}} f, f\rangle\big) \\ &\quad = \mathrm{Re}\big(\langle K^\dagger K f, F_{\mathbb{I}^c} f\rangle + \langle K^\dagger F_{\mathbb{I}} f, F_{\mathbb{I}} f\rangle\big) \\ &\quad = \mathrm{Re}\langle K^\dagger K f, F_{\mathbb{I}^c} f\rangle + \mathrm{Re}\langle K^\dagger F_{\mathbb{I}} f, F_{\mathbb{I}} f\rangle \end{aligned}$$

$$= \mathrm{Re}\overline{\langle F_{\mathbb{I}^c} f, K^\dagger K f\rangle} + \langle K^\dagger F_{\mathbb{I}} f, F_{\mathbb{I}} f\rangle$$
$$= \mathrm{Re}\sum_{j\in\mathbb{I}^c}\langle K^\dagger K f, f_j\rangle\langle g_j, f\rangle + \left\langle \sum_{j\in\mathbb{I}}\langle f, g_j\rangle K^\dagger f_j, \sum_{j\in\mathbb{I}^c}\langle f, g_j\rangle f_j\right\rangle.$$

Now, combining Lemmas 2.1 and 2.2 we have

$$\begin{aligned}
&\mathrm{Re}\sum_{j\in\mathbb{I}}\langle f, g_j\rangle\langle f_j, K^\dagger K f\rangle + \left\langle \sum_{j\in\mathbb{I}^c}\langle f, g_j\rangle K^\dagger f_j, \sum_{j\in\mathbb{I}^c}\langle f, g_j\rangle f_j\right\rangle\\
&\quad = \mathrm{Re}\langle (KK^\dagger F_{\mathbb{I}} + F_{\mathbb{I}^c}^* K^\dagger F_{\mathbb{I}^c}) f, f\rangle\\
&\quad = \mathrm{Re}\langle (KK^\dagger (K - F_{\mathbb{I}^c}) + F_{\mathbb{I}^c}^* K^\dagger F_{\mathbb{I}^c}) f, f\rangle\\
&\quad = \langle Kf, f\rangle - \mathrm{Re}\langle KK^\dagger F_{\mathbb{I}^c} f, f\rangle + \langle F_{\mathbb{I}^c}^* K^\dagger F_{\mathbb{I}^c} f, f\rangle\\
&\quad = \langle K^{\frac{1}{2}} f, K^{\frac{1}{2}} f\rangle - \mathrm{Re}\langle K^{\frac{1}{2}} K^{\frac{1}{2}} K^\dagger F_{\mathbb{I}^c} f, f\rangle + \langle (K^{\frac{1}{2}} K^\dagger F_{\mathbb{I}^c})^* (K^{\frac{1}{2}} K^\dagger F_{\mathbb{I}^c}) f, f\rangle\\
&\quad = \frac{3}{4}\|K^{\frac{1}{2}} f\|^2 + \left\langle \frac{1}{2}K^{\frac{1}{2}} f - K^{\frac{1}{2}} K^\dagger S_{\mathbb{I}^c} f, \frac{1}{2}K^{\frac{1}{2}} f - K^{\frac{1}{2}} K^\dagger F_{\mathbb{I}^c} f\right\rangle\\
&\quad \geqslant \frac{3}{4}\|K^{\frac{1}{2}} f\|^2
\end{aligned}$$

for every $f \in \mathscr{H}$. This completes the proof. □

In the following we give an inequality for Parseval K-frames, where a scalar $\lambda \in [0,1]$ is involved.

THEOREM 2.8. *Suppose $K \in B(\mathscr{H})$ and $\{f_j\}_{j\in\mathbb{J}}$ is a Parseval K-frame for $\mathscr{H}$. Then for any $\lambda \in [0,1]$, for all $\mathbb{I} \subset \mathbb{J}$ and all $f \in \mathscr{H}$, we have*

$$\begin{aligned}
&2\lambda\left(\mathrm{Re}\sum_{j\in\mathbb{I}^c}\langle f, f_j\rangle\overline{\langle KK^* f, f_j\rangle}\right) + \left\|\sum_{j\in\mathbb{I}}\langle f, g_j\rangle f_j\right\|^2\\
&\quad = 2(1-\lambda)\left(\mathrm{Re}\sum_{j\in\mathbb{I}}\langle f, g_j\rangle\overline{\langle KK^* f, f_j\rangle}\right) + \left\|\sum_{j\in\mathbb{I}^c}\langle f, f_j\rangle f_j\right\|^2 + (2\lambda - 1)\|KK^* f\|^2\\
&\quad \geqslant (2\lambda - \lambda^2)\|KK^* f\|^2.
\end{aligned}$$

Proof. For every $\mathbb{I} \subset \mathbb{J}$, by (1.3) we have $S_{\mathbb{I}} + S_{\mathbb{I}^c} = KK^*$. Taking $S_{\mathbb{I}}$ and $S_{\mathbb{I}^c}$ instead of U and V respectively in Lemma 2.4 yields

$$\begin{aligned}
&2\lambda\left(\mathrm{Re}\sum_{j\in\mathbb{I}^c}\langle f, f_j\rangle\overline{\langle KK^* f, f_j\rangle}\right) + \left\|\sum_{j\in\mathbb{I}}\langle f, g_j\rangle f_j\right\|^2\\
&\quad = \lambda\left(\langle S_{\mathbb{I}^c} KK^* f, f\rangle + \langle S_{\mathbb{I}^c} f, KK^* f\rangle\right) + \|S_{\mathbb{I}} f\|^2\\
&\quad = \lambda\left(\langle S_{\mathbb{I}^c} KK^* f, f\rangle + \langle KK^* S_{\mathbb{I}^c} f, f\rangle\right) + \langle S_{\mathbb{I}} S_{\mathbb{I}} f, f\rangle\\
&\quad = \langle S_{\mathbb{I}^c} S_{\mathbb{I}^c} f, f\rangle + (1-\lambda)\left(\langle KK^* S_{\mathbb{I}} f, f\rangle + \langle S_{\mathbb{I}} KK^* f, f\rangle\right) + (2\lambda - 1)\|KK^* f\|^2
\end{aligned}$$

$$= 2(1-\lambda)\left(\mathrm{Re}\sum_{j\in\mathbb{I}}\langle f,g_j\rangle\overline{\langle KK^*f,f_j\rangle}\right)+\left\|\sum_{j\in\mathbb{I}^c}\langle f,f_j\rangle f_j\right\|^2+(2\lambda-1)\|KK^*f\|^2$$
$$\geqslant (2\lambda-\lambda^2)\|KK^*f\|^2$$

for every $f\in\mathscr{H}$ and the proof is finished. □

REMARK 2.9. If taking $\lambda=\frac{1}{2}$ in Theorem 2.8, then we get the inequality in Theorem 1.4.

At the end of the paper, we make a remark on the right-hand inequality of (1.8) shown in Theorem 1.5.

Suppose $K\in B(\mathscr{H})$ and $\{f_j\}_{j\in\mathbb{J}}$ is a Parseval K-frame for $\mathscr{H}$. If $S_{\mathbb{I}}$ commutes with $S_{\mathbb{I}^c}$ for every $\mathbb{I}\subset\mathbb{J}$, then $S_{\mathbb{I}^c}S_{\mathbb{I}}\geqslant 0$ and

$$0\leqslant S_{\mathbb{I}}S_{\mathbb{I}^c}=S_{\mathbb{I}}(KK^*-S_{\mathbb{I}})=S_{\mathbb{I}}KK^*-S_{\mathbb{I}}^2.$$

Therefore,

$$\begin{aligned}S_{\mathbb{I}}^2+S_{\mathbb{I}^c}^2&=S_{\mathbb{I}}^2+(KK^*-S_{\mathbb{I}})^2\\&=S_{\mathbb{I}}^2+(KK^*)^2-KK^*S_{\mathbb{I}}-S_{\mathbb{I}}KK^*+S_{\mathbb{I}}^2\\&=(KK^*)^2+(S_{\mathbb{I}}^2-S_{\mathbb{I}}KK^*)+(S_{\mathbb{I}}^2-KK^*S_{\mathbb{I}})\\&=(KK^*)^2-(S_{\mathbb{I}}KK^*-S_{\mathbb{I}}^2)-S_{\mathbb{I}^c}S_{\mathbb{I}}\leqslant(KK^*)^2.\end{aligned}$$

Hence for each $f\in\mathscr{H}$ we have

$$\begin{aligned}\left\|\sum_{j\in\mathbb{I}}\langle f,f_j\rangle f_j\right\|^2+\left\|\sum_{j\in\mathbb{I}^c}\langle f,f_j\rangle f_j\right\|^2&=\langle S_{\mathbb{I}}f,S_{\mathbb{I}}f\rangle+\langle S_{\mathbb{I}^c}f,S_{\mathbb{I}^c}f\rangle\\&=\langle(S_{\mathbb{I}}^2+S_{\mathbb{I}^c}^2)f,f\rangle\\&\leqslant\langle(KK^*)^2f,f\rangle=\|KK^*f\|^2.\end{aligned}$$

For every $f\in\mathscr{H}$, since

$$\begin{aligned}\|KK^*f\|^2-\left(2\|K\|^2\|K^*f\|^2-\frac{1}{2}\|KK^*f\|^2\right)&=\frac{3}{2}\|KK^*f\|^2-2\|K\|^2\|K^*f\|^2\\&\leqslant\frac{3}{2}\|K\|^2\|K^*f\|^2-2\|K\|^2\|K^*f\|^2\\&=-\frac{1}{2}\|K\|^2\|K^*f\|^2\leqslant 0,\end{aligned}$$

showing that the upper bound condition for the middle term of (1.8) obtained in this case is better than the original one.

REFERENCES

[1] F. ARABYANI NEYSHABURI, A. AREFIJAMAAL, *Some constructions of K-frames and their duals*, Rocky Mountain J. Math. **47** (2017), 1749–1764.

[2] F. ARABYANI NEYSHABURI, GH. MOHAJERI MINAEI, E. ANJIDANI, *On some equalities and inequalities for K-frames*, www.arxiv.org, math.FA/1705.10155v1.

[3] R. BALAN, P. G. CASAZZA, D. EDIDIN, G. KUTYNIOK, *A new identity for Parseval frames*, Proc. Amer. Math. Soc. **135** (2007), 1007–1015.

[4] J. J. BENEDETTO, A. M. POWELL, O. YILMAZ, *Sigma-Delta $(\Sigma\Delta)$ quantization and finite frames*, IEEE Trans. Inform. Theory **52** (2006), 1990–2005.

[5] P. G. CASAZZA, *The art of frame theory*, Taiwanese J. Math. **4** (2000), 129–201.

[6] O. CHRISTENSEN, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003.

[7] I. DAUBECHIES, A. GROSSMANN, Y. MEYER, *Painless nonorthogonal expansions*, J. Math. Phys. **27** (1986), 1271–1283.

[8] R. J. DUFFIN, A. C. SCHAEFFER, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341–366.

[9] H. G. FEICHTINGER, T. WERTHER, *Atomic systems for subspaces*, in: L. Zayed (Ed.), Proceedings SampTA 2001, Orlando, FL, 2001, pp. 163–165.

[10] L. GĂVRUŢA, *Frames for operators*, Appl. Comput. Harmon. Anal. **32** (2012), 139–144.

[11] P. GĂVRUŢA, *On some identities and inequalities for frames in Hilbert spaces*, J. Math. Anal. Appl. **321** (2006), 469–478.

[12] X. X. GUO, *Canonical dual K-Bessel sequences and dual K-Bessel generators for unitary systems of Hilbert spaces*, J. Math. Anal. Appl. **444** (2016), 598–609.

[13] J. Z. LI, Y. C. ZHU, *Some equalities and inequalities for g-Bessel sequences in Hilbert spaces*, Appl. Math. Lett. **25** (2012), 1601–1607.

[14] T. STROHMER, R. HEATH, *Grassmannian frames with applications to coding and communication*, Appl. Comput. Harmon. Anal. **14** (2003), 257–275.

[15] W. SUN, *Asymptotic properties of Gabor frame operators as sampling density tends to infinity*, J. Funct. Anal. **258** (2010), 913–932.

[16] Z. Q. XIANG, Y. M. LI, *Frame sequences and dual frames for operators*, Science Asia **42** (2016), 222–230.

[17] X. C. XIAO, Y. C. ZHU, L. GĂVRUŢA, *Some properties of K-frames in Hilbert spaces*, Results Math. **63** (2013), 1243–1255.

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