

A FAMILY OF MEROMORPHIC FUNCTIONS INVOLVING GENERALIZED MITTAG–LEFFLER FUNCTION

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Abstract. We introduce a new family of meromorphic functions defined by the second order differential subordination involving generalized Mittag-Leffler function. Certain convolution properties of the family are discussed.

1. Introduction

The Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad (z, \alpha, \beta \in \mathbb{C}; \operatorname{Re}(\alpha) > 0), \quad (1.1)$$

where $\Gamma(\cdot)$ is classical gamma function. It was first introduced in 1903 by Swedish mathematician Mittag-Leffler for $\beta = 1$.

The Mittag-Leffler function $E_{\alpha,\beta}(z)$, as well as its various further generalizations, arise naturally in the solution of fractional differential equations and fractional integro-differential equations which are associated with (for example) the kinetic equation, random walks, Lévy flights, super-diffusive transport problems and in the study of complex systems. In particular, it is an explicit formula for the resolvent of Riemann-Liouville fractional integrals by Hille and Tamarkin. The more properties and applications of the Mittag-Leffler functions, together with their generalizations, can be found in a number of recent works [1] to [3], and [8] to [12].

In [12], Yin and Huang considered a generalized Mittag-Leffler function as follows:

$$E_{\alpha,\beta,q}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(\alpha n + \beta)} \quad (z, \alpha, \beta \in \mathbb{C}, q \in (0, +\infty); \operatorname{Re}(\alpha) > 0), \quad (1.2)$$

where $\Gamma_q(x)$ is classical q-gamma function defined by

$$\Gamma_q(x) = \frac{q!q^x}{x(x+1)\cdots(x+q)}.$$

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It is known that $\lim_{q \rightarrow +\infty} \Gamma_q(x) = \Gamma(x)$. The logarithmic derivative of q -gamma function

$$\psi_q(x) = \frac{d}{dx} \log \Gamma_q(x) = \frac{\Gamma'_q(x)}{\Gamma_q(x)}$$

is known as generalized digamma function. Its derivatives $\psi_q^{(n)}(x)$ are known as the generalized polygamma function. These functions have the following representations of series

$$\psi_q^{(n)}(x) = \sum_{k=0}^q \frac{(-1)^{n-1} n!}{(x+k)^{n+1}}. \tag{1.3}$$

Let $\Sigma(p)$ denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p} z^{n-p} \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}), \tag{1.4}$$

which are analytic in the punctured open unit disk $U_0 = \{z \in \mathbb{C} : 0 < |z| < 1\}$. The class $\Sigma(p)$ is closed under the Hadamard product (or convolution)

$$(f_1 * f_2)(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p,1} a_{n-p,2} z^{n-p} = (f_2 * f_1)(z),$$

where

$$f_j(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p,j} z^{n-p} \in \Sigma(p) \quad (j = 1, 2).$$

Now, for $f \in \Sigma(p)$, we consider the following operator $L_{\alpha,\beta}^q : \Sigma(p) \rightarrow \Sigma(p)$ associated with the generalized Mittag-Leffler function $E_{\alpha,\beta,q}(z)$:

$$\begin{aligned} L_{\alpha,\beta}^q f(z) &= (\Gamma_q(\beta) z^{-p} E_{\alpha,\beta,q}(z)) * f(z) \\ &= z^{-p} + \sum_{n=1}^{\infty} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha n + \beta)} a_{n-p} z^{n-p}, \end{aligned} \tag{1.5}$$

where $z, \alpha, \beta \in \mathbb{C}$, $q \in (0, +\infty)$ and $\text{Re}(\alpha) > 0$.

For functions f and g analytic in U , we say that f is subordinate to g , written $f \prec g$, if g is univalent in U , $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let P be the class of functions h with $h(0) = 1$, which are analytic and convex univalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

In this paper we introduce and investigate the following subclass of $\Sigma(p)$.

DEFINITION.. A function $f \in \Sigma(p)$ is said to be in the class $G_{\alpha,\beta,q}(\lambda; h)$ if it satisfies the second order differential subordination

$$\frac{\lambda - 1}{p} z^{p+1} \left(L_{\alpha,\beta}^q f(z) \right)' + \frac{\lambda}{p(p+1)} z^{p+2} \left(L_{\alpha,\beta}^q f(z) \right)'' \prec h(z), \tag{1.6}$$

where $\lambda, \alpha, \beta \in \mathbb{C}$, $q \in (0, +\infty)$, $\text{Re}(\alpha) > 0$ and $h \in P$.

Let A be the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in U . A function $f \in A$ is said to be in the class $S^*(\gamma)$ if $\text{Re} \left(\frac{z f'(z)}{f(z)} \right) > \gamma$ ($z \in U$) for some γ ($\gamma < 1$). When $0 \leq \gamma < 1$, $S^*(\gamma)$ is the class of starlike functions of order γ in U . A function $f \in A$ is said to be prestarlike of order γ in U if

$$\frac{z}{(1-z)^{2(1-\gamma)}} * f(z) \in S^*(\gamma) \quad (\gamma < 1).$$

We denote this class by $R(\gamma)$ (see [6]). It is obvious that a function $f \in A$ is in the class $R(0)$ if and only if f is convex univalent in U and $R(\frac{1}{2}) = S^*(\frac{1}{2})$.

The study of the Mittag-Leffler function and its various generalizations is a recent interesting topic in geometric function theory. In this paper we shall make a further contribution to the subject by showing some convolution properties for meromorphic functions involving the generalized Mittag-Leffler function.

The following lemmas will be used in our present investigation.

LEMMA 1. ([5]) *Let g be analytic in U and h be analytic and convex univalent in U with $h(0) = g(0)$. If*

$$g(z) + \frac{1}{\mu} z g'(z) \prec h(z), \tag{1.7}$$

where $\text{Re} \mu \geq 0$ and $\mu \neq 0$, then

$$g(z) \prec \tilde{h}(z) = \mu z^{-\mu} \int_0^z t^{\mu-1} h(t) dt \prec h(z)$$

and \tilde{h} is the best dominant of (1.7).

LEMMA 2. ([6]) *Let $\gamma < 1$, $f \in S^*(\gamma)$ and $g \in R(\gamma)$. Then, for analytic function F in U ,*

$$\frac{g * (fF)}{g * f}(U) \subset \overline{co}(F(U)),$$

where $\overline{co}(F(U))$ denotes the closed convex hull of $F(U)$.

LEMMA 3. ([4]) *Let $g(z) = 1 + \sum_{n=m}^{\infty} b_n z^n$ ($m \in \mathbb{N}$) be analytic in U . If $\text{Re}(g(z)) > 0$ ($z \in U$), then*

$$\text{Re}(g(z)) \geq \frac{1 - |z|^m}{1 + |z|^m} \quad (z \in U).$$

2. Properties of the class $G_{\alpha,\beta,q}(\lambda;h)$

In this section we shall derive several convolution properties for functions in the class $G_{\alpha,\beta,q}(\lambda;h)$.

THEOREM 1. *Let $\lambda < 0$, $\gamma > 0$ and $f \in G_{\alpha,\beta,q}(\lambda; \gamma h + 1 - \gamma)$. If $\gamma \leq \gamma_0$, where*

$$\gamma_0 = \frac{1}{2} \left(1 + \frac{p+1}{\lambda} \int_0^1 \frac{u^{-\frac{p+1}{\lambda}-1}}{1+u} du \right)^{-1}, \tag{2.1}$$

then $f \in G_{\alpha,\beta,q}(0;h)$. The bound γ_0 is sharp when $h(z) = \frac{1}{1-z}$.

Proof. Let

$$g(z) = -\frac{1}{p}z^{p+1} \left(L_{\alpha,\beta}^q f(z) \right)' \tag{2.2}$$

for $f \in G_{\alpha,\beta,q}(\lambda; \gamma h + 1 - \gamma)$ with $\lambda < 0$ and $\gamma > 0$. Then we have

$$\begin{aligned} g(z) - \frac{\lambda}{p+1}zg'(z) &= \frac{\lambda-1}{p}z^{p+1} \left(L_{\alpha,\beta}^q f(z) \right)' + \frac{\lambda}{p(p+1)}z^{p+2} \left(L_{\alpha,\beta}^q f(z) \right)'' \\ &< \gamma h(z) + 1 - \gamma. \end{aligned}$$

Hence an application of Lemma 1 yields

$$g(z) < -\frac{\gamma(p+1)}{\lambda}z^{\frac{p+1}{\lambda}} \int_0^z t^{-\frac{p+1}{\lambda}-1}h(t)dt + 1 - \gamma = (h * \psi)(z), \tag{2.3}$$

where

$$\psi(z) = -\frac{\gamma(p+1)}{\lambda}z^{\frac{p+1}{\lambda}} \int_0^z \frac{t^{-\frac{p+1}{\lambda}-1}}{1-t}dt + 1 - \gamma. \tag{2.4}$$

If $0 < \gamma \leq \gamma_0$, where $\gamma_0 (> 1)$ is given by (2.1), then it follows from (2.3) that

$$\begin{aligned} \operatorname{Re}(\psi(z)) &= -\frac{\gamma(p+1)}{\lambda} \int_0^1 u^{-\frac{p+1}{\lambda}-1} \operatorname{Re} \left(\frac{1}{1-uz} \right) du + 1 - \gamma \\ &> -\frac{\gamma(p+1)}{\lambda} \int_0^1 \frac{u^{-\frac{p+1}{\lambda}-1}}{1+u} du + 1 - \gamma \\ &\geq \frac{1}{2} \quad (z \in U). \end{aligned}$$

Now, by using the Herglotz representation for the function ψ , from (2.2) and (2.3) we arrive at

$$-\frac{1}{p}z^{p+1} \left(L_{\alpha,\beta}^q f(z) \right)' < (h * \psi)(z) < h(z)$$

because h is convex univalent in U . This shows that $f \in G_{\alpha,\beta,q}(0;h)$.

For the function $h(z) = \frac{1}{1-z}$ and $f \in \Sigma(p)$ defined by

$$-\frac{1}{p}z^{p+1} \left(L_{\alpha,\beta}^q f(z) \right)' = -\frac{\gamma(p+1)}{\lambda}z^{\frac{p+1}{\lambda}} \int_0^z \frac{t^{-\frac{p+1}{\lambda}-1}}{1-t}dt + 1 - \gamma,$$

it is easy to verify that

$$\frac{\lambda-1}{p}z^{p+1} \left(L_{\alpha,\beta}^q f(z) \right)' + \frac{\lambda}{p(p+1)}z^{p+2} \left(L_{\alpha,\beta}^q f(z) \right)'' = \gamma h(z) + 1 - \gamma.$$

Thus $f \in G_{\alpha,\beta,q}(\lambda; \gamma h + 1 - \gamma)$. Also, for $\gamma > \gamma_0$, we have

$$\operatorname{Re} \left\{ -\frac{1}{p}z^{p+1} \left(L_{\alpha,\beta}^q f(z) \right)' \right\} \longrightarrow -\frac{\gamma(p+1)}{\lambda} \int_0^1 \frac{u^{-\frac{p+1}{\lambda}-1}}{1+u} du + 1 - \gamma < \frac{1}{2} \quad (z \rightarrow -1),$$

which implies that $f \notin G_{\alpha,\beta,q}(0;h)$. Hence the bound γ_0 cannot be increased when $h(z) = \frac{1}{1-z}$. The proof of the theorem is completed. \square

THEOREM 2. Let $f \in G_{\alpha,\beta,q}(\lambda;h)$, $g \in \Sigma(p)$ and $\text{Re}(z^p g(z)) > \frac{1}{2}$ ($z \in U$). Then $f * g \in G_{\alpha,\beta,q}(\lambda;h)$.

Proof. For $f \in G_{\alpha,\beta,q}(\lambda;h)$ and $g \in \Sigma(p)$, we have

$$\begin{aligned} & \frac{\lambda - 1}{p} z^{p+1} \left(L_{\alpha,\beta}^q (f * g)(z) \right)' + \frac{\lambda}{p(p+1)} z^{p+2} \left(L_{\alpha,\beta}^q (f * g)(z) \right)'' \\ &= \frac{\lambda - 1}{p} (z^p g(z)) * \left(z^{p+1} (L_{\alpha,\beta}^q f(z))' \right) + \frac{\lambda}{p(p+1)} (z^p g(z)) * \left(z^{p+2} (L_{\alpha,\beta}^q f(z))'' \right) \\ &= (z^p g(z)) * \psi(z), \end{aligned} \tag{2.5}$$

where

$$\psi(z) = \frac{\lambda - 1}{p} z^{p+1} \left(L_{\alpha,\beta}^q f(z) \right)' + \frac{\lambda}{p(p+1)} z^{p+2} \left(L_{\alpha,\beta}^q f(z) \right)'' \prec h(z). \tag{2.6}$$

In view of the conditions of the theorem, the function $z^p g(z)$ has the Herglotz representation:

$$z^p g(z) = \int_{|x|=1} \frac{d\mu(x)}{1 - xz} \quad (z \in U), \tag{2.7}$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x| = 1$ and $\int_{|x|=1} d\mu(x) = 1$. Since the function h is convex univalent in U , it follows from (2.5), (2.6) and (2.7) that

$$\begin{aligned} & \frac{\lambda - 1}{p} z^{p+1} \left(L_{\alpha,\beta}^q (f * g)(z) \right)' + \frac{\lambda}{p(p+1)} z^{p+2} \left(L_{\alpha,\beta}^q (f * g)(z) \right)'' \\ &= \int_{|x|=1} \psi(xz) d\mu(x) \prec h(z). \end{aligned}$$

This shows that $f * g \in G_{\alpha,\beta,q}(\lambda;h)$ and the theorem is proved. \square

THEOREM 3. Let $f \in G_{\alpha,\beta,q}(\lambda;h)$ be given by (1.4) and let

$$s_k(z) = z^{-p} + \sum_{n=1}^{k-1} a_{n-p} z^{n-p} \quad (k \in \mathbb{N} \setminus \{1\}).$$

Then the function

$$p_k(z) = z^{-p-1} \int_0^z u^p s_k(u) du$$

is also in the class $G_{\alpha,\beta,q}(\lambda;h)$.

Proof. For $f \in G_{\alpha,\beta,q}(\lambda;h)$, we have

$$p_k(z) = z^{-p} + \sum_{n=1}^{k-1} \frac{a_{n-p}}{n+1} z^{n-p} = (f * g_k)(z) \quad (k \in \mathbb{N} \setminus \{1\}), \tag{2.8}$$

where

$$g_k(z) = z^{-p} + \sum_{n=1}^{k-1} \frac{z^{n-p}}{n+1} \in \Sigma(p).$$

Further, for $k \in \mathbb{N} \setminus \{1\}$, it is known from [7] that

$$\operatorname{Re} \{z^p g_k(z)\} = \operatorname{Re} \left\{ 1 + \sum_{n=1}^{k-1} \frac{z^n}{n+1} \right\} > \frac{1}{2} \quad (z \in U). \tag{2.9}$$

In view of (2.8) and (2.9), an application of Theorem 2 leads to $p_k(z) \in G_{\alpha,\beta,q}(\lambda;h)$. The proof of the theorem is completed. \square

THEOREM 4. *Let $f \in G_{\alpha,\beta,q}(\lambda;h)$, $g \in \Sigma(p)$ and $z^{p+1}g(z) \in R(\gamma)$ ($\gamma < 1$). Then $f * g \in G_{\alpha,\beta,q}(\lambda;h)$.*

Proof. From (2.5) we can write

$$\begin{aligned} & \frac{\lambda - 1}{p} z^{p+1} \left(L_{\alpha,\beta}^q (f * g)(z) \right)' + \frac{\lambda}{p(p+1)} z^{p+2} \left(L_{\alpha,\beta}^q (f * g)(z) \right)'' \\ &= \frac{(z^{p+1}g(z)) * (z\psi(z))}{(z^{p+1}g(z)) * z}, \end{aligned} \tag{2.10}$$

where the function ψ is defined as in (2.6).

Since the function h is convex univalent in U ,

$$\psi(z) \prec h(z), \quad z^{p+1}g(z) \in R(\gamma) \quad \text{and} \quad z \in S^*(\gamma) \quad (\gamma < 1),$$

from (2.10) and Lemma 2, we obtain the desired result. The proof of the theorem is completed. \square

Taking $\gamma = 0$ and $\gamma = \frac{1}{2}$ in Theorem 4, we have the following.

COROLLARY. *Let $f \in G_{\alpha,\beta,q}(\lambda;h)$ and let $g \in \Sigma(p)$ satisfy either of the following conditions:*

(i) $z^{p+1}g(z)$ is convex univalent in U

or

(ii) $z^{p+1}g(z) \in S^*\left(\frac{1}{2}\right)$.

Then $f * g \in G_{\alpha,\beta,q}(\lambda;h)$.

THEOREM 5. *Let $\lambda \leq 0$ and*

$$f_j(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p,j} z^{n-p} \in G_{\alpha,\beta,q}(\lambda;h_j) \quad (j = 1, 2), \tag{2.11}$$

where

$$h_j(z) = \gamma_j + (1 - \gamma_j) \frac{1+z}{1-z} \quad \text{and} \quad \gamma_j < 1. \tag{2.12}$$

If $f \in \Sigma(p)$ is defined by

$$\left(L_{\alpha,\beta}^q f(z)\right)' = -\frac{1}{p} \left(\left(L_{\alpha,\beta}^q f_1(z)\right)' * \left(L_{\alpha,\beta}^q f_2(z)\right)' \right), \tag{2.13}$$

then $f \in G_{\alpha,\beta,q}(\lambda;h)$, where

$$h(z) = \gamma + (1 - \gamma) \frac{1+z}{1-z} \tag{2.14}$$

and γ is given by

$$\gamma = \begin{cases} 1 - 4(1 - \gamma_1)(1 - \gamma_2) \left(1 + \frac{p+1}{\lambda} \int_0^1 \frac{u^{-\frac{p+1}{\lambda}-1}}{1+u} du \right) & (\lambda < 0), \\ 1 - 4(1 - \gamma_1)(1 - \gamma_2) & (\lambda = 0). \end{cases} \tag{2.15}$$

The bound γ is the best possible.

Proof. We consider the case when $\lambda < 0$. By setting

$$H_j(z) = \frac{\lambda - 1}{p} z^{p+1} \left(L_{\alpha,\beta}^q f_j(z)\right)' + \frac{\lambda}{p(p+1)} z^{p+2} \left(L_{\alpha,\beta}^q f_j(z)\right)'' \quad (j = 1, 2)$$

for f_j ($j = 1, 2$) given by (2.11), we find that

$$H_j(z) = 1 + \sum_{n=1}^{\infty} b_{n,j} z^n \prec \gamma_j + (1 - \gamma_j) \frac{1+z}{1-z} \quad (j = 1, 2) \tag{2.16}$$

and

$$\left(L_{\alpha,\beta}^q f_j(z)\right)' = \frac{p(p+1)}{\lambda} z^{\frac{(1-\lambda)(p+1)}{\lambda}} \int_0^z t^{-\frac{p+1}{\lambda}-1} H_j(t) dt \quad (j = 1, 2). \tag{2.17}$$

Now, if $f \in \Sigma(p)$ is defined by (2.13), we find from (2.17) that

$$\begin{aligned} \left(L_{\alpha,\beta}^q f(z)\right)' &= -\frac{1}{p} \left(\left(L_{\alpha,\beta}^q f_1(z)\right)' * \left(L_{\alpha,\beta}^q f_2(z)\right)' \right) \\ &= -\frac{1}{p} \left(\frac{p(p+1)}{\lambda} z^{-p-1} \int_0^1 u^{-\frac{p+1}{\lambda}-1} H_1(uz) du \right) \\ &\quad * \left(\frac{p(p+1)}{\lambda} z^{-p-1} \int_0^1 u^{-\frac{p+1}{\lambda}-1} H_2(uz) du \right) \\ &= \frac{p(p+1)}{\lambda} z^{-p-1} \int_0^1 u^{-\frac{p+1}{\lambda}-1} H(uz) du, \end{aligned} \tag{2.18}$$

where

$$H(z) = -\frac{p+1}{\lambda} \int_0^1 u^{-\frac{p+1}{\lambda}-1} (H_1 * H_2)(uz) du. \tag{2.19}$$

Also, by using (2.16) and the Herglotz theorem, we see that

$$\operatorname{Re} \left\{ \left(\frac{H_1(z) - \gamma_1}{1 - \gamma_1} \right) * \left(\frac{1}{2} + \frac{H_2(z) - \gamma_2}{2(1 - \gamma_2)} \right) \right\} > 0 \quad (z \in U),$$

which leads to

$$\operatorname{Re}\{(H_1 * H_2)(z)\} > \gamma_0 = 1 - 2(1 - \gamma_1)(1 - \gamma_2) \quad (z \in U).$$

According to Lemma 3, we have

$$\operatorname{Re}\{(H_1 * H_2)(z)\} \geq \gamma_0 + (1 - \gamma_0) \frac{1 - |z|}{1 + |z|} \quad (z \in U). \tag{2.20}$$

Now it follows from (2.18) to (2.20) that

$$\begin{aligned} & \operatorname{Re} \left\{ \frac{\lambda - 1}{p} z^{p+1} \left(L_{\alpha, \beta}^q f(z) \right)' + \frac{\lambda}{p(p+1)} z^{p+2} \left(L_{\alpha, \beta}^q f(z) \right)'' \right\} \\ &= \operatorname{Re}\{H(z)\} \\ &= -\frac{p+1}{\lambda} \int_0^1 u^{-\frac{p+1}{\lambda}-1} \operatorname{Re}\{(H_1 * H_2)(uz)\} du \\ &\geq -\frac{p+1}{\lambda} \int_0^1 u^{-\frac{p+1}{\lambda}-1} \left(\gamma_0 + (1 - \gamma_0) \frac{1 - u|z|}{1 + u|z|} \right) du \\ &> \gamma_0 - \frac{(p+1)(1 - \gamma_0)}{\lambda} \int_0^1 u^{-\frac{p+1}{\lambda}-1} \frac{1 - u}{1 + u} du \\ &= 1 - 4(1 - \gamma_1)(1 - \gamma_2) \left(1 + \frac{p+1}{\lambda} \int_0^1 \frac{u^{-\frac{p+1}{\lambda}-1}}{1 + u} du \right) \\ &= \gamma, \end{aligned}$$

which proves that $f \in G_{\alpha, \beta, q}(\lambda; h)$ for the function h given by (2.14).

In order to show that the bound γ is sharp, we take the functions $f_j \in \Sigma(p)$ ($j = 1, 2$) defined by

$$\left(L_{\alpha, \beta}^q f_j(z) \right)' = \frac{p(p+1)}{\lambda} z^{\frac{(1-\lambda)(p+1)}{\lambda}} \int_0^z t^{-\frac{p+1}{\lambda}-1} \left(\gamma_j + (1 - \gamma_j) \frac{1+t}{1-t} \right) dt \quad (j = 1, 2), \tag{2.21}$$

for which we have

$$\begin{aligned} H_j(z) &= \frac{\lambda - 1}{p} z^{p+1} \left(L_{\alpha, \beta}^q f_j(z) \right)' + \frac{\lambda}{p(p+1)} z^{p+2} \left(L_{\alpha, \beta}^q f_j(z) \right)'' \\ &= \gamma_j + (1 - \gamma_j) \frac{1+z}{1-z} \quad (j = 1, 2) \end{aligned}$$

and

$$(H_1 * H_2)(z) = 1 + 4(1 - \gamma_1)(1 - \gamma_2) \frac{z}{1 - z}.$$

Hence, for the function f given by (2.13), we have

$$\begin{aligned} & \frac{\lambda - 1}{p} z^{p+1} \left(L_{\alpha, \beta}^q f(z) \right)' + \frac{\lambda}{p(p+1)} z^{p+2} \left(L_{\alpha, \beta}^q f(z) \right)'' \\ &= -\frac{p+1}{\lambda} \int_0^1 u^{-\frac{p+1}{\lambda}-1} \left(1 + 4(1-\gamma_1)(1-\gamma_2) \frac{uz}{1-uz} \right) du \rightarrow \gamma \quad (\text{as } z \rightarrow -1). \end{aligned}$$

Finally, for the case when $\lambda = 0$, the proof of the theorem is simple and we choose to omit the details involved. \square

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