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COMPLETE CONVERGENCE AND COMPLETE MOMENT CONVERGENCE FOR ARRAYS OF ROWWISE WIDELY ORTHANT DEPENDENT RANDOM VARIABLES AND AN APPLICATION

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Abstract. Some general results on complete convergence and complete moment convergence for arrays of rowwise widely orthant dependent random variables is established. As an application, the complete consistency for the estimators in non-parametric model is established.

1. Introduction

Many statistical procedures depend on such sums as $\sum_{i=1}^{n} a_{ni}X_{ni}$, so the study of the limit properties of this type of weighted sums of random variables is of great interest. There are many papers concern the limit properties of weighted sums of random variables, one important topic of which is compete convergence, the concept of which was first introduced by Hsu and Robbins [1] as follows:

A sequence $\{U_n, n \geqslant 1\}$ of random variables converges completely to a constant C if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|U_n - C| > \varepsilon) < \infty.$$

From the Borel-Cantelli lemma, one can obtain that $U_n \to C$ almost surely.

Let $\{U_n, n \geqslant 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$, $q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n{}^{-1}|U_n| - \varepsilon\}_{+}^{q} < \infty \text{ for all } \varepsilon > 0,$$

then the result above was defined as complete moment convergence by Chow [2]. It is easy to check that complete moment convergence implies complete convergence. Consequently, complete moment convergence is much stronger than complete convergence.

The concept of widely orthant dependent random variables was introduced by Wang et al. [3] as follows.

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DEFINITION 1.1. A finite collection of random variables $X_1, X_2, \cdots, X_n$ is said to be widely upper orthant dependent (WUOD) if there exists a finite real number $g_U(n)$ such that for all finite real numbers $x_i, 1 \leqslant i \leqslant n$,

$$P(X_1 > x_1, X_2 > x_2, \cdots, X_n > x_n) \leqslant g_U(n) \prod_{i=1}^{n} P(X_i > x_i). \tag{1.1}$$

A finite collection of random variables $X_1, X_2, \cdots, X_n$ is said to be widely lower orthant dependent (WLOD) if there exists a finite real number $g_L(n)$ such that for all finite real numbers $x_i, 1 \leqslant i \leqslant n$,

$$P(X_1 \leqslant x_1, X_2 \leqslant x_2, \cdots, X_n \leqslant x_n) \leqslant g_L(n) \prod_{i=1}^{n} P(X_i \leqslant x_i). \tag{1.2}$$

If $X_1, X_2, \cdots, X_n$ are both WUOD and WLOD, then $X_1, X_2, \cdots, X_n$ are said to be widely orthant dependent (WOD), and $g_U(n)$, $g_L(n)$ are called dominating coefficients. A sequence of random variables $\{X_n, n \geqslant 1\}$ is said to be WOD if every finite subcollection is WOD.

An array of random variables $\{X_{ni}, 1 \leqslant i \leqslant k_n, n \geqslant 1\}$ is said to be rowwise WOD if for every $n \geqslant 1$, $\{X_{ni}, 1 \leqslant i \leqslant k_n\}$ are WOD.

With various dominating coefficients, the WOD structure reveals many other dependence structures. Wang et al. [3] offered some examples to show that WOD random variables contain negatively dependent random variables, positively dependent random variables, and some other classes of dependent random variables, also they presented some examples to show that the opposite is not true. By letting $x_i \to -\infty$ in (1.1) and $x_i \to \infty$ in (1.2) for each $1 \leqslant i \leqslant n$, it is easy to show that $g_U(n) \geqslant 1$, $g_L(n) \geqslant 1$. If both (1.1) and (1.2) hold with $g_U(n) = g_L(n) = M$ for each $n \geqslant 1$, where $M \geqslant 1$ is a constant, then the random variables are called extended negatively dependent (END), which was introduced by Liu [4]. If both (1.1) and (1.2) hold with $g_U(n) = g_L(n) = 1$ for all $n \geqslant 1$, then the random variables are called negatively orthant depenent (NOD), which was introduced by Lehmann [5] (cf. also Joag-Dev and Proschan [6]). Joag-Dev and Proschan [6] also pointed out that negatively associated (NA) random variables must be NOD and NOD is not necessarily NA, so NA random variables are WOD.

Recently, Sung et al. [7] established the following complete convergence for NOD random variables.

THEOREM 1.1. *Let $\{X_{ni}, 1 \leqslant i \leqslant k_n, n \geqslant 1\}$ be an array of rowwise NOD random variables, $\{c_n, n \geqslant 1\}$ be a sequence of positive constants, and $\{b_n, n \geqslant 1\}$ be a sequence of positive constants such that $\lim_{n\to\infty} b_n = \infty$. Suppose that*

(i) $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} P(|X_{ni}| > \varepsilon) < \infty \ \forall \varepsilon > 0$,

(ii) $\sum_{n=1}^{\infty} c_n \left(\sum_{i=1}^{k_n} P(|X_{ni}| > 1/b_n) \right)^{N_1} < \infty$ *for some* $N_1 > 0$,

(iii) $b_n \sum_{i=1}^{k_n} EX_{ni}^2 I(|X_{ni}| \leqslant 1/b_n) \to 0$, $n \to \infty$, *and*

(iv) $\sum_{n=1}^{\infty} c_n \exp\{-N_2 b_n\} < \infty$ *for some* $N_2 > 0$.

Then for all $\varepsilon > 0$*,*

$$\sum_{n=1}^{\infty} c_n P\left(\left|\sum_{i=1}^{k_n}(X_{ni} - EX_{ni}I(|X_{ni}| \leqslant 1/b_n))\right| > \varepsilon\right) < \infty.$$

The result above is very general and meaningful. As applications of the result, Sung et al. [7] presented some corollaries which improve and extend the existing ones. Qiu et al. [8] extended Theorem 1.1 to END random variables and presented more applications of the result. The aim of this paper is not only to extend the result of Theorem 1.1 for NOD random variables as well as the corresponding result in Qiu et al. [8] for END random variables to the case of WOD random variables, but also to establish the complete moment convergence, which is much stronger than complete convergence. As applications, we further investigate complete consistency for the estimators in non-parametric model. These results generalize and improve some existing ones for dependent or independent random variables.

Next let us recall the concept of stochastic domination, which is weaker than that of identical distribution.

DEFINITION 1.2. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_n| > x) \leqslant CP(|X| > x)$$

for all $x \geqslant 0$ and $n \geqslant 1$.

Throughout the paper, let $g(n) = \max\{g_U(n), g_L(n)\}$. Let C be a positive constant whose value may be different in different places. The symbol $\lfloor x \rfloor$ denotes the integer part of x. let $c(f)$ denote the set of continuity points of the function f on A and $\|x\|$ denote the Euclidean norm of $x \in \mathbb{R}^m$. Let $\log x = \ln\max(x, e)$ and $I(A)$ be the indicator function of the set A.

2. Main results

2.1. Complete convergence and complete moment convergence

In this subsection, the main results will be presented. The first one concerns the complete convergence for arrays of WOD random variables.

THEOREM 2.1. *Let* $\{X_{ni}, 1 \leqslant i \leqslant k_n, n \geqslant 1\}$ *be an array of rowwise WOD random variables,* $\{c_n, n \geqslant 1\}$ *be a sequence of positive constants, and* $\{b_n, n \geqslant 1\}$ *be a sequence of positive constants such that* $\lim_{n\to\infty} b_n = \infty$*. Suppose that*

(i) $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} P(|X_{ni}| > \varepsilon) < \infty \ \forall \varepsilon > 0$*;*

(ii) $\sum_{n=1}^{\infty} c_n g(n) \left(\sum_{i=1}^{k_n} P(|X_{ni}| > 1/b_n)\right)^{N_1} < \infty$ *for some* $N_1 > 0$*;*

(iii) $b_n \sum_{i=1}^{k_n} EX_{ni}^2 I(|X_{ni}| \leqslant 1/b_n) \to 0,\ n \to \infty$;

(iv) $\sum_{n=1}^{\infty} c_n g(n) \exp\{-N_2 b_n\} < \infty$ *for some* $N_2 > 0$.

Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n P\left(\left|\sum_{i=1}^{k_n}(X_{ni} - EX_{ni}I(|X_{ni}| \leqslant 1/b_n))\right| > \varepsilon\right) < \infty. \tag{2.1}$$

THEOREM 2.2. *Under the conditions of Theorem 2.1, if* $\sum_{i=1}^{k_n} EX_{ni}I(|X_{ni}| \leqslant 1/b_n) \to 0$ *as* $n \to \infty$, *then for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n P\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| > \varepsilon\right) < \infty.$$

Based on the result above, we can obtain the following result for WOD random variables. The result can be applied to prove the complete consistency of weighed estimators in either nonparametric or semi-parametric regression models.

COROLLARY 2.1. *Let* $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *be an array of rowwise WOD random variables such that* $EX_{ni} = 0$ *and* $\{X_{ni}\}$ *are stochastically dominated by a random variable* X *satisfying* $E|X|^{2p} < \infty$ *for some* $p \geqslant 1$. *Let the dominating coefficients* $g(n) = O(n^t)$ *for some* $t \geqslant 0$. *Assume that* $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *is an array of constants satisfying*

$$\max_{1 \leqslant i \leqslant n} |a_{ni}| = O(n^{-1/p}), \tag{2.2}$$

$$\sum_{i=1}^{n} a_{ni}^2 = o((\log n)^{-1}). \tag{2.3}$$

Then $\sum_{i=1}^{n} a_{ni}X_{ni}$ *converges completely to zero.*

REMARK 2.1. Sung [9] obtained the corresponding result for NOD random variables. It is deserved to mention that the coefficients $g(n)$ are only required to be polynomial increasing, and the moment condition is independent of the coefficients $g(n)$. Since WOD structure contains NOD structure as a special case, Corollary 2.1 extends the corresponding result of Sung [9] from NOD settings to WOD settings.

THEOREM 2.3. *Let* $q > 0$ *and* $\{X_{ni}, 1 \leqslant i \leqslant k_n, n \geqslant 1\}$ *be an array of rowwise WOD random variables,* $\{c_n, n \geqslant 1\}$ *be a sequence of positive constants, and* $\{b_n, n \geqslant 1\}$ *be a sequence of positive constants such that* $\lim_{n\to\infty} b_n = \infty$. *Suppose that conditions (ii) and (iii) in Theorem 2.1 and the following conditions hold:*

(a) $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} E|X_{ni}|^q I(|X_{ni}| > \varepsilon) < \infty \ \forall\, \varepsilon > 0$;

(b) $\sum_{n=1}^{\infty} c_n g(n) b_n^{-N_3} < \infty$ *for some* $N_3 > 0$;

(c) $b_n^s \sum_{i=1}^{k_n} E|X_{ni}|^s I(|X_{ni}| > 1/b_n) \to 0$ *for some* $s > 0$.

Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n E\left\{\left|\sum_{i=1}^{k_n}(X_{ni} - EX_{ni}I(|X_{ni}| \leqslant 1/b_n))\right| - \varepsilon\right\}_+^q < \infty. \tag{2.4}$$

THEOREM 2.4. *Under the conditions of Theorem 2.3, if* $\sum_{i=1}^{k_n} EX_{ni}I(|X_{ni}| \leqslant 1/b_n) \to 0$ *as* $n \to \infty$, *then for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n E\left\{\left|\sum_{i=1}^{k_n} X_{ni}\right| - \varepsilon\right\}_+^q < \infty.$$

COROLLARY 2.2. *Let* $\alpha > 1/2$, $\alpha p > 1$. *Let* $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *be an array of WOD random variables stochastically dominated by a random variable* X *satisfying* $E|X|^p < \infty$. *Assume further that* $EX_{ni} = 0$ *if* $p > 1$ *and* $g(n) = O(n^t)$ *for some* $t \geqslant 0$. *Then for any* $\varepsilon > 0$ *and* $0 < q < p$,

$$\sum_{n=1}^{\infty} n^{\alpha p - \alpha q - 2} E\left\{\left|\sum_{i=1}^{n} X_{ni}\right| - \varepsilon n^{\alpha}\right\}_+^q < \infty. \tag{2.5}$$

REMARK 2.2. Chow [2] obtained the result

$$\sum_{n=1}^{\infty} n^{\alpha p - \alpha - 2} E\left\{\left|\sum_{i=1}^{n} X_i\right| - \varepsilon n^{\alpha}\right\}_+ < \infty$$

for a sequence of independent and identically distributed random variables with $EX_1 = 0$, $\alpha > \frac{1}{2}$, $p \geqslant 1$, $\alpha p > 1$ and $E\{|X_1|^p + |X_1|\log(1+|X_1|)\} < \infty$. Compared to this classical result, we have the following improvements or generalizations:

(1) The result of Chow [2] was considered for one-indexed independent and identically distributed random variables, while Corollary 2.2 is established for double-indexed WOD random variables with stochastic domination, which is much weaker than that of Chow [2];

(2) The value of p is extended from $p \geqslant 1$ to any $p > 0$ and the exponent q is extended from $q \equiv 1$ to any $0 < q < p$;

(3) For $p = 1$, the moment condition in Chow [2] is improved to $E|X| < \infty$ and the zero mean assumption is no more needed in Corollary 2.2.

2.2. Consistency for estimators in non-parametric model

In this subsection, we will investigate the complete consistency for weighted estimators in non-parametric model for WOD random errors based on the result established in section 2.1.

Consider the following nonparametric regression model:

$$Y_{ni} = f(x_{ni}) + \varepsilon_{ni}, \quad i = 1, 2, \cdots, n, \; n \geqslant 1, \tag{2.6}$$

where x_{ni} are fixed design points from A, where $A \subset \mathbb{R}^m$ is a given compact set for some $m \geqslant 1$, $f(\cdot)$ is an unknown regression function defined on A, and ε_{ni} are random errors. A natural estimator of $f(\cdot)$ is known as the following weighted regression estimator:

$$\hat{f}_n(x) = \sum_{i=1}^{n} \omega_{ni}(x) Y_{ni}, \tag{2.7}$$

where $x \in A$, and $\omega_{ni}(x) = \omega_{ni}(x; x_{n1}, x_{n2}, \cdots, x_{nn})$, $i = 1, 2, \cdots, n$ are weight functions.

The above estimator was first proposed by Stone [10] and adapted by Georgiev [11] and then studied by many authors. For more details one can refer to Roussas [12], Roussas et al. [13], Fan [14], Liang and Jing [15], Wang et al. [16] and so on.

Before presenting the result of complete consistency for the estimator (2.7), the following assumptions on weight functions $\omega_{ni}(x)$ is needed:

$$(H_1) \sum_{i=1}^{n} \omega_{ni}(x) \to 1, \quad n \to \infty; \quad (H_2) \sum_{i=1}^{n} |\omega_{ni}(x)| \leqslant C < \infty \; \forall\, n;$$

$$(H_3) \sum_{i=1}^{n} |\omega_{ni}(x)| \cdot |f(x_{ni}) - f(x)| I(\|x_{ni} - x\| > a) \to 0, \quad n \to \infty, \; \forall\, a > 0.$$

According to the assumptions above, we obtain the following result on complete consistency of the nonparametric regression estimator $\hat{f}_n(x)$.

THEOREM 2.5. *Suppose that* (H_1)–(H_3) *hold. Let* $\{\varepsilon_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *be an array of rowwise zero mean WOD random errors stochastically dominated by a random variable* ε *satisfying* $E|\varepsilon|^{2p} < \infty$ *for some* $p \geqslant 1$. *Assume that* $g(n) = O(n^t)$ *for some* $t \geqslant 0$. *If*

$$\max_{1 \leqslant i \leqslant n} |\omega_{ni}(x)| = O(n^{-1/p}) \tag{2.8}$$

and

$$\sum_{i=1}^{n} (\omega_{ni}(x))^2 = o((\log n)^{-1}). \tag{2.9}$$

Then for all $x \in c(f)$,

$$\hat{f}_n(x) \to f(x) \;\; \textit{completely}. \tag{2.10}$$

REMARK 2.3. Condition (2.9) can be easily satisfied. For example, if $\max_{1 \leqslant i \leqslant n} |\omega_{ni}(x)| = O(n^{-1/p})$ for some $1 \leqslant p < 2$, then it is easy to see that

$$\sum_{i=1}^{n} (\omega_{ni}(x))^2 \leqslant C n^{1-2/p} = o((\log n)^{-1}).$$

Moreover, it differs from the result in Wang et al. [16] that the moment condition in Theorem 2.5 is independent of the coefficients $g(n)$.

3. Proofs of the main results

We first give some lemmas which are essential in proving our main results. From Wang et al. [3], one can easily obtain the following important properties for WOD random variables.

LEMMA 3.1. *Let random variables $X_1, X_2, \cdots, X_n$ be WOD.*

(i) If $f_1, f_2, \cdots, f_n$ are all nondecreasing (or nonincreasing) functions, then random variables $f_1(X_1), f_2(X_2), \cdots, f_n(X_n)$ are WOD.

(ii) For each $n \geqslant 1$, one has that

$$E\left(\prod_{i=1}^{n} X_i^+\right) \leqslant g(n)\prod_{i=1}^{n} EX_i^+.$$

Inspired by Sung et al. [7], we can obtain the following exponential inequality for WOD random variables.

LEMMA 3.2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of WOD random variables with $EX_n = 0$ and $|X_n| \leqslant d_n, n \geqslant 1$, where $\{d_n, n \geqslant 1\}$ is a sequence of positive constants. Then for any $t \in \mathbb{R}$,*

$$E\exp\left\{t\sum_{i=1}^{n} X_i\right\} \leqslant g(n)\exp\left\{\frac{t^2}{2}\sum_{i=1}^{n} e^{td_i} EX_i^2\right\}.$$

Proof. Noting that for all x, $1+x \leqslant e^x \leqslant 1+x+\frac{x^2}{2}e^{|x|}$, we have that for each $1 \leqslant i \leqslant n$,

$$\begin{aligned} Ee^{tX_i} &\leqslant 1+tEX_i+\frac{t^2}{2}E\left(X_i^2 e^{t|X_i|}\right) \\ &= 1+\frac{t^2}{2}E\left(X_i^2 e^{t|X_i|}\right) \\ &\leqslant 1+\frac{t^2}{2}e^{td_i}EX_i^2 \leqslant \exp\left\{\frac{t^2}{2}e^{td_i}EX_i^2\right\}. \end{aligned}$$

Then it follows from Lemma 3.1 that

$$\begin{aligned} E\exp\left\{t\sum_{i=1}^{n} X_i\right\} &= E\prod_{i=1}^{n}\exp\{tX_i\} \leqslant g(n)\prod_{i=1}^{n} E\exp\{tX_i\} \\ &\leqslant g(n)\prod_{i=1}^{n}\exp\left\{\frac{t^2}{2}e^{td_i}EX_i^2\right\} = g(n)\exp\left\{\frac{t^2}{2}\sum_{i=1}^{n} e^{td_i}EX_i^2\right\}. \quad \square \end{aligned}$$

The next one is the Fuk-Nagaev type inequality for WOD random variables, which can be found in Wang et al. [16].

LEMMA 3.3. *Let $0 < \gamma \leqslant 2$. Let $\{X_n, n \geqslant 1\}$ be a sequence of WOD random variables with $EX_n = 0$ for each $n \geqslant 1$ when $1 \leqslant \gamma \leqslant 2$. Then for all $n \geqslant 1$, $x > 0$ and $y > 0$,*

$$P\left(\left|\sum_{i=1}^{n} X_i\right| \geqslant x\right) \leqslant \sum_{i=1}^{n} P(|X_i| \geqslant y) + 2g(n)\exp\left\{\frac{x}{y}-\frac{x}{y}\ln\left(1+\frac{xy^{\gamma-1}}{\sum_{i=1}^{n} E|X_i|^\gamma}\right)\right\}.$$

The last lemma is an important property for stochastic domination. One can refer to Wu [17] for example.

LEMMA 3.4. *Let $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of rowwise random variables stochastically dominated by a random variable X. Then for any $a > 0$ and $b > 0$,*

$$E|X_{ni}|^a I(|X_{ni}| \leqslant b) \leqslant C_1 \left[E|X|^a I(|X| \leqslant b) + b^a P(|X| > b)\right],$$

$$E|X_{ni}|^a I(|X_{ni}| > b) \leqslant C_2 E|X|^a I(|X| > b),$$

where C_1 and C_2 are positive constants.

With the lemmas above accounted for, we will present the proofs of the results.

Proof of Theorem 2.1. The set of all natural numbers can be partitioned into two subsets

$$\mathbb{N}_1 = \left\{ n : \sum_{i=1}^{k_n} P(|X_{ni}| > 1/b_n) \leqslant 1 \right\}, \text{ and } \mathbb{N}_2 = \mathbb{N} - \mathbb{N}_1.$$

Therefore, applying condition (ii), we have

$$\begin{aligned} &\sum_{n\in\mathbb{N}_2} c_n P\left(\left|\sum_{i=1}^{k_n}(X_{ni} - EX_{ni}I(|X_{ni}| \leqslant 1/b_n))\right| > \varepsilon\right) \\ &\leqslant \sum_{n\in\mathbb{N}_2} c_n \leqslant \sum_{n\in\mathbb{N}_2} c_n \left(\sum_{i=1}^{k_n} P(|X_{ni}| > 1/b_n)\right)^{N_1} < \infty. \end{aligned}$$

Hence it suffices to prove

$$\sum_{n\in\mathbb{N}_1} c_n P\left(\left|\sum_{i=1}^{k_n}(X_{ni} - EX_{ni}I(|X_{ni}| \leqslant 1/b_n))\right| > \varepsilon\right) < \infty, \ \forall \varepsilon > 0. \tag{3.1}$$

Since $b_n \to \infty$ as $n \to \infty$, there exists a positive constant M such that $1/b_n < \varepsilon/[4(N_1 + 1)]$ for all $n > M$. For fixed $n \geqslant 1$, denote that

$$\begin{aligned} X_{ni}(1) &= -1/b_n I(X_{ni} < -1/b_n) + X_{ni}I(|X_{ni}| \leqslant 1/b_n) + 1/b_n I(X_{ni} > 1/b_n), \\ X_{ni}(2) &= -1/b_n I(X_{ni} < -1/b_n), \ X_{ni}(3) = 1/b_n I(X_{ni} > 1/b_n), \\ X_{ni}(4) &= X_{ni}I(1/b_n < |X_{ni}| \leqslant \varepsilon/[4(N_1+1)]). \end{aligned}$$

From Lemma 3.1, it is easy to show that $\{X_{ni}(l) - EX_{ni}(l), 1 \leqslant i \leqslant k_n, n \geqslant 1\}$, $l = 1, 2, 3$, are all arrays of rowwise WOD random variables. Therefore,

$$\begin{aligned} &\sum_{n\in\mathbb{N}_1} c_n P\left(\sum_{i=1}^{k_n}(X_{ni} - EX_{ni}I(|X_{ni}| \leqslant 1/b_n)) > \varepsilon\right) \\ &\leqslant \sum_{n\in\mathbb{N}_1} c_n P\left(\bigcup_{i=1}^{k_n}\{|X_{ni}| > \varepsilon/[4(N_1+1)]\}\right) \\ &\quad + \sum_{n\in\mathbb{N}_1} c_n P\left(\sum_{i=1}^{k_n}(X_{ni}I(|X_{ni}| \leqslant \varepsilon/[4(N_1+1)]) - EX_{ni}I(|X_{ni}| \leqslant 1/b_n)) > \varepsilon\right) \\ &=: I_1 + I_2. \end{aligned}$$

From (i), we have

$$I_1 \leqslant \sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} P\left(|X_{ni}| > \varepsilon/[4(N_1+1)]\right) < \infty.$$

For I_2, noting that $I_2 < \infty$ for $n \in \mathbb{N}_1$ and $n \leqslant M$, so we only need to consider the case $n \in \mathbb{N}_1$ and $n > M$. It is to show that

$$\begin{aligned}
&\sum_{i=1}^{k_n} (X_{ni} I(|X_{ni}| \leqslant \varepsilon/[4(N_1+1)]) - EX_{ni} I(|X_{ni}| \leqslant 1/b_n)) \\
&= \sum_{i=1}^{k_n} (X_{ni}(1) - EX_{ni}(1)) + \sum_{i=1}^{k_n} (-X_{ni}(2) + EX_{ni}(2)) + \sum_{i=1}^{k_n} (-X_{ni}(3) + EX_{ni}(3)) + \sum_{i=1}^{k_n} X_{ni}(4).
\end{aligned}$$

Hence, we have that

$$\begin{aligned}
I_2 &\leqslant C + \sum_{n \in \mathbb{N}_1, n > M} c_n P\left(\sum_{i=1}^{k_n} (X_{ni}(1) - EX_{ni}(1)) > \varepsilon/4\right) \\
&\quad + \sum_{n \in \mathbb{N}_1, n > M} c_n P\left(\sum_{i=1}^{k_n} (-X_{ni}(2) + EX_{ni}(2)) > \varepsilon/4\right) \\
&\quad + \sum_{n \in \mathbb{N}_1, n > M} c_n P\left(\sum_{i=1}^{k_n} (-X_{ni}(3) + EX_{ni}(3)) > \varepsilon/4\right) \\
&\quad + \sum_{n \in \mathbb{N}_1, n > M} c_n P\left(\sum_{i=1}^{k_n} X_{ni}(4) > \varepsilon/4\right) \\
&=: C + I_{21} + I_{22} + I_{23} + I_{24}.
\end{aligned}$$

It follows from the definition of $X_{ni}(1)$ that $|X_{ni}(1) - EX_{ni}(1)| \leqslant 2/b_n$ and $(X_{ni}(1))^2 = 1/b_n^2 I(|X_{ni}| > 1/b_n) + X_{ni}^2 I(|X_{ni}| \leqslant 1/b_n)$. Note that $1/b_n \sum_{i=1}^{k_n} P(|X_{ni}| > 1/b_n) = o(1)$ for any $n \in \mathbb{N}_1$. Applying Lemma 3.2 with $t = 4(N_2+1)b_n/\varepsilon$, from Markov's inequality and condition (iii) one has that for any $n \in \mathbb{N}_1$ and $n > M$,

$$\begin{aligned}
&P\left(\sum_{i=1}^{k_n} (X_{ni}(1) - EX_{ni}(1)) > \varepsilon/4\right) \\
&\leqslant g(n) \exp\left\{-\frac{\varepsilon}{4} t + \frac{t^2}{2} e^{\frac{2t}{b_n}} \sum_{i=1}^{k_n} E(X_{ni}(1) - EX_{ni}(1))^2\right\} \\
&\leqslant g(n) \exp\left\{-\frac{\varepsilon}{4} t + \frac{t^2}{2} e^{\frac{2t}{b_n}} \sum_{i=1}^{k_n} E(X_{ni}(1))^2\right\} \\
&\leqslant g(n) \exp\left\{-(N_2+1)b_n + \frac{8(N_2+1)^2}{\varepsilon^2} \exp\left\{\frac{8(N_2+1)}{\varepsilon}\right\} b_n o(1)\right\} \\
&\leqslant g(n) \exp\{-(N_2+1)b_n + o(1)b_n\} \\
&\leqslant C g(n) \exp\{-N_2 b_n\}.
\end{aligned}$$

Consequently, from condition (iv) one has that

$$I_{21} \leqslant C \sum_{n\in\mathbb{N}_1, n>M} c_n g(n)\exp\{-N_2 b_n\} < \infty.$$

For I_{22}, from the definition of $X_{ni}(2)$ one can also have that $|X_{ni}(2) - EX_{ni}(2)| \leqslant 2/b_n$. Applying Lemma 3.2 again with $t = 4(N_2+1)b_n/\varepsilon$, we have by Markov's inequality that for all $n \in \mathbb{N}_1$ and $n > M$,

$$\begin{aligned}
&P\left(\sum_{i=1}^{k_n}(-X_{ni}(2)+EX_{ni}(2)) > \varepsilon/4\right)\\
&\leqslant g(n)\exp\left\{-\frac{\varepsilon}{4}t + \frac{t^2}{2}e^{\frac{2t}{b_n}}\sum_{i=1}^{k_n}E(-X_{ni}(2)+EX_{ni}(2))^2\right\}\\
&\leqslant g(n)\exp\left\{-\frac{\varepsilon}{4}t + \frac{t^2}{2}e^{\frac{2t}{b_n}}\sum_{i=1}^{k_n}\frac{1}{b_n^2}P(|X_{ni}| > 1/b_n)\right\}\\
&\leqslant g(n)\exp\{-(N_2+1)b_n + o(1)b_n\}\\
&\leqslant Cg(n)\exp\{-N_2 b_n\}.
\end{aligned}$$

Hence, from condition (iv) again one can obtain that

$$I_{22} \leqslant C \sum_{n\in\mathbb{N}_1, n>M} c_n g(n)\exp\{-N_2 b_n\} < \infty.$$

Similar to the proof of $I_{22} < \infty$, we can obtain $I_{23} < \infty$.

Finally, we will prove $I_{24} < \infty$. It follows from the definition of WOD random variables that for $n \in \mathbb{N}_1$,

$$\begin{aligned}
&P\left(\sum_{i=1}^{k_n}X_{ni}I(1/b_n < |X_{ni}| \leqslant \varepsilon/[4(N_1+1)]) > \varepsilon/4\right)\\
&\leqslant P\left(\text{there exist at least } \lfloor N_1+1\rfloor's\ X_{ni} \text{ such that } X_{ni} > 1/b_n\right)\\
&= P\left\{\bigcup_{1\leqslant j_1<\cdots<j_{\lfloor N_1+1\rfloor}\leqslant k_n}(X_{nj_1} > 1/b_n, \cdots, X_{nj_{\lfloor N_1+1\rfloor}} > 1/b_n)\right\}\\
&\leqslant g(n)\sum_{j_1,\ldots,j_{\lfloor N_1+1\rfloor}}\prod_{i=1}^{\lfloor N_1+1\rfloor}P(X_{nj_i} > 1/b_n)\\
&\leqslant g(n)\left(\sum_{i=1}^{k_n}P(|X_{ni}| > 1/b_n)\right)^{\lfloor N_1+1\rfloor}\\
&\leqslant g(n)\left(\sum_{i=1}^{k_n}P(|X_{ni}| > 1/b_n)\right)^{N_1},
\end{aligned}$$

which together with condition (ii) obtains that

$$I_{24} \leqslant \sum_{n\in\mathbb{N}_1, n>M} c_n g(n) \left(\sum_{i=1}^{k_n} P(|X_{ni}| > 1/b_n) \right)^{N_1} < \infty.$$

Consequently, we have proved $I_2 < \infty$ and thus

$$\sum_{n\in\mathbb{N}_1} c_n P\left(\sum_{k=1}^{k_n} (X_{nk} - EX_{nk} I(|X_{nk}| \leqslant 1/b_n)) > \varepsilon \right) < \infty. \tag{3.2}$$

Noting that $\{-X_{ni}, 1 \leqslant i \leqslant k_n, n \geqslant 1\}$ is also an array of rowwise WOD random variables, replacing X_{ni} by $-X_{ni}$ for each $1 \leqslant i \leqslant k_n$, $n \geqslant 1$ in (3.2), we have that

$$\sum_{n\in\mathbb{N}_1} c_n P\left(\sum_{k=1}^{k_n} (-X_{nk} + EX_{nk} I(|X_{nk}| \leqslant 1/b_n)) > \varepsilon \right) < \infty. \tag{3.3}$$

(3.1) follows from (3.2) and (3.3) immediately. The proof is completed. □

Proof of Theorem 2.2. Noting that $\sum_{i=1}^{k_n} EX_{ni} I(|X_{ni}| \leqslant 1/b_n) \to 0$, we obtain that for any n large enough,

$$\left| \sum_{i=1}^{k_n} EX_{ni} I(|X_{ni}| \leqslant 1/b_n) \right| \leqslant \varepsilon/2.$$

Hence, it follows from Theorem 2.1 that

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n P\left(\left| \sum_{i=1}^{k_n} X_{ni} \right| > \varepsilon \right) \\
&\leqslant C \sum_{n=1}^{\infty} c_n P\left(\left| \sum_{i=1}^{k_n} (X_{ni} - EX_{ni} I(|X_{ni}| \leqslant 1/b_n)) \right| + \left| \sum_{i=1}^{k_n} EX_{ni} I(|X_{ni}| \leqslant 1/b_n) \right| > \varepsilon \right) \\
&\leqslant C \sum_{n=1}^{\infty} c_n P\left(\left| \sum_{i=1}^{k_n} (X_{ni} - EX_{ni} I(|X_{ni}| \leqslant 1/b_n)) \right| > \varepsilon/2 \right) < \infty. \quad \square
\end{aligned}$$

Proof of Corollary 2.1. Letting $c_n = 1$, $b_n = \log n$, $k_n = n$, $g(n) = O(n^t)$ and replacing X_{ni} by $a_{ni} X_{ni}$ for each $1 \leqslant i \leqslant n$, $n \geqslant 1$ in Theorem 2.2. In order to prove Corollary 2.1, we only need to check that all the conditions of Theorem 2.2 hold. For condition (i), it follows from (2.2) and Lemma 3.4 that

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{i=1}^{n}P(|a_{ni}X_{ni}|>\varepsilon)\\
&\leqslant \varepsilon^{-1}\sum_{n=1}^{\infty}\sum_{i=1}^{n}E|a_{ni}X_{ni}|I(|a_{ni}X_{ni}|>\varepsilon)\\
&\leqslant C\sum_{n=1}^{\infty}n^{1-1/p}E|X|I(|X|>n^{1/p}\varepsilon)\\
&= C\sum_{n=1}^{\infty}n^{1-1/p}\sum_{j=n}^{\infty}E|X|I(j^{1/p}\varepsilon<|X|\leqslant (j+1)^{1/p}\varepsilon)\\
&= C\sum_{j=1}^{\infty}E|X|I(j^{1/p}\varepsilon<|X|\leqslant (j+1)^{1/p}\varepsilon)\sum_{n=1}^{j}n^{1-1/p}\\
&\leqslant C\sum_{j=1}^{\infty}j^{2-1/p}E|X|I(j^{1/p}\varepsilon<|X|\leqslant (j+1)^{1/p}\varepsilon)\\
&\leqslant CE|X|^{2p}<\infty.
\end{aligned}
$$

For condition (ii), taking $N_1>t+1$, then it follows from Markov's inequality, (2.2) and Lemma 3.4 that

$$
\begin{aligned}
&\sum_{n=1}^{\infty}g(n)\left(\sum_{i=1}^{n}P(|a_{ni}X_{ni}|>(\log n)^{-1})\right)^{N_1}\\
&\leqslant C\sum_{n=1}^{\infty}n^{t}\left(\sum_{i=1}^{n}(\log n)^{2p}E|a_{ni}X_{ni}|^{2p}\right)^{N_1}\\
&\leqslant C\sum_{n=1}^{\infty}n^{t}\left(n^{-1}(\log n)^{2p}E|X|^{2p}\right)^{N_1}\\
&\leqslant C\sum_{n=1}^{\infty}n^{t-N_1}(\log n)^{2pN_1}<\infty.
\end{aligned}
$$

For condition (iii), from (2.3) and Lemma 3.4 again, one can obtain that

$$\log n\sum_{i=1}^{n}Ea_{ni}^2X_{ni}^2I(|a_{ni}X_{ni}|\leqslant (\log n)^{-1})\leqslant C\log n\sum_{i=1}^{n}a_{ni}^2EX^2\to 0,\ \ n\to\infty.$$

Condition (iv) holds obviously by taking $N_2>t+1$. Hence it remains to prove

$$\sum_{i=1}^{n}Ea_{ni}X_{ni}I(|a_{ni}X_{ni}|\leqslant (\log n)^{-1})\to 0.$$

From $EX_{ni}=0$, (2.2) and Lemma 3.4 one can easily get that

$$\begin{aligned}\left|\sum_{i=1}^{n} Ea_{ni}X_{ni}I(|a_{ni}X_{ni}|\leqslant(\log n)^{-1})\right| &= \left|\sum_{i=1}^{n} Ea_{ni}X_{ni}I(|a_{ni}X_{ni}|>(\log n)^{-1})\right| \\ &\leqslant \sum_{i=1}^{n} E|a_{ni}X_{ni}|I(|a_{ni}X_{ni}|>(\log n)^{-1}) \\ &\leqslant (\log n)^{2p-1}\sum_{i=1}^{n} E|a_{ni}X_{ni}|^{2p}I(|a_{ni}X_{ni}|>(\log n)^{-1}) \\ &\leqslant C(\log n)^{2p-1}\sum_{i=1}^{n}|a_{ni}|^{2p}E|X|^{2p} \\ &\leqslant Cn^{-1}(\log n)^{2p-1}\to 0,\ \ n\to\infty.\end{aligned}$$

Consequently, all the conditions of Theorem 2.2 are satisfied and the desired result follows from Theorem 2.2 immediately. $\square$

Proof of Theorem 2.3. Denote for any $n\geqslant 1$ that

$$S_n=\sum_{i=1}^{k_n}(X_{ni}-EX_{ni}I(|X_{ni}|\leqslant 1/b_n)),$$

then we have that

$$\begin{aligned}&\sum_{n=1}^{\infty} c_nE\left\{\left|\sum_{i=1}^{k_n}(X_{ni}-EX_{ni}I(|X_{ni}|\leqslant 1/b_n))\right|-\varepsilon\right\}_+^q \\ &=\sum_{n=1}^{\infty} c_n\int_0^1 P\left(|S_n|>\varepsilon+t^{1/q}\right)dt+\sum_{n=1}^{\infty} c_n\int_1^{\infty} P\left(|S_n|>\varepsilon+t^{1/q}\right)dt \\ &=: J_1+J_2.\end{aligned}$$

To prove $J_1<\infty$, we first prove (2.1) holds. According to the conditions of Theorem 2.3, it suffices to show that conditions (i) and (iv) of Theorem 2.1 are satisfied. For all $\varepsilon>0$, it follows from Markov's inequality and condition (a) that

$$\sum_{n=1}^{\infty} c_n\sum_{i=1}^{k_n} P(|X_{ni}|>\varepsilon)\leqslant\varepsilon^{-q}\sum_{n=1}^{\infty} c_n\sum_{i=1}^{k_n} E|X_{ni}|^qI(|X_{ni}|>\varepsilon)<\infty.$$

Hence, condition (i) of Theorem 2.1 holds.

For condition (iv), noting that $\exp\{-N_2b_n\}=o(b_n^{-N_3})$ for all $N_2>0$ and $N_3>0$, from condition (b) one can easily obtain that

$$\sum_{n=1}^{\infty} c_ng(n)\exp\{-N_2b_n\}\leqslant C\sum_{n=1}^{\infty} c_ng(n)b_n^{-N_3}<\infty.$$

Therefore, (2.1) holds true under the assumptions of Theorem 2.3. It follows from Theorem 2.1 that

$$J_1 \leqslant \sum_{n=1}^{\infty} c_n P(|S_n| > \varepsilon) < \infty.$$

In the following, we will prove $J_2 < \infty$. One can easily obtain that

$$\begin{aligned} J_2 &\leqslant \sum_{n=1}^{\infty} c_n \int_1^{\infty} P\left(\max_{1\leqslant i\leqslant k_n} |X_{ni}| > t^{1/q}\right) dt \\ &\quad + \sum_{n=1}^{\infty} c_n \int_1^{\infty} P\left(\left|\sum_{i=1}^{k_n} (X_{ni} I(|X_{ni}| \leqslant t^{1/q}) - EX_{ni} I(|X_{ni}| \leqslant 1/b_n))\right| > t^{1/q}\right) dt \\ &=: J_{21} + J_{22}. \end{aligned}$$

From condition (a) and the fact that $E|X| = \int_0^\infty P(|X| > t)dt$, we obtain that

$$J_{21} \leqslant \sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} \int_0^{\infty} P(|X_{ni}| I(|X_{ni}| > 1) > t^{1/q}) dt = \sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} E|X_{ni}|^q I(|X_{ni}| > 1) < \infty.$$

Denote for each $1 \leqslant i \leqslant k_n$ and $n \geqslant 1$ that

$$\begin{aligned} U_{ni} &= -t^{1/q} I(X_{ni} < -t^{1/q}) + X_{ni} I(|X_{nk}| \leqslant t^{1/q}) + t^{1/q} I(X_{ni} > t^{1/q}), \\ V_{ni} &= t^{1/q} I(X_{ni} < -t^{1/q}) - t^{1/q} I(X_{ni} > t^{1/q}). \end{aligned}$$

Therefore,

$$\begin{aligned} &\left|\sum_{i=1}^{k_n} (X_{ni} I(|X_{ni}| \leqslant t^{1/q}) - EX_{ni} I(|X_{ni}| \leqslant 1/b_n))\right| \\ &\leqslant \left|\sum_{i=1}^{k_n} (U_{ni} - EU_{ni})\right| + \left|\sum_{i=1}^{k_n} (V_{ni} - EV_{ni})\right| + \sum_{i=1}^{k_n} E|X_{ni}| I(1/b_n < |X_{ni}| \leqslant t^{1/q}). \end{aligned}$$

Moreover, from Markov's inequality and condition (c), we have that

$$\sup_{t\geqslant 1} t^{-1/q} \sum_{i=1}^{k_n} E|X_{ni}| I(1/b_n < |X_{ni}| \leqslant t^{1/q}) \leqslant \sum_{i=1}^{k_n} P(|X_{ni}| > 1/b_n) \to 0, \ \ n \to \infty,$$

which implies that for any $t \geqslant 1$ and all n large enough,

$$\sum_{i=1}^{k_n} E|X_{ni}| I(1/b_n < |X_{ni}| \leqslant t^{1/q}) < \frac{1}{3} t^{1/q}.$$

Therefore,

$$\begin{aligned} J_{22} &\leqslant C \sum_{n=1}^{\infty} c_n \int_1^{\infty} P\left(\left|\sum_{i=1}^{k_n} (V_{ni} - EV_{ni})\right| > \frac{1}{3} t^{1/q}\right) dt \\ &\quad + C \sum_{n=1}^{\infty} c_n \int_1^{\infty} P\left(\left|\sum_{i=1}^{k_n} (U_{ni} - EU_{ni})\right| > \frac{1}{3} t^{1/q}\right) dt \\ &=: J_{23} + J_{24}. \end{aligned}$$

It follows from condition (a) that

$$\begin{aligned} J_{23} &\leqslant C\sum_{n=1}^{\infty} c_n \int_1^{\infty} t^{-1/q} \sum_{i=1}^{k_n} E|V_{ni}|dt \\ &= C\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} \int_1^{\infty} P(|X_{ni}| > t^{1/q})dt \\ &\leqslant C\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} E|X_{ni}|^q I(|X_{ni}| > 1) < \infty. \end{aligned}$$

For J_{24}, let $\gamma = 2$, $x = \frac{1}{3}t^{1/q}$, $y = \frac{1}{3\lambda}t^{1/q}$ with $\lambda > \max\{q, N_3, N_3/s, q/s\}$ in Lemma 3.3, one can get that

$$\begin{aligned} J_{24} \leqslant\; & C\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} \int_1^{\infty} P\left(|U_{ni} - EU_{ni}| > \frac{1}{3\lambda}t^{1/q}\right) dt \\ & + C\sum_{n=1}^{\infty} c_n g(n) \int_1^{\infty} \left(1 + \frac{t^{2/q}}{9\lambda \sum_{i=1}^{k_n} E(U_{ni} - EU_{ni})^2}\right)^{-\lambda} dt \\ =:\; & J_{25} + J_{26}. \end{aligned}$$

It follows from condition (c) and $\lim_{n\to\infty} b_n = \infty$ that for all n large enough (such that at least $1/b_n \leqslant 1$),

$$\begin{aligned} & \sup_{t\geqslant 1} \max_{1\leqslant i\leqslant k_n} t^{-1/q}|EU_{ni}| \leqslant \sup_{t\geqslant 1} \max_{1\leqslant i\leqslant k_n} t^{-1/q}E|U_{ni}| \\ \leqslant\; & \sup_{t\geqslant 1} \max_{1\leqslant i\leqslant k_n} \{t^{-1/q}E|X_{ni}|I(|X_{ni}| \leqslant 1/b_n) \\ & + t^{-1/q}E|X_{ni}|I(1/b_n < |X_{ni}| \leqslant t^{1/q}) + P(|X_{ni}| > t^{1/q})\} \\ \leqslant\; & 1/b_n + 2\sum_{i=1}^{k_n} P(|X_{ni}| > 1/b_n) \to 0, \quad n \to \infty. \end{aligned}$$

Hence from $|U_{ni}| \leqslant |X_{ni}|$ and condition (a), one can obtain that

$$\begin{aligned} J_{25} &\leqslant C\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} \int_1^{\infty} P\left(|U_{ni}| > \frac{1}{4\lambda}t^{1/q}\right) dt \\ &\leqslant C\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} \int_1^{\infty} P\left(|X_{ni}| > \frac{1}{4\lambda}t^{1/q}\right) dt \\ &\leqslant C\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} E|X_{ni}|^q I(|X_{ni}| > \frac{1}{4\lambda}) < \infty. \end{aligned}$$

It follows from the definition of U_{ni} that $U_{ni}^2 = t^{2/q}I(|X_{ni}| > t^{1/q}) + X_{ni}^2 I(|X_{ni}| \leqslant t^{1/q})$.

Hence, by C_r inequality we have that

$$\begin{aligned}
J_{26} &\leqslant C\sum_{n=1}^{\infty} c_n g(n)\int_1^{\infty} t^{-2\lambda/q}\left(\sum_{i=1}^{k_n} E(U_{ni}-EU_{ni})^2\right)^{\lambda} dt \\
&\leqslant C\sum_{n=1}^{\infty} c_n g(n)\int_1^{\infty} t^{-2\lambda/q}\left(\sum_{i=1}^{k_n} EU_{ni}^2\right)^{\lambda} dt \\
&\leqslant C\sum_{n=1}^{\infty} c_n g(n)\int_1^{\infty} t^{-2\lambda/q}\left(\sum_{i=1}^{k_n} EX_{ni}^2 I(|X_{ni}|\leqslant 1/b_n)\right)^{\lambda} dt \\
&\quad +C\sum_{n=1}^{\infty} c_n g(n)\int_1^{\infty} t^{-2\lambda/q}\left(\sum_{i=1}^{k_n} EX_{ni}^2 I(1/b_n<|X_{ni}|\leqslant t^{1/q})\right)^{\lambda} dt \\
&\quad +C\sum_{n=1}^{\infty} c_n g(n)\int_1^{\infty} \left(\sum_{i=1}^{k_n} P(|X_{ni}|>t^{1/q})\right)^{\lambda} dt \\
&=: J_{27}+J_{28}+J_{29}.
\end{aligned}$$

It follows from condition (iii) that $b_n\sum_{i=1}^{k_n} EX_{n_i}^2 I(|X_{n_i}|\leqslant 1/b_n)<1$ for all n large enough. Noting that $\lambda>N_3$ and $\lambda>q$, we have by condition (b) that

$$\begin{aligned}
J_{27} &= C\sum_{n=1}^{\infty} c_n g(n) b_n{}^{-\lambda}\left(b_n\sum_{i=1}^{k_n} EX_{ni}^2 I(|X_{ni}|\leqslant 1/b_n)\right)^{\lambda}\int_1^{\infty} t^{-2\lambda/q} dt \\
&\leqslant C\sum_{n=1}^{\infty} c_n g(n) b_n^{-\lambda}<\infty.
\end{aligned}$$

Now we will estimate J_{28}. Recall that $\lambda>\max\{q,N_3,N_3/s,q/s\}$. It follows from condition (c) that $b_n^s\sum_{i=1}^{k_n} E|X_{ni}|^s I(|X_{ni}|>1/b_n)<1$ for all n large enough. If $0<s\leqslant 2$, from condition (b) we obtain that

$$\begin{aligned}
J_{28} &\leqslant C\sum_{n=1}^{\infty} c_n g(n) b_n^{-s\lambda}\int_1^{\infty} t^{-s\lambda/q}\left(b_n^s\sum_{i=1}^{k_n} E|X_{ni}|^s I(1/b_n<|X_{ni}|\leqslant t^{1/q})\right)^{\lambda} dt \\
&\leqslant C\sum_{n=1}^{\infty} c_n g(n) b_n^{-s\lambda}\int_1^{\infty} t^{-s\lambda/q} dt<\infty,
\end{aligned}$$

providing that $\lambda>N_3/s$ and $\lambda>q/s$. If $s>2$, we also have

$$\begin{aligned}
J_{28} &\leqslant C\sum_{n=1}^{\infty} c_n g(n) b_n^{-2\lambda}\int_1^{\infty} t^{-2\lambda/q}\left(b_n^s\sum_{i=1}^{k_n} E|X_{ni}|^s I(1/b_n<|X_{ni}|\leqslant t^{1/q})\right)^{\lambda} dt \\
&\leqslant C\sum_{n=1}^{\infty} c_n g(n) b_n^{-2\lambda}\int_1^{\infty} t^{-2\lambda/q} dt<\infty,
\end{aligned}$$

providing that $\lambda > N_3$ and $\lambda > q$.

Finally, we will prove $J_{29} < \infty$. Note that $1/b_n < 1 \leqslant t^{1/q}$ for all n large enough. Therefore, we have by condition (b) again and $\lambda > N_3/s$, $\lambda > q/s$ that

$$J_{29} \leqslant C\sum_{n=1}^{\infty} c_n g(n) b_n^{-s\lambda} \int_1^{\infty} t^{-s\lambda/q} \left(b_n^s \sum_{i=1}^{k_n} E|X_{ni}|^s I(|X_{ni}| > 1/b_n) \right)^{\lambda} dt$$
$$\leqslant C\sum_{n=1}^{\infty} c_n g(n) b_n^{-s\lambda} \int_1^{\infty} t^{-s\lambda/q} dt < \infty.$$

Consequently, the proof of the theorem is completed. □

Proof of Theorem 2.4. By some similar arguments as the proof of Theorem 2.2, one can obtain the desired result immediately from (2.4) and $\sum_{i=1}^{k_n} EX_{ni}I(|X_{ni}| \leqslant 1/b_n) \to 0$. The detail is omitted. □

Proof of Corollary 2.2. Let $c_n = n^{\alpha p-2}$, $k_n = n$, $b_n = n^r$, $0 < r < \min\{\alpha - 1/p, 2\alpha - 1\}$ and replace X_{ni} by $n^{-\alpha}X_{ni}$ for each $1 \leqslant i \leqslant n$, $n \geqslant 1$ in Theorem 2.4. It suffices to check that all the conditions of Theorem 2.4 hold.

For condition (a), it follows from Lemma 3.4 that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{i=1}^{n} n^{-\alpha q} E|X_{ni}|^q I(|X_{ni}| > n^{\alpha}\varepsilon) \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1} E|X|^q I(|X| > n^{\alpha}\varepsilon) \\
&= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1} \sum_{j=n}^{\infty} E|X|^q I(j^{\alpha}\varepsilon < |X| \leqslant (j+1)^{\alpha}\varepsilon) \\
&= C\sum_{j=1}^{\infty} E|X|^q I(j^{\alpha}\varepsilon < |X| \leqslant (j+1)^{\alpha}\varepsilon) \sum_{n=1}^{j} n^{\alpha p-\alpha q-1} \\
&\leqslant C\sum_{j=1}^{\infty} j^{\alpha p-\alpha q} E|X|^q I(j^{\alpha}\varepsilon < |X| \leqslant (j+1)^{\alpha}\varepsilon) \\
&\leqslant C\sum_{j=1}^{\infty} E|X|^p I(j^{\alpha}\varepsilon < |X| \leqslant (j+1)^{\alpha}\varepsilon) \leqslant CE|X|^p < \infty.
\end{aligned}$$

Condition (b) holds trivially by choosing N_3 sufficiently large such that $\alpha p + t - 2 - rN_3 < -1$. For condition (c), note that $r < \alpha - 1/p$. Therefore, by choosing $s = p$, we can obtain from Lemma 3.4 that

$$n^{rp} \sum_{i=1}^{n} n^{-\alpha p} E|X_{ni}|^p I(|X_{ni}| > n^{\alpha - r}) \leqslant CE|X|^p n^{1-(\alpha-r)p} \to 0, \quad n \to \infty.$$

For condition (ii), choosing N_1 large enough such that $\alpha p + t - 2 - [(\alpha - r)p - 1]N_1 <$

-1, one can obtain from Lemma 3.4 and Markov's inequality that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} g(n)\left(\sum_{i=1}^{n} P(|n^{-\alpha} X_{ni}|>n^{-r})\right)^{N_1} \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p+t-2}\left(\sum_{i=1}^{n} P(|X|>n^{\alpha-r})\right)^{N_1} \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p+t-2-[(\alpha-r)p-1]N_1}(E|X|^p)^{N_1}<\infty.
\end{aligned}$$

For condition (iii), if $p \geqslant 2$, then

$$n^r \sum_{i=1}^{n} E|n^{-\alpha} X_{ni}|^2 I(|X_{ni}| \leqslant n^{\alpha-r}) \leqslant Cn^{r-2\alpha+1} \to 0, \ \ n \to \infty.$$

If $p<2$, it follows from $0<r<\alpha-1/p$ that $1-\alpha p+(p-1)r<1/p-\alpha<0$ if $p>1$ or $1-\alpha p+(p-1)r \leqslant 1-\alpha p<0$ if $0<p\leqslant 1$. Hence,

$$\begin{aligned}
n^r \sum_{i=1}^{n} E|n^{-\alpha} X_{ni}|^2 I(|n^{-\alpha} X_{ni}| \leqslant n^{-r}) &\leqslant n^{r(p-1)} \sum_{i=1}^{n} E|n^{-\alpha} X_{ni}|^p \\
&\leqslant Cn^{1-\alpha p+(p-1)r} \to 0, \ \ n\to\infty.
\end{aligned}$$

Finally, we still need to prove that

$$\sum_{i=1}^{n} n^{-\alpha} EX_{ni} I(|X_{ni}| \leqslant n^{\alpha-r}) \to 0, \ \ n\to\infty.$$

Note from $0<r<\alpha-1/p$ that $1-\alpha p+(p-1)r<0$. If $p>1$, we have by $EX_{ni}=0$ and Lemma 3.4 that

$$\begin{aligned}
\left|\sum_{i=1}^{n} n^{-\alpha} EX_{ni} I(|X_{ni}| \leqslant n^{\alpha-r})\right| &= \left|\sum_{i=1}^{n} n^{-\alpha} EX_{ni} I(|X_{ni}| > n^{\alpha-r})\right| \\
&\leqslant n^{-\alpha}\sum_{i=1}^{n} E|X_{ni}| I(|X_{ni}| > n^{\alpha-r}) \\
&\leqslant n^{-\alpha}\sum_{i=1}^{n} n^{(\alpha-r)(1-p)} E|X_{ni}|^p I(|X_{ni}| > n^{\alpha-r}) \\
&\leqslant Cn^{1-\alpha p+(p-1)r} E|X|^p I(|X|>n^{\alpha-r}) \to 0, \ \ n\to\infty;
\end{aligned}$$

and if $0<p\leqslant 1$, we also have by Lemma 3.4 that

$$\begin{aligned}
\left|\sum_{i=1}^{n} n^{-\alpha} EX_{ni} I(|X_{ni}| \leqslant n^{\alpha-r})\right| &\leqslant n^{-\alpha}\sum_{i=1}^{n} E|X_{ni}| I(|X_{ni}| \leqslant n^{\alpha-r}) \\
&\leqslant n^{-\alpha}\sum_{i=1}^{n} n^{(\alpha-r)(1-p)} E|X_{ni}|^p I(|X_{ni}| \leqslant n^{\alpha-r}) \\
&\leqslant Cn^{1-\alpha p+(p-1)r} E|X|^p \to 0, \ \ n\to\infty.
\end{aligned}$$

Consequently, the desired result follows immediately from Theorem 2.4. The proof is completed. □

Proof of Theorem 2.5. For any $a > 0$ and $x \in c(f)$, we obtain from (2.6) and (2.7) that

$$|E\hat{f}_n(x) - f(x)| \leqslant \sum_{i=1}^{n} |\omega_{ni}(x)| \cdot |f(x_{ni}) - f(x)| I(\|x_{ni} - x\| \leqslant a)$$
$$+ \sum_{i=1}^{n} |\omega_{ni}(x)| \cdot |f(x_{ni}) - f(x)| I(\|x_{ni} - x\| > a)$$
$$+ |f(x)| \cdot \left| \sum_{i=1}^{n} \omega_{ni}(x) - 1 \right|. \tag{3.4}$$

It follows from $x \in c(f)$ that for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for any x' satisfying $\|x' - x\| < \delta$, $|f(x') - f(x)| < \varepsilon$. Therefore, taking $0 < a < \delta$ in (3.4), one can easily obtain from conditions $(H_1) - (H_3)$ that

$$\lim_{n\to\infty} |E\hat{f}_n(x) - f(x)| = 0.$$

Noting that $\omega_{ni}(x) = \omega_{ni}(x)^+ - \omega_{ni}(x)^-$, we may assume without loss of generality that $\omega_{ni}(x) \geqslant 0$ for each $1 \leqslant i \leqslant n$ and $n \geqslant 1$. Consequently, to prove (2.10), it suffices to prove

$$\hat{f}_n(x) - E\hat{f}_n(x) = \sum_{i=1}^{n} \omega_{ni}(x)\varepsilon_{ni} \to 0 \text{ completely,}$$

which is a direct result of Corollary 2.1 with $X_{ni} = \varepsilon_{ni}$ and $a_{ni} = \omega_{ni}(x)$. The proof is completed. □

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