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ON INNER AND OUTER RADII IN MINKOWSKI SPACES

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Abstract. Some sharp bounds for the inner radius and the outer radius of the unit ball of a Minkowski space with respect to its isoperimetrix are known. To find more such bounds is a challenging problem. Related to this, we derive new relations between inner and outer radii as well as cross-section measures for the Holmes-Thompson and Busemann measures.

1. Introduction

This paper refers to the geometry of finite dimensional real Banach spaces, also called *Minkowski spaces*, and to classical convexity. More precisley, the notions of different cross-section measures from classical convexity are used to obtain new results on unit balls and isoperimetrices of Minkowski spaces.

We recall that a *convex body* K in $\mathbb{R}^d, d \geqslant 2$, is a compact, convex set with nonempty interior, and that K is said to be *centered* if it is symmetric with respect to the origin o of $\mathbb{R}^d$. As usual, S^{d-1} denotes the standard Euclidean unit sphere in $\mathbb{R}^d$. We write λ_i for the i*-dimensional Lebesgue measure* (*volume*) in $\mathbb{R}^d$, where $1 \leqslant i \leqslant d$, and instead of λ_d we simply write λ. We denote by $u^\perp$ the $(d-1)$-dimensional subspace orthogonal to $u \in S^{d-1}$, and by l_u the 1-subspace parallel to u.

For a convex body $K \subset \mathbb{R}^d$ we denote by $\lambda_{d-1}(K, u^\perp)$ and $\lambda_1(K, u)$ the $(d-1)$-*dimensional* and 1-*dimensional inner cross-section measures of* K, i.e., the maximal measure of a hyperplane section of K normal to $u \in S^{d-1}$, and the *maximal chord length* of K in the direction u, respectively. Furthermore, $\lambda_1(K|l_u)$ denotes the *width* of K at u, and $\lambda_{d-1}(K|u^\perp)$ the $(d-1)$-*dimensional outer cross-section measure* or *brightness* of K at $u \in S^{d-1}$, where $K|u^\perp$ is the orthogonal projection of K onto $u^\perp$. These notions can be found in the monograph [3]. In [11] and [16] the following results for cross-section measures were derived.

For a convex body K in $\mathbb{R}^d$, $d \geqslant 2$, and every direction $u \in S^{d-1}$ we have

$$\lambda(K) \leqslant \lambda_{d-1}(K|u^\perp)\lambda_1(K,u) \leqslant d\lambda(K),$$

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and both sides are sharp.

On the other hand, for each $u \in S^{d-1}$ a convex body K in $\mathbb{R}^d$, $d \geqslant 2$, satisfies

$$\lambda(K) \leqslant \lambda_{d-1}(K, u^{\perp})\lambda_1(K|l_u) \leqslant d\lambda(K),$$

again with sharpness on both sides.

Our main purpose is to establish connections between cross-section measures and inner/outer radii for centered convex bodies. The inner radius and outer radius of the unit ball with respect to its isoperimetrix in Minkowski spaces (for Holmes-Thompson and Busemann measures) will be used to obtain these connections. Thus, our main results will be related to finite dimensional real Banach spaces as well.

For a convex body K in $\mathbb{R}^d$, the *polar body* K° of K is defined by

$$K^\circ = \{y \in \mathbb{R}^d : \langle x, y\rangle \leqslant 1, x \in K\}.$$

We identify $\mathbb{R}^d$ and its *dual space* $\mathbb{R}^{d*}$ by using the standard basis. In that case, λ_i and λ_i^* coincide in $\mathbb{R}^d$. The symbol ε_i stands for the volume of the standard Euclidean unit ball in $\mathbb{R}^i$.

For a convex body K in $\mathbb{R}^d$ and $u \in S^{d-1}$, the *support function* of K is defined by

$$h_K(u) = \sup\{\langle u, y\rangle : y \in K\},$$

and with o as an interior point of K its *radial function* $\rho_K(u)$ is defined by

$$\rho_K(u) = \max\{\alpha \geqslant 0 : \alpha u \in K\}.$$

It is well known that

$$\rho_{K^\circ}(u) = \frac{1}{h_K(u)}, \ u \in S^{d-1}.$$

If K is a centered convex body, then $2\rho_K(u) = \lambda_1(K \cap l_u)$, and $2h_K(u) = \lambda_1(K|l_u)$ for any $u \in S^{d-1}$.

The *projection body* ΠK of a convex body K in $\mathbb{R}^d$ is defined by $h_{\Pi K}(u) = \lambda_{d-1}(K|u^{\perp})$ for each $u \in S^{d-1}$ (see [3, Chapter 4]). Note that any projection body is a *zonoid* (i.e., a limit of vector sums of segments) centered at the origin. In particular, if K is a polytope, then its projection body is a *zonotope* centered at the origin (see [15] and [4] for many properties and applications of this interesting class of convex bodies). We also refer to [1], [6], [7], and [12] for affine isoperimetric inequalilties related to projection bodies. The *intersection body* IK of a convex body $K \subset \mathbb{R}^d$ is defined by $\rho_{IK}(u) = \lambda_{d-1}(K \cap u^{\perp})$ for each $u \in S^{d-1}$ (cf. [5] and [3, Chapter 8]). If K is a centered convex body, then IK is also a centered convex body (see [2]).

We write $(\mathbb{R}^d, ||\cdot||) = \mathbb{M}^d$ for a d*-dimensional real Banach space*, i.e., a *Minkowski space* with *unit ball* B which is a centered convex body; see [17]. The *unit sphere of* $\mathbb{M}^d$ is the boundary ∂B of the unit ball.

2. Isoperimetrices and inner/outer radii in Minkowski spaces

A Minkowski space $\mathbb{M}^d$ possesses a Haar measure μ, and this measure is unique up to multiplying the Lebesgue measure by a constant, i.e., $\mu = \sigma_B \lambda$.

The following notions are well known; see [17, Chapter 5]. The *d-dimensional Holmes-Thompson volume* of a convex body K in $\mathbb{M}^d$ is defined by

$$\mu_B^{HT}(K) = \frac{\lambda(K)\lambda(B^\circ)}{\varepsilon_d}, \text{ i.e., } \sigma_B = \frac{\lambda(B^\circ)}{\varepsilon_d},$$

and the *d-dimensional Busemann volume* of K is defined by

$$\mu_B^{Bus}(K) = \frac{\varepsilon_d}{\lambda(B)}\lambda(K), \text{ i.e., } \sigma_B = \frac{\varepsilon_d}{\lambda(B)} \text{ (and } \mu_B^{Bus}(B) = \varepsilon_d).$$

In order to define the Minkowski surface area of a convex body, one has to define σ_B similarly in $\mathbb{M}^{d-1}$. That is, for the Holmes-Thompson measure we have $\sigma_B(u) = \lambda_{d-1}((B \cap u^\perp)^\circ)/\varepsilon_{d-1}$, and for the Busemann measure $\sigma_B(u) = \varepsilon_{d-1}/\lambda_{d-1}(B \cap u^\perp)$ (see [17, pp. 150-151]). The *Minkowski surface area* of K can be also defined in terms of mixed volumes (see [14] for notation and more about mixed volumes) by

$$\mu_B(\partial K) = dV(K[d-1], I_B),$$

where I_B is that convex body whose support function is $\sigma_B(u)$. For the Holmes--Thompson measure, I_B is given by $I_B^{HT} = \Pi(B^\circ)/\varepsilon_{d-1}$ (cf. [17, p. 150 and p. 157] for detailed explanation), and therefore it is a centered zonoid. For the Busemann measure we have $I_B^{Bus} = \varepsilon_{d-1}(IB)^\circ$ (see again [17, pp. 150-151]). Among all homothetic images of I_B a unique one is specified, which is called the *isoperimetrix* $\hat{I}_B$ and is determined by $\mu_B(\partial \hat{I}_B) = d\mu_B(\hat{I}_B)$. The *isoperimetrix for the Holmes-Thompson measure* is defined by

$$\hat{I}_B^{HT} = \frac{\varepsilon_d}{\lambda(B^\circ)} I_B^{HT} = \frac{\varepsilon_d}{\varepsilon_{d-1}} \frac{1}{\lambda(B^\circ)} \Pi B^\circ, \tag{1}$$

and the *isoperimetrix for the Busemann measure* by

$$\hat{I}_B^{Bus} = \frac{\lambda(B)}{\varepsilon_d} I_B^{Bus} = \frac{\varepsilon_{d-1}}{\varepsilon_d} \lambda(B)(IB)^\circ; \tag{2}$$

see [17, Chapter 5].

If K and L are convex bodies in $\mathbb{M}^d$, *the inner radius* of K with respect to L is defined by $r(K,L) := \max\{\alpha : \exists x \in \mathbb{M}^d \text{ with } \alpha L \subseteq K + x\}$, and the *outer radius* of K with respect to L is defined by $R(K,L) := \min\{\alpha : \exists x \in \mathbb{M}^d \text{ with } \alpha L \supseteq K + x\}$.

One should notice that $r(K, \hat{I}_B)$ and $R(K, \hat{I}_B)$ can also be defined in terms of the support functions of the involved sets. In particular, if K is a centered convex body, then $r(K, \hat{I}_B)$ is the maximum value of α such that $\alpha \leqslant h_K(u)/h_{\hat{I}_B}(u)$ for all $u \in S^{d-1}$. Similarly, $R(K, \hat{I}_B)$ is the minimum value of α such that $\alpha \geqslant h_K(u)/h_{\hat{I}_B}(u)$ for all $u \in S^{d-1}$ (see [13] and [18]).

3. Cross-section measures and inner/outer radii

For a convex body K, we denote by $w_B(K)$ and $D_B(K)$ the *Minkowskian thickness* (i.e., $w_B(K) = \min_{u \in S^{d-1}} \frac{2w(K,u)}{w(B,u)}$, where $w(K,u)$ is the Euclidean width of K in the direction u) and the *Minkowskian diameter* (i.e., the maximum of this Minkowskian width function of K), respectively.

One can easily see that $r(\hat{I}_B^{HT}, B) = \frac{1}{R(B, \hat{I}_B^{HT})}$ and $R(\hat{I}_B^{HT}, B) = \frac{1}{r(B, \hat{I}_B^{HT})}$. Also, it is easy to establish that if K is a centered convex body in $\mathbb{M}^d$, then $r(K,B) = \frac{w_B(K)}{2}$ and $R(K,B) = \frac{D_B(K)}{2}$.

We recall that some sharp bounds for $r(B, \hat{I}_B^{HT})$ and $R(B, \hat{I}_B^{HT})$ are known (see [8], [9], and also the next section). Below we show the connection between cross-section measures and the outer radius $R(B, \hat{I}_B^{HT})$.

Our first theorem refers to cross-section measures of polars of unit balls and outer radii of isoperimetrices for the Holmes-Thompson measure.

THEOREM 1. *Let B be the unit ball of $\mathbb{M}^d$. Then*

a) $R(B, \hat{I}_B^{HT}) \geqslant 1$ if and only if $\min_{u \in S^{d-1}} \frac{\lambda_{d-1}(B^\circ | u^\perp)\lambda_1(B^\circ \cap l_u)}{\lambda(B^\circ)} \leqslant \frac{2\varepsilon_{d-1}}{\varepsilon_d}$.

b) $R(B, \hat{I}_B^{HT}) \leqslant 1$ if and only if $\frac{\lambda_{d-1}(B^\circ | u^\perp)\lambda_1(B^\circ \cap l_u)}{\lambda(B^\circ)} \geqslant \frac{2\varepsilon_{d-1}}{\varepsilon_d}$ for all $u \in S^{d-1}$.

Proof. As we know,

$$\frac{2}{R(B, \hat{I}_B^{HT})} = 2r(\hat{I}_B^{HT}, B) = w_B(\hat{I}_B^{HT}).$$

We can expand $w_B(\hat{I}_B^{HT})$ as follows:

$$\begin{aligned} w_B(\hat{I}_B^{HT}) &= \min_{u \in S^{d-1}} \frac{2w(\hat{I}_B^{HT}, u)}{w(B,u)} = \min_{u \in S^{d-1}} \frac{2h_{\hat{I}_B^{HT}}(u)}{h_B(u)} \\ &= \min_{u \in S^{d-1}} \frac{2\varepsilon_d}{\lambda(B^\circ)} h_{I_B^{HT}}(u)\rho_{B^\circ}(u) = \min_{u \in S^{d-1}} \frac{2\varepsilon_d}{\varepsilon_{d-1}} \frac{h_{\Pi B^\circ}(u)\rho_{B^\circ}(u)}{\lambda(B^\circ)} \\ &= \min_{u \in S^{d-1}} \frac{\varepsilon_d}{\varepsilon_{d-1}} \frac{\lambda_{d-1}(B^\circ | u^\perp)\lambda_1(B^\circ \cap l_u)}{\lambda(B^\circ)}. \end{aligned}$$

Therefore,

$$\frac{2\varepsilon_{d-1}}{\varepsilon_d} = R(B, \hat{I}_B^{HT}) \min_{u \in S^{d-1}} \frac{\lambda_{d-1}(B^\circ | u^\perp)\lambda_1(B^\circ \cap l_u)}{\lambda(B^\circ)}.$$

Hence, the results follow. □

From Theorem 1 the following result can be easily deduced:

COROLLARY 2. *Let B be the unit ball of $\mathbb{M}^d$. Then* $\dfrac{\lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ\cap l_u)}{\lambda(B^\circ)} \geqslant \dfrac{2\varepsilon_{d-1}}{\varepsilon_d}$ *for all $u \in S^{d-1}$ if and only if $B \subseteq \hat{I}_B^{HT}$.*

We can also use Theorem 1 to get a characterization of ellipsoids.

THEOREM 3. *Let B be the unit ball of $\mathbb{M}^d$. Then*

$$\max_{u\in S^{d-1}} \frac{\lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ\cap l_u)}{\lambda(B^\circ)} = \frac{2\varepsilon_{d-1}}{\varepsilon_d}$$

if and only if B is an ellipsoid.

Proof. As mentioned above,

$$\frac{2}{r(B,\hat{I}_B^{HT})} = 2R(\hat{I}_B^{HT},B) = D_B(\hat{I}_B^{HT}).$$

Then, from the expansion of $D_B(\hat{I}_B^{HT})$ similar to $w_B(\hat{I}_B^{HT})$, one gets

$$D_B(\hat{I}_B^{HT}) = \max_{u\in S^{d-1}} \frac{\varepsilon_d}{\varepsilon_{d-1}} \frac{\lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ\cap l_u)}{\lambda(B^\circ)}.$$

Thus,

$$\frac{2\varepsilon_{d-1}}{\varepsilon_d} = r(B,\hat{I}_B^{HT}) \max_{u\in S^{d-1}} \frac{\lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ\cap l_u)}{\lambda(B^\circ)}.$$

It is known that $r(B,\hat{I}_B^{HT}) \leqslant 1$ with equality if and only if B is an ellipsoid. Hence we have the result. □

It is known that there exists $u \in S^{d-1}$ such that

$$\frac{\lambda_{d-1}(B|u^\perp)\lambda 1(B\cap l_u)}{\lambda(B)} \geqslant \frac{2\varepsilon_{d-1}}{\varepsilon_d},$$

with equality for all $u \in S^{d-1}$ if and only if B is an ellipsoid (cf. [8]).

Also, some sharp bounds for $r(B,\hat{I}_B^{Bus})$ and $R(B,\hat{I}_B^{Bus})$ are known (see [8], [9], or our next section). Below we discuss the connection between cross-section measures and $r(B,\hat{I}_B^{Bus})$ as well as $R(B,\hat{I}_B^{Bus})$.

THEOREM 4. *Let B be the unit ball of $\mathbb{M}^d$. Then*

a) $R(B,\hat{I}_B^{Bus}) \geqslant 1$ *if and only if* $\max_{u\in S^{d-1}} \dfrac{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)} \geqslant \dfrac{2\varepsilon_{d-1}}{\varepsilon_d}$.

b) $R(B,\hat{I}_B^{Bus}) \leqslant 1$ *if and only if* $\dfrac{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)} \leqslant \dfrac{2\varepsilon_{d-1}}{\varepsilon_d}$ *for all $u \in S^{d-1}$.*

Proof. As we know,

$$\frac{2}{R(B,\hat{I}_B^{Bus})} = 2r(\hat{I}_B^{Bus},B) = w_B(\hat{I}_B^{Bus}),$$

and $w_B(\hat{I}_B^{Bus})$ can be expanded as follows:

$$\begin{aligned} w_B(\hat{I}_B^{Bus}) &= \min_{u\in S^{d-1}} \frac{2h_{\hat{I}_B^{Bus}}(u)}{h_B(u)} = \min_{u\in S^{d-1}} \frac{2\lambda(B)\varepsilon_{d-1}h_{(IB)^\circ}(u)}{\varepsilon_d h_B(u)} \\ &= \frac{2\varepsilon_{d-1}}{\varepsilon_d} \min_{u\in S^{d-1}} \frac{\lambda(B)}{\rho_{IB}(u)h_B(u)} = \frac{2\varepsilon_{d-1}}{\varepsilon_d} \min_{u\in S^{d-1}} \frac{2\lambda(B)}{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\varepsilon_d}{2\varepsilon_{d-1}} &= R(B,\hat{I}_B^{Bus}) \min_{u\in S^{d-1}} \frac{\lambda(B)}{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)} \\ &= R(B,\hat{I}_B^{Bus}) \frac{\lambda(B)}{\max_{u\in S^{d-1}} \lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}, \end{aligned}$$

and hence

$$\max_{u\in S^{d-1}} \frac{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)} = R(B,\hat{I}_B^{Bus}) \frac{2\varepsilon_{d-1}}{\varepsilon_d}.$$

Thus, our results are confirmed. □

The next result can be easily deduced from Theorem 4.

COROLLARY 5. *Let B be the unit ball of $\mathbb{M}^d$. Then* $\frac{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)} \leqslant \frac{2\varepsilon_{d-1}}{\varepsilon_d}$ *for all $u \in S^{d-1}$ if and only if $B \subseteq \hat{I}_B^{Bus}$.*

Now we combine inner radii of isoperimetrices for the Busemann measure with cross-section measures.

THEOREM 6. *If B is the unit ball of $\mathbb{M}^d$, then*

a) $r(B,\hat{I}_B^{Bus}) \leqslant 1$ if and only if $\min_{u\in S^{d-1}} \frac{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)} \leqslant \frac{2\varepsilon_{d-1}}{\varepsilon_d}$,

b) $r(B,\hat{I}_B^{Bus}) \geqslant 1$ if and only if $\frac{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)} \geqslant \frac{2\varepsilon_{d-1}}{\varepsilon_d}$ for all $u \in S^{d-1}$.

Proof. By expanding $D_B(\hat{I}_B^{Bus})$ (similar to $w_B(\hat{I}_B^{Bus})$), we obtain

$$D_B(\hat{I}_B^{Bus}) = \frac{2\varepsilon_{d-1}}{\varepsilon_d} \max_{u\in S^{d-1}} \frac{2\lambda(B)}{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}.$$

Since $D_B(\hat{I}_B^{Bus}) = \frac{2}{r(B,\hat{I}_B^{Bus})}$, we get

$$\frac{\varepsilon_d}{2\varepsilon_{d-1}} = \max_{u\in S^{d-1}} \frac{\lambda(B)}{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)} r(B,\hat{I}_B^{Bus}).$$

Hence, the results follow from the following equality:

$$\min_{u\in S^{d-1}} \frac{\lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u)}{\lambda(B)} = r(B,\hat{I}_B^{Bus})\frac{2\varepsilon_{d-1}}{\varepsilon_d}. \quad \square$$

COROLLARY 7. *Let B be the unit ball of $\mathbb{M}^d$. Then* $\frac{\lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u)}{\lambda(B)} \geqslant \frac{2\varepsilon_{d-1}}{\varepsilon_d}$ *for all $u\in S^{d-1}$ if and only if $\hat{I}_B^{Bus}\subseteq B$.*

The following proposition refers to cross-section measures of centered convex bodies.

PROPOSITION 8. *Let B be a centered convex body in $\mathbb{R}^d$. Then*

$$a)\ \max_{u\in S^{d-1}} \lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u) \leqslant \max_{u\in S^{d-1}} \lambda_{d-1}(B|u^{\perp})\lambda_1(B\cap l_u),$$

$$b)\ \min_{u\in S^{d-1}} \lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u) \leqslant \min_{u\in S^{d-1}} \lambda_{d-1}(B|u^{\perp})\lambda_1(B\cap l_u).$$

Proof. a) In [10] it was proved that $R(B,\hat{I}_B^{Bus})\cdot r(B^{\circ},\hat{I}_{B^{\circ}}^{HT})\leqslant 1$. Thus, by using the equalities

$$R(B,\hat{I}_B^{Bus}) = \frac{\varepsilon_d}{2\varepsilon_{d-1}}\max_{u\in S^{d-1}} \frac{\lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u)}{\lambda(B)},$$

$$r(B^{\circ},\hat{I}_{B^{\circ}}^{HT}) = \frac{2\varepsilon_{d-1}}{\varepsilon_d}\frac{\lambda(B)}{\max_{u\in S^{d-1}}\lambda_{d-1}(B|u^{\perp})\lambda_1(B\cap l_u)},$$

the result follows.

b) In [10] it was also proved that $R(B^{\circ},\hat{I}_{B^{\circ}}^{HT})\cdot r(B,\hat{I}_B^{Bus})\leqslant 1$. Thus, by

$$R(B^{\circ},\hat{I}_{B^{\circ}}^{HT}) = \frac{2\varepsilon_{d-1}}{\varepsilon_d}\min_{u\in S^{d-1}} \frac{\lambda(B)}{\lambda_{d-1}(B|u^{\perp})\lambda_1(B\cap l_u)}$$

and

$$r(B,\hat{I}_B^{Bus}) = \frac{\varepsilon_d}{2\varepsilon_{d-1}}\frac{\min_{u\in S^{d-1}}\lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u)}{\lambda(B)},$$

we have the result. $\square$

4. Inclusion results

One of the challenging open problems of Minkowski Geometry is the question whether for $d \geqslant 3$ the homothety of the unit ball and the normalized solution of the isoperimetric problem implies that the unit ball must be an ellipsoid (see [3], [14], and [17]). For Minkowski planes, apart from ellipses, other curves (such as Radon curves) have this property as well. Related to this, we clarify now the inclusions between the unit ball B and ΠB°, and between B and $(IB)^{\circ}$. We start with the inclusions referring to the projection body.

PROPOSITION 9. *If B is a Minkowskian ball of $\mathbb{M}^d$ with $\lambda(B)=1$, then the following exact inclusions hold:*

$$B^{\circ} \subseteq 2\Pi B \subseteq dB^{\circ}.$$

Proof. By the definitions of the inner and outer radii we have

$$r(B,\hat{I}_B^{HT})\hat{I}_B^{HT} \subseteq B \subseteq R(B,\hat{I}_B^{HT})\hat{I}_B^{HT}.$$

We recall the following exact bounds for the inner radius and the outer radius:

$$\frac{2\varepsilon_{d-1}}{d\varepsilon_d} \leqslant r(B,\hat{I}_B^{HT}) \leqslant 1,$$

$$R(B,\hat{I}_B^{HT}) \leqslant \frac{2\varepsilon_{d-1}}{\varepsilon_d}.$$

Therefore, using (1) we get

$$\frac{2\varepsilon_{d-1}}{d\varepsilon_d}\frac{\varepsilon_d}{\lambda(B^{\circ})}\frac{\Pi B^{\circ}}{\varepsilon_{d-1}} \subseteq B \subseteq \frac{2\varepsilon_{d-1}}{\varepsilon_d}\frac{\varepsilon_d}{\lambda(B^{\circ})}\frac{\Pi B^{\circ}}{\varepsilon_{d-1}}.$$

Thus

$$\frac{2}{d}\Pi B^{\circ} \subseteq \lambda(B^{\circ})B \subseteq 2\Pi B^{\circ}.$$

Hence, the results follow by setting B to be B°, and $\lambda(B)=1$. □

Analogously, we obtain for the intersection body

PROPOSITION 10. *Let B be a Minkowskian ball of $\mathbb{M}^d$ with $\lambda(B)=1$. Then the following exact inclusions hold:*

$$B^{\circ} \subseteq 2IB \subseteq dB^{\circ}.$$

Proof. Again we have

$$r(B,\hat{I}_B^{Bus})\hat{I}_B^{Bus} \subseteq B \subseteq R(B,\hat{I}_B^{Bus})\hat{I}_B^{Bus}.$$

From the exact bounds

$$\frac{\varepsilon_d}{2\varepsilon_{d-1}} \leqslant r(B,\hat{I}_B^{Bus}), \quad R(B,\hat{I}_B^{Bus}) \leqslant \frac{d\varepsilon_d}{2\varepsilon_{d-1}}$$

and (2) we have

$$\frac{\varepsilon_d}{2\varepsilon_{d-1}}\frac{\lambda(B)}{\varepsilon_d}\varepsilon_{d-1}(IB)^{\circ} \subseteq B \subseteq \frac{d\varepsilon_d}{2\varepsilon_{d-1}}\frac{\lambda(B)}{\varepsilon_d}\varepsilon_{d-1}(IB)^{\circ}.$$

Thus

$$\frac{\lambda(B)}{2}(IB)^{\circ} \subseteq B \subseteq \frac{d\lambda(B)}{2}(IB)^{\circ}.$$

Setting $\lambda(B) = 1$, we have

$$\frac{2}{d}IB \subseteq B^{\circ} \subseteq 2IB,$$

and the results follow. □

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