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SOME INEQUALITIES FOR THE L_p–CURVATURE IMAGES

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Abstract. Lutwak introduced the notion of L_p-curvature image and proved an inequality for volumes of convex body and its L_p-curvature image. In this article, based on the L_p-affine surface area and L_p-dual affine surface area, we establish the affine isoperimetric inequalities, cyclic inequalities and a monotonic inequality for L_p-curvature images.

1. Introduction and main results

Let K be a convex body if K is a compact, convex subset in n-dimensional Euclidean space $\mathbb{R}^n$ with non-empty interior. The set of all convex bodies in $\mathbb{R}^n$ is written as $\mathscr{K}^n$. Let $\mathscr{K}_o^n$ denote the set of convex bodies containing the origin in their interiors, and $\mathscr{K}_c^n$ denote the set of convex bodies with centroid at the origin. Besides, $\mathscr{S}_o^n$ denotes the set of star bodies (with respect to the origin) and $\mathscr{S}_c^n$ denotes the set of star bodies whose centroid lies at the origin in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$ and $V(K)$ denote the n-dimensional volume of the body K. For the standard unit ball B in $\mathbb{R}^n$, write $V(B)=\omega_n$.

In 1996, Lutwak introduced the notion of L_p-curvature function of convex body (see [12, 13]). For $K\in\mathscr{K}_o^n$ and real $p\geqslant 1$, the L_p-curvature function, $f_p(K,\cdot)$: $S^{n-1}\to\mathbb{R}$, is defined by

$$\frac{dS_p(K,\cdot)}{dS}=f_p(K,\cdot), \tag{1.1}$$

where the L_p-surface area measure $S_p(K,\cdot)$ of K is absolutely continuous with respect to spherical Lebesgue measure S. Here, we write $\mathscr{F}_o^n$ ($\mathscr{F}_c^n$) as the subset of $\mathscr{K}_o^n$ ($\mathscr{K}_c^n$) that has a positive continuous curvature function.

By the L_p-curvature function, Lutwak in [12] gave the notion of L_p-curvature image as follows: For each $K\in\mathscr{F}_o^n$ and real $p\geqslant 1$, let $\Lambda_pK\in\mathscr{S}_o^n$ denote the L_p-curvature image of K, and define

$$\rho(\Lambda_pK,\cdot)^{n+p}=\frac{V(\Lambda_pK)}{\omega_n}f_p(K,\cdot). \tag{1.2}$$

Associated with the L_p-curvature images, Lutwak ([12]) obtained the following result.

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THEOREM 1.A. *For $K, L \in \mathscr{F}_c^n$, $p \geqslant 1$, then*

$$V(\Lambda_p K)V(K)^{\frac{p-n}{p}} \leqslant \omega_n^{\frac{2p-n}{p}}, \tag{1.3}$$

with equality for $n = p > 1$ if and only if K and L are dilates, for $n \neq p > 1$ if and only if $K = L$, for $n \neq p = 1$ if and only if K is a translation of L.

Later, Wang etc. ([25]) continuously studied the L_p-curvature images for convex bodies and established the following polar dual forms of Theorem 1.A:

THEOREM 1.B. *For $K \in \mathscr{F}_o^n$, $p \geqslant 1$ and $\Lambda_p K \in \mathscr{K}_o^n$, then*

$$V(\Lambda_p K)V(K^*)^{\frac{n-p}{p}} \leqslant \omega_n^{\frac{n}{p}}, \tag{1.4}$$

with equality if and only if K is an ellipsoid. Here K^* denotes the polar of K.

THEOREM 1.C. *For $K \in \mathscr{F}_c^n$ and $p \geqslant 1$, then*

$$V(\Lambda_p^* K)V(K)^{\frac{n-p}{p}} \leqslant \omega_n^{\frac{n}{p}}, \tag{1.5}$$

with equality for $p > 1$ if and only if K and $\Lambda_p^* K$ are dilates, and for $p = 1$ if and only if K and $\Lambda_p^* K$ are homothetic. Here $\Lambda_p^* K$ denotes the polar of $\Lambda_p K$.

For more studies of the L_p-curvature images, the interested readers may refer to the following articles [8, 14, 15, 16].

In this paper, associated with the notions of L_p-affine surface area and L_p-dual affine surface area, we continuously research the L_p-curvature images. Firstly, we establish the following L_p-affine surface area forms of Theorems 1.A and 1.C.

THEOREM 1.1. *For $K \in \mathscr{F}_o^n$ and $p \geqslant 1$, if $\Lambda_p K \in \mathscr{K}_c^n$, then*

$$\Omega_p(\Lambda_p K)\Omega_p(K)^{\frac{p-n}{p}} \leqslant (n\omega_n)^{\frac{2p-n}{p}}, \tag{1.6}$$

with equality if and only if $\Lambda_p K$ is an ellipsoid.

THEOREM 1.2. *If $K \in \mathscr{F}_c^n$ and $p \geqslant 1$, then*

$$\Omega_p(\Lambda_p^* K)\Omega_p(K)^{\frac{n-p}{p}} \leqslant (n\omega_n)^{\frac{n}{p}}, \tag{1.7}$$

with equality if and only if $\Lambda_p K$ is an ellipsoid.

In Theorems 1.1–1.2, $\Omega_p(K)$ denotes the L_p-affine surface area of $K \in \mathscr{K}_o^n$.

Further, we establish the cyclic inequalities of L_p-curvature images for the L_p-affine surface area and L_p-dual affine surface area, respectively.

THEOREM 1.3. *If $K \in \mathscr{F}_o^n$ and $1 \leqslant p < q < r$, then*

$$\Omega_q(\Lambda_q K)^{(n+q)(r-p)} \leqslant \Omega_p(\Lambda_p K)^{(n+p)(r-q)}\Omega_r(\Lambda_r K)^{(n+r)(q-p)}. \tag{1.8}$$

THEOREM 1.4. *If $K \in \mathscr{F}_o^n$ and $1 \leqslant p < q < r$, then*

$$\widetilde{\Omega}_q(\Lambda_q K)^{(n+q)(r-p)} \leqslant \widetilde{\Omega}_p(\Lambda_p K)^{(n+p)(r-q)} \widetilde{\Omega}_r(\Lambda_r K)^{(n+r)(q-p)}, \tag{1.9}$$

with equality if and only if $\Lambda_p K$, $\Lambda_q K$ and $\Lambda_r K$ are dilates. Here, $\widetilde{\Omega}_p(K)$ denotes the L_p-dual affine surface area of $K \in \mathscr{S}_o^n$.

Finally, combined with another type of L_p-affine surface area, we give a monotonic inequality for L_p-curvature images.

THEOREM 1.5. *If $K \in \mathscr{F}_o^n$ and $1 \leqslant p < q$, then*

$$\left[\frac{\omega_n^n \widetilde{\Omega}_{-p}(\Lambda_p K)^{n-p}}{n^{n-p} V(\Lambda_p K)^n V(K)^{n-p}}\right]^{\frac{1}{p}} \leqslant \left[\frac{\omega_n^n \widetilde{\Omega}_{-q}(\Lambda_q K)^{n-q}}{n^{n-q} V(\Lambda_q K)^n V(K)^{n-q}}\right]^{\frac{1}{q}}, \tag{1.10}$$

with equality if and only if $\Lambda_p K$ and $\Lambda_q K$ are dilates. Here, $\widetilde{\Omega}_{-p}(K)$ denotes the L_p-dual affine surface area of $K \in \mathscr{S}_o^n$.

Please see the next section for the above interrelated background materials. The proofs of Theorems 1.1–1.5 will be completed in Section 3.

2. Preliminaries

2.1. Polar bodies and Blaschke-Santaló inequality

If $E \subseteq \mathbb{R}^n$ is a nonempty subset, the polar set of E, E^*, is defined by (see [5, 17])

$$E^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1,\ y \in E\}. \tag{2.1}$$

From this, it is easy to get that $(K^*)^* = K$ for all $K \in \mathscr{K}_o^n$.

From definition (2.1). we know that if $K \in \mathscr{K}_o^n$, the support and radial functions of K^*, the polar body of K, have the following relationship (see [5])

$$h(K^*, \cdot) = \frac{1}{\rho(K, \cdot)}, \qquad \rho(K^*, \cdot) = \frac{1}{h(K, \cdot)}. \tag{2.2}$$

Besides, the polar bodies of convex bodies satisfy the following properties (see [5]): If $K \in \mathscr{K}_o^n$, $\phi \in GL(n)$, then

$$(\phi K)^* = \phi^{-\tau} K^*. \tag{2.3}$$

In particular, for $\lambda > 0$,

$$(\lambda K)^* = \frac{1}{\lambda} K^*. \tag{2.4}$$

For a geometric body and its polar body, Lutwak extended the Blaschke-Santaló inequality as follows (see [5, 17]): *If $K \in \mathscr{S}_c^n$, then*

$$V(K)V(K^*) \leqslant \omega_n^2, \tag{2.5}$$

with equality if and only if K is an ellipsoid.

2.2. L_p-mixed volume

Suppose that $\mathbb{R}$ is the set of real numbers. If $K \in \mathscr{K}^n$, the support function of K, $h_K = h(K,\cdot)\colon \mathbb{R}^n \to \mathbb{R}$, is defined by (see [4])

$$h(K,x) = \max\{x\cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x\cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$.

If $K,L \in \mathscr{K}_o^n$, for $p \geqslant 1$, the L_p-mixed volume of K and L is given by (see [11])

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h(L,u)^p dS_p(K,u). \tag{2.6}$$

Associated with formula (2.6) and $dS_p(K,u) = h(K,u)^{1-p}dS(K,u)$ for $u \in S^{n-1}$, if $K = L$, then

$$V_p(K,K) = \frac{1}{n}\int_{S^{n-1}} h(K,u)^p dS_p(K,u) = \frac{1}{n}\int_{S^{n-1}} h(K,u)dS(K,u) = V(K). \tag{2.7}$$

2.3. L_p-dual mixed volume

For K is a compact star shaped (about the origin) in $\mathbb{R}^n$, the radial function ρ_K of K, $\rho_K = \rho(K,\cdot)\colon \mathbb{R}^n\backslash\{0\} \to [0,+\infty)$, is defined by (see [5])

$$\rho(K,x) = \max\{\lambda \geqslant 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n\backslash\{0\},$$

if ρ_K is positive and continuous, then called K is a star body.

If $K,L \in \mathscr{S}_o^n$, $p \geqslant 1$, the L_p-dual mixed volume of K and L is given by (see [12])

$$\widetilde{V}_{-p}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n+p}\rho(L,u)^{-p}dS(u). \tag{2.8}$$

Another kind of L_p-dual mixed volume was introduced as follows (see [6, 7]): If $K,L \in \mathscr{S}_o^n$ and $p > 0$, the L_p-dual mixed volume of K and L is given by

$$\widetilde{V}_p(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-p}\rho(L,u)^p dS(u). \tag{2.9}$$

Here the integral expression is with respect to spherical Lebesgue measure S on S^{n-1}.

From (2.8) and (2.9), we easily know that

$$\widetilde{V}_{-p}(K,K) = \widetilde{V}_p(K,K) = V(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^n dS(u). \tag{2.10}$$

Associated with (1.2), (2.6) and (2.8), Lutwak ([12]) gave the following result. If $K \in \mathscr{F}_o^n$, and $p \geqslant 1$, then for any $Q \in \mathscr{S}_o^n$,

$$V_p(K,Q^*) = \frac{\omega_n}{V(\Lambda_p K)}\widetilde{V}_{-p}(\Lambda_p K,Q). \tag{2.11}$$

2.4. L_p-affine surface area

In 1996, associated with L_p-mixed volume (2.6), Lutwak ([12]) defined the L_p-affine surface area as follows: For $K \in \mathscr{K}_o^n$ and $p \geqslant 1$, the L_p-affine surface area, $\Omega_p(K)$, of K is defined by

$$n^{-\frac{p}{n}}\Omega_p(K)^{\frac{n+p}{n}} = \inf\{nV_p(K,Q^*)V(Q)^{\frac{p}{n}} : Q \in \mathscr{S}_o^n\}. \tag{2.12}$$

From definitions (2.12) and (1.2), the following formula can be obtained (see [12]): For $K \in \mathscr{F}_o^n$, and $p \geqslant 1$, then

$$\Omega_p(K) = n\omega_n^{\frac{n}{n+p}} V(\Lambda_p K)^{\frac{p}{n+p}}. \tag{2.13}$$

Regarding the studies of L_p-affine surface areas, many results have been found in these articles (see [9, 10, 12, 18, 23, 24, 26, 27, 28, 29, 30, 31]).

2.5. Two L_p-dual affine surface areas

In 2008, Wang and He (see [21]) gave the definition of L_p-dual affine surface area. Further, Wang and Feng ([3]) made the appropriate improvement as follows: For $K \in \mathscr{S}_o^n$, $n \neq p \geqslant 1$, the L_p-dual affine surface area, $\widetilde{\Omega}_{-p}(K)$, of K is defined by

$$n^{\frac{p}{n}}\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}} = \inf\{n\widetilde{V}_{-p}(K,Q)V(Q^*)^{-\frac{p}{n}} : Q \in \mathscr{S}_c^n\}. \tag{2.14}$$

Afterwards, Wang and Wang ([20], also see [22]) defined another L_p-dual affine surface area as follows: For $K \in \mathscr{S}_o^n$ and $p > 0$, then the L_p-dual affine surface area, $\widetilde{\Omega}_p(K)$, of K is defined by

$$n^{-\frac{p}{n}}\widetilde{\Omega}_p(K)^{\frac{n+p}{n}} = \sup\{n\widetilde{V}_p(K,Q^*)V(Q)^{\frac{p}{n}} : Q \in \mathscr{S}_c^n\}. \tag{2.15}$$

For the studies of above two type of L_p-dual affine surface areas, some results have been obtained in these articles (see [2, 19, 25, 32]).

3. Proofs of Theorems

In this part, we will give the proofs of Theorems 1.1–1.5. In order to prove Theorem 1.1, we need the following lemmas.

LEMMA 3.1. ([25]) *If $K \in \mathscr{F}_o^n$, $p \geqslant 1$ and $\phi \in GL(n)$, then*

$$\Lambda_p \phi K = |det\phi|^{\frac{1}{p}} \phi^{-\tau}\Lambda_p K. \tag{3.1}$$

LEMMA 3.2. ([12]) *If $K \in \mathscr{K}_o^n$, $p \geqslant 1$ and $\phi \in GL(n)$, then*

$$\Omega_p(\phi K) = |det\phi|^{\frac{n-p}{n+p}}\Omega_p(K). \tag{3.2}$$

According to Lemma 3.2, we immediately obtain that:

LEMMA 3.3. *If $K \in \mathscr{K}_o^n$, $p \geqslant 1$ and $c > 0$, then*

$$\Omega_p(cK) = c^{\frac{n(n-p)}{n+p}} \Omega_p(K). \tag{3.3}$$

LEMMA 3.4. ([12]) *If $K \in \mathscr{K}_c^n$, $p \geqslant 1$, then*

$$\Omega_p(K) \leqslant n\omega_n^{\frac{2p}{n+p}} V(K)^{\frac{n-p}{n+p}}, \tag{3.4}$$

with equality if and only if K is an ellipsoid.

Proof of Theorem 1.1. From (2.12), for any $Q \in \mathscr{S}_o^n$, we obtain

$$\Omega_p(\Lambda_p K)^{\frac{n+p}{n}} \leqslant n^{\frac{n+p}{n}} V_p(\Lambda_p K, Q^*) V(Q)^{\frac{p}{n}}.$$

Let $Q = \Lambda_p^* K$, since $\Lambda_p K \in \mathscr{S}_c^n$, associated with (2.5) and (2.7), we get

$$\begin{aligned} \Omega_p(\Lambda_p K)^{\frac{n+p}{n}} &\leqslant n^{\frac{n+p}{n}} V(\Lambda_p K) V(\Lambda_p^* K)^{\frac{p}{n}} \\ &= n^{\frac{n+p}{n}} V(\Lambda_p K)^{\frac{p}{n}} V(\Lambda_p^* K)^{\frac{p}{n}} V(\Lambda_p K)^{\frac{n-p}{n}} \\ &\leqslant n^{\frac{n+p}{n}} \omega_n^{\frac{2p}{n}} V(\Lambda_p K)^{\frac{n-p}{n}}, \end{aligned}$$

i.e.,

$$V(\Lambda_p K)^{\frac{p-n}{n}} \Omega_p(\Lambda_p K)^{\frac{n+p}{n}} \leqslant n^{\frac{n+p}{n}} \omega_n^{\frac{2p}{n}}. \tag{3.5}$$

From (2.13), we have

$$V(\Lambda_p K) = n^{-\frac{n+p}{p}} \omega_n^{-\frac{n}{p}} \Omega_p(K)^{\frac{n+p}{p}}. \tag{3.6}$$

This together with (3.5) yields

$$\Omega_p(\Lambda_p K) \Omega_p(K)^{\frac{p-n}{p}} \leqslant (n\omega_n)^{\frac{2p-n}{p}},$$

i.e., inequality (1.6) is obtained.

Now, we give the equality condition of inequality (1.6). For unit ball B, we know $V(B) = \omega_n$, $\Omega_p(B) = n\omega_n$. If $\Lambda_p K = B$ in left part of (3.5), we get

$$V(B)^{\frac{p-n}{n}} \Omega_p(B)^{\frac{n+p}{n}} = (\omega_n)^{\frac{p-n}{n}} (n\omega_n)^{\frac{n+p}{n}} = n^{\frac{n+p}{n}} \omega_n^{\frac{2p}{n}}. \tag{3.7}$$

Thus, if $\Lambda_p K = B$, then equality holds in (3.5).

Further, for $\phi \in GL(n)$, according to (3.5) and using (3.1), (3.2) and (3.3), we have

$$\begin{aligned} &V(\Lambda_p \phi K)^{\frac{p-n}{n}} \Omega_p(\Lambda_p \phi K)^{\frac{n+p}{n}} \\ &= V(|det\phi|^{\frac{1}{p}} \phi^{-\tau} \Lambda_p K)^{\frac{p-n}{n}} \Omega_p(|det\phi|^{\frac{1}{p}} \phi^{-\tau} \Lambda_p K)^{\frac{n+p}{n}} \\ &= |det\phi|^{\frac{p-n}{p}} |det\phi^{-\tau}|^{\frac{p-n}{n}} V(\Lambda_p K)^{\frac{p-n}{n}} |det\phi|^{\frac{n-p}{p}} |det\phi^{-\tau}|^{\frac{n-p}{n}} \Omega_p(\Lambda_p K)^{\frac{n+p}{n}} \\ &= V(\Lambda_p K)^{\frac{p-n}{n}} \Omega_p(\Lambda_p K)^{\frac{n+p}{n}}. \end{aligned}$$

This means that the left side of (3.5) is affine invariance. Let E denote the ellipsoid and take $E = \phi B$ in left part of (3.5), we see that if $\Lambda_p K$ is an ellipsoid, then equality holds in (3.5).

Conversely, if equality holds in (3.5), by (2.13), we get

$$\Omega_p(\Lambda_p K) = (n\omega_n)^{\frac{2p-n}{p}} \Omega_p(K)^{\frac{n-p}{p}} = n\omega_n^{\frac{2p}{n+p}} V(\Lambda_p K)^{\frac{n-p}{n+p}}. \tag{3.8}$$

This combining with the equality condition of (3.4), we see that $\Lambda_p K$ must be an ellipsoid.

Because of (3.5) and (1.6) are equivalent, thus, equality holds in inequality (1.6) if and only if $\Lambda_p K$ is an ellipsoid. □

According to the (1.4) and (2.13), we immediately get the following result.

LEMMA 3.5. ([12]) *If $K \in \mathscr{K}_c^n$, then*

$$\Omega_p(K) \leqslant n\omega_n^{\frac{2n}{n+p}} V(K^*)^{\frac{p-n}{n+p}}, \tag{3.9}$$

with equality if and only if K is an ellipsoid.

Proof of Theorem 1.2. From (2.12), we get

$$\Omega_p(\Lambda_p^* K)^{\frac{n+p}{n}} \leqslant n^{\frac{n+p}{n}} V_p(\Lambda_p^* K, Q^*) V(Q)^{\frac{p}{n}}.$$

Let $Q = \Lambda_p K$, associated with (2.5) and (2.7), we see that

$$\begin{aligned}\Omega_p(\Lambda_p^* K)^{\frac{n+p}{n}} &\leqslant n^{\frac{n+p}{n}} V(\Lambda_p^* K) V(\Lambda_p K)^{\frac{p}{n}} \\ &\leqslant n^{\frac{n+p}{n}} \omega_n^2 V(\Lambda_p K)^{\frac{p-n}{n}},\end{aligned}$$

i.e.,

$$V(\Lambda_p K)^{\frac{n-p}{n}} \Omega_p(\Lambda_p^* K)^{\frac{n+p}{n}} \leqslant n^{\frac{n+p}{n}} \omega_n^2. \tag{3.10}$$

This and (2.13) give inequality (1.7).

Similar to the deduction of equality condition of inequality (3.5), we know that equality holds in (3.10) if and only if $\Lambda_p K$ is an ellipsoid.

Since (3.10) and (1.7) are equivalent, thus, equality holds in (1.7) if and only if $\Lambda_p K$ is an ellipsoid. □

Proof of Theorem 1.3. For $1 \leqslant p < q < r$ and any $Q_1, Q_3 \in \mathscr{S}_o^n$, there exists $Q_2 \in \mathscr{S}_o^n$ such that

$$\rho(Q_2, \cdot)^{q(r-p)} = \rho(Q_1, \cdot)^{p(r-q)} \rho(Q_3, \cdot)^{r(q-p)}. \tag{3.11}$$

Then for any $u \in S^{n-1}$, this yields

$$\rho(Q_2, u)^n = \rho(Q_1, u)^{\frac{np(r-q)}{q(r-p)}} \rho(Q_3, u)^{\frac{nr(q-p)}{q(r-p)}}.$$

Since $1 \leqslant p < q < r$, then $\frac{q(r-p)}{p(r-q)} > 1$, according to the Hölder's integral inequality and formula (2.10), we get

$$\begin{aligned}
&V(Q_1)^{\frac{p(r-q)}{q(r-p)}} V(Q_3)^{\frac{r(q-p)}{q(r-p)}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \left(\rho(Q_1,u)^{\frac{np(r-q)}{q(r-p)}}\right)^{\frac{q(r-p)}{p(r-q)}} dS(u)\right]^{\frac{p(r-q)}{q(r-p)}} \\
&\quad \times \left[\frac{1}{n}\int_{S^{n-1}} \left(\rho(Q_3,u)^{\frac{nr(q-p)}{q(r-p)}}\right)^{\frac{q(r-p)}{r(q-p)}} dS(u)\right]^{\frac{r(q-p)}{q(r-p)}} \\
&\geqslant \frac{1}{n}\int_{S^{n-1}} \rho(Q_1,u)^{\frac{np(r-q)}{q(r-p)}} \rho(Q_3,u)^{\frac{nr(q-p)}{q(r-p)}} dS(u) \\
&= \frac{1}{n}\int_{S^{n-1}} \rho(Q_2,u)^n dS(u) = V(Q_2).
\end{aligned}$$

i.e.,

$$V(Q_2)^{q(r-p)} \leqslant V(Q_1)^{p(r-q)} V(Q_3)^{r(q-p)}. \tag{3.12}$$

Since for any $1 \leqslant p < q < r$ and $\Lambda_p K, \Lambda_r K \in \mathscr{K}_o^n$, by (1.1) and L_p-Minkowskis existence theorem (see [1] or Theorem 9.2.3 of [5]), we know that there exists $\Lambda_q K \in \mathscr{K}_o^n$ such that

$$f_q(\Lambda_q K, u) = f_p(\Lambda_p K, u)^{\frac{r-q}{r-p}} f_r(\Lambda_r K, u)^{\frac{q-p}{r-p}}. \tag{3.13}$$

Associated with (3.11) and (3.13), we see that for any $u \in S^{n-1}$,

$$\rho(Q_2,u)^{-q} f_q(\Lambda_q K, u) = \left[\rho(Q_1,u)^{-p} f_p(\Lambda_p K, u)\right]^{\frac{r-q}{r-p}} \left[\rho(Q_3,u)^{-r} f_r(\Lambda_r K, u)\right]^{\frac{q-p}{r-p}}.$$

Since $1 \leqslant p < q < r$, then $0 < \frac{r-q}{r-p} < 1$, according to the Hölder's integral inequality and using (2.2) and (2.6), we get

$$\begin{aligned}
&V_p(\Lambda_p K, Q_1^*)^{\frac{r-q}{r-p}} V_r(\Lambda_r K, Q_3^*)^{\frac{q-p}{r-p}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \left((\rho(Q_1,u)^{-p} f_p(\Lambda_p K,u))^{\frac{r-q}{r-p}}\right)^{\frac{r-p}{r-q}} dS(u)\right]^{\frac{r-q}{r-p}} \\
&\quad \times \left[\frac{1}{n}\int_{S^{n-1}} \left((\rho(Q_3,u)^{-r} f_r(\Lambda_r K,u))^{\frac{q-p}{r-p}}\right)^{\frac{r-p}{q-p}} dS(u)\right]^{\frac{q-p}{r-p}} \\
&\geqslant \frac{1}{n}\int_{S^{n-1}} \left((\rho(Q_1,u)^{-p} f_p(\Lambda_p K,u)\right)^{\frac{r-q}{r-p}} \\
&\quad \times \left((\rho(Q_3,u)^{-r} f_r(\Lambda_r K,u)\right)^{\frac{q-p}{r-p}} dS(u) \\
&= V_q(\Lambda_q K, Q_2^*),
\end{aligned}$$

i.e.,

$$V_q(\Lambda_q K, Q_2^*)^{r-p} \leqslant V_p(\Lambda_p K, Q_1^*)^{r-q} V_r(\Lambda_r K, Q_3^*)^{q-p}. \tag{3.14}$$

Hence, combined with (3.12) and (3.14), we get

$$\left(V_q(\Lambda_q K, Q_2^*)V(Q_2)^{\frac{q}{n}}\right)^{r-p} \leqslant \left(V_p(\Lambda_p K, Q_1^*)V(Q_1)^{\frac{p}{n}}\right)^{r-q}\left(V_r(\Lambda_r K, Q_3^*)V(Q_3)^{\frac{r}{n}}\right)^{q-p}.$$

This together with (2.12) yields

$$\Omega_q(\Lambda_q K)^{(n+q)(r-p)} \leqslant \Omega_p(\Lambda_p K)^{(n+p)(r-q)}\Omega_r(\Lambda_r K)^{(n+r)(q-p)}.$$

This gives (1.8). □

Proof of Theorem 1.4. By (2.15), we have

$$\widetilde{\Omega}_p(\Lambda_p K)^{\frac{n+p}{np}} = \sup\{n^{\frac{n+p}{np}}\widetilde{V}_p(\Lambda_p K, Q^*)^{\frac{1}{p}}V(Q)^{\frac{1}{n}} : Q \in \mathscr{S}_c^n\}. \tag{3.15}$$

Since $1 \leqslant p < q < r$ and $\Lambda_p K, \Lambda_r K \in \mathscr{S}_o^n$, there exists $\Lambda_q K \in \mathscr{S}_o^n$ such that

$$\rho(\Lambda_q K, \cdot)^{(n-q)(r-p)} = \rho(\Lambda_p K, \cdot)^{(n-p)(r-q)}\rho(\Lambda_r K, \cdot)^{(n-r)(q-p)}. \tag{3.16}$$

Associated with (3.16), we see that for any $Q \in \mathscr{S}_o^n$ and $u \in S^{n-1}$,

$$\begin{aligned}&\rho(\Lambda_q K, u)^{(n-q)}\rho(Q^*, u)^q \\ =&\left[\rho(\Lambda_p K, u)^{(n-p)}\rho(Q^*, u)^p\right]^{\frac{r-q}{r-p}}\left[\rho(\Lambda_r K, u)^{(n-r)}\rho(Q^*, u)^r\right]^{\frac{q-p}{r-p}}.\end{aligned}$$

Notice that $p < q < r$ implies $0 < \frac{r-q}{r-p} < 1$, according to the Hölder's integral inequality and (2.9), we have

$$\begin{aligned}&\widetilde{V}_p(\Lambda_p K, Q^*)^{\frac{r-q}{r-p}}\widetilde{V}_r(\Lambda_r K, Q^*)^{\frac{q-p}{r-p}} \\ =&\left[\frac{1}{n}\int_{S^{n-1}}\left((\rho(\Lambda_p K, u)^{n-p}\rho(Q^*, u)^p)^{\frac{r-q}{r-p}}\right)^{\frac{r-p}{r-q}}dS(u)\right]^{\frac{r-q}{r-p}} \\ &\times\left[\frac{1}{n}\int_{S^{n-1}}\left((\rho(\Lambda_r K, u)^{n-r}\rho(Q^*, u)^r)^{\frac{q-p}{r-p}}\right)^{\frac{r-p}{q-p}}dS(u)\right]^{\frac{q-p}{r-p}} \\ \geqslant&\frac{1}{n}\int_{S^{n-1}}\left(\rho(\Lambda_p K, u)^{(n-p)}\rho(Q^*, u)^p\right)^{\frac{r-q}{r-p}}\left(\rho(\Lambda_r K, u)^{(n-r)}\rho(Q^*, u)^r\right)^{\frac{q-p}{r-p}}dS(u) \\ =&\widetilde{V}_q(\Lambda_q K, Q^*),\end{aligned}$$

i.e.,

$$\widetilde{V}_q(\Lambda_q K, Q^*)^{r-p} \leqslant \widetilde{V}_p(\Lambda_p K, Q^*)^{r-q}\widetilde{V}_r(\Lambda_r K, Q^*)^{q-p}. \tag{3.17}$$

From the equality condition of Hölder's integral inequality, we see that equality holds in (3.17) if and only if $\Lambda_p K$ and $\Lambda_r K$ are dilates. This together with (3.16) shows that equality holds in (3.17) if and only if $\Lambda_p K$, $\Lambda_q K$ and $\Lambda_r K$ are dilates.

This together with (3.15) yields

$$\left[\widetilde{\Omega}_q(\Lambda_q K)^{\frac{n+q}{nq}}\right]^{q(r-p)} \leqslant \left[\widetilde{\Omega}_p(\Lambda_p K)^{\frac{n+p}{np}}\right]^{p(r-q)} \left[\widetilde{\Omega}_r(\Lambda_r K)^{\frac{n+r}{nr}}\right]^{r(q-p)},$$

i.e.,

$$\widetilde{\Omega}_q(\Lambda_q K)^{(n+q)(r-p)} \leqslant \widetilde{\Omega}_p(\Lambda_p K)^{(n+p)(r-q)} \widetilde{\Omega}_r(\Lambda_r K)^{(n+r)(q-p)}.$$

This gives (1.9).

According to the equality condition of (3.17), we know that equality holds in (1.9) if and only if $\Lambda_p K$, $\Lambda_q K$ and $\Lambda_r K$ are dilates. □

LEMMA 3.6. ([12]) *If $K, L \in \mathscr{K}_o^n$, $1 \leqslant p < q$, then*

$$\left[\frac{V_p(K,L)}{V(K)}\right]^{\frac{1}{p}} \leqslant \left[\frac{V_q(K,L)}{V(K)}\right]^{\frac{1}{q}}, \tag{3.18}$$

with equality if and only if K and L are dilates.

Proof of Theorem 1.5. According to (2.14), we have

$$\widetilde{\Omega}_{-p}(\Lambda_p K)^{\frac{n-p}{n}} = \inf\{n^{\frac{n-p}{n}} \widetilde{V}_{-p}(\Lambda_p K, Q) V(Q^*)^{-\frac{p}{n}} : Q \in \mathscr{S}_c^n\}.$$

This together with (2.11), we see that for any $Q \in \mathscr{S}_c^n$,

$$\widetilde{\Omega}_{-p}(\Lambda_p K)^{\frac{n-p}{n}} = \inf\{n^{\frac{n-p}{n}} \frac{V(\Lambda_p K)}{\omega_n} V_p(K, Q^*) V(Q^*)^{-\frac{p}{n}} : Q \in \mathscr{S}_c^n\}.$$

Hence, by Lemma 3.6, we get for $1 \leqslant p < q$,

$$\begin{aligned}
\left[\frac{\omega_n^n \widetilde{\Omega}_{-p}(\Lambda_p K)^{n-p}}{n^{n-p} V(\Lambda_p K)^n V(K)^{n-p}}\right]^{\frac{1}{p}} &= \inf\left\{\left[\frac{V_p(K,Q^*)}{V(K)}\right]^{\frac{n}{p}} V(K) V(Q^*)^{-1} : Q \in \mathscr{S}_c^n\right\} \\
&\leqslant \inf\left\{\left[\frac{V_q(K,Q^*)}{V(K)}\right]^{\frac{n}{q}} V(K) V(Q^*)^{-1} : Q \in \mathscr{S}_c^n\right\} \\
&= \left[\frac{\omega_n^n \widetilde{\Omega}_{-q}(\Lambda_q K)^{n-q}}{n^{n-q} V(\Lambda_q K)^n V(K)^{n-q}}\right]^{\frac{1}{q}}.
\end{aligned}$$

This gives (1.10).

By the equality condition of Lemma 3.6, we know that equality holds in (1.10) if and only if $\Lambda_p K$ and $\Lambda_q K$ are dilates. □

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