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LYAPUNOV–TYPE INEQUALITIES FOR FRACTIONAL DIFFERENTIAL EQUATIONS UNDER MULTI–POINT BOUNDARY CONDITIONS

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Abstract. In this work, we establish new Lyapunov-type inequalities for fractional differential equations under multi-point boundary conditions.

1. Introduction

The well-known result of Lyapunov [9] states that if $u(t)$ is a nontrivial solution of the differential system

$$\begin{aligned} &u''(t)+r(t)u(t)=0, \quad t\in(a,b),\\ &u(a)=0=u(b), \end{aligned} \tag{1.1}$$

where $r(t)$ is a continuous function defined in $[a,b]$, then

$$\int_a^b |r(t)|dt > \frac{4}{b-a}, \tag{1.2}$$

and the constant 4 cannot be replaced by a larger number.

Lyapunov inequality (1.2) is a useful tool in various branches of mathematics including disconjugacy, oscillation theory, and eigenvalue problems. Many improvements and generalizations of the inequality (1.2) have appeared in the literature. A thorough literature review of continuous and discrete Lyapunov-type inequalities and their applications can be found in the survey articles by Cheng [3], Brown and Hinton [1] and Tiryaki [12].

The study of Lyapunov-type inequalities for the differential equation depends on a fractional differential operator was initiated by Rui A. C. Ferreira [4]. He first obtained a Lyapunov-type inequality when the differential equation depends on the Riemann-Liouville fractional derivative, the main result is as follows.

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THEOREM 1.1. *If the following fractional boundary value problem*

$$(D_{a^+}^{\alpha}u)(t)+q(t)u(t)=0, \quad a<t<b,\ 1<\alpha\leqslant 2, \tag{1.3}$$
$$u(a)=0=u(b), \tag{1.4}$$

has a nontrivial solution, where q is a real and continuous function, then

$$\int_a^b |q(s)|ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{1.5}$$

Recently, some Lyapunov-type inequalities were obtained for different fractional boundary value problems. In this direction, we refer to Ferreira [5], Jleli and Samet [6,7], O'Regan and Samet [10], Rong and Bai [11], Wang, Liang and Xia [13] and Cabrera, Sadarangani, and Samet [2].

For example, Cabrera, Sadarangani, and Samet [2] obtain some Lyapunov-type inequalities for a higher-order nonlocal fractional boundary value problem, they give the following Lyapunov inequalities.

THEOREM 1.2. *If the fractional boundary value problem*

$$(D_{a^+}^{\alpha}u)(t)+q(t)u(t)=0, \quad a<t<b,\ 2<\alpha\leqslant 3, \tag{1.6}$$
$$u(a)=u'(a)=0, \quad u'(b)=\beta u(\xi), \tag{1.7}$$

has a nontrivial solution, where q is a real and continuous function, $a<\xi<b, 0\leqslant \beta(\xi-a)^{\alpha-1}<(\alpha-1)(b-a)^{\alpha-2}$, then

$$\int_a^b (b-s)^{\alpha-2}(s-a)|q(s)|ds \geqslant \left(1+\frac{\beta(b-a)^{\alpha-1}}{(\alpha-1)(b-a)^{\alpha-2}-\beta(\xi-a)^{\alpha-1}}\right)^{-1}\Gamma(\alpha). \tag{1.8}$$

THEOREM 1.3. *If the fractional boundary value problem*

$$(D_{a^+}^{\alpha}u)(t)+q(t)u(t)=0, \quad a<t<b,\ 2<\alpha\leqslant 3, \tag{1.9}$$
$$u(a)=u'(a)=0, \quad u'(b)=\beta u(\xi), \tag{1.10}$$

has a nontrivial solution, where q is a real and continuous function, $a<\xi<b, 0\leqslant \beta(\xi-a)^{\alpha-1}<(\alpha-1)(b-a)^{\alpha-2}$, then

$$\int_a^b |q(s)|ds \geqslant \frac{\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{(b-a)^{\alpha-1}(\alpha-2)^{\alpha-2}}\left(1+\frac{\beta(b-a)^{\alpha-1}}{(\alpha-1)(b-a)^{\alpha-2}-\beta(\xi-a)^{\alpha-1}}\right)^{-1}. \tag{1.11}$$

Motivated by [2], in this paper, we study the problem of finding some Lyapunov-type inequalities for the fractional differential equations with multi-point boundary conditions.

$$(D_{a^+}^{\alpha}u)(t)+q(t)u(t)=0, \quad a<t<b,\ 1<\alpha\leqslant 2, \tag{1.12}$$

$$u(a)=0, \quad (D_{a^+}^{\beta}u)(b)=\sum_{i=1}^{m-2} b_i(D_{a^+}^{\beta}u)(\xi_i), \tag{1.13}$$

where $D_{a^+}^{\alpha}$ denotes the standard Riemann-Liouville fractional derivative of order α, $\alpha > \beta + 1$, $0 \leqslant \beta < 1$, $a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b$, $b_i \geqslant 0 (i = 1, 2, \cdots, m-2), 0 \leqslant \sum_{i=1}^{m-2} b_i(\xi_i - a)^{\alpha-\beta-1} < (b-a)^{\alpha-\beta-1}$ and $q : [a,b] \to \mathbb{R}$ is a continuous function.

2. Preliminaries

In this section, we recall the concepts of the Riemann-Liouville fractional integral, the Riemann-Liouville fractional derivative of order $\alpha \geqslant 0$.

DEFINITION 2.1. [8] Let $\alpha \geqslant 0$ and f be a real function defined on $[a,b]$. The Riemann-Liouville fractional integral of order α is defined by $(I_{a^+}^0 f) \equiv f$ and

$$(I_{a^+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0,\ t \in [a,b].$$

DEFINITION 2.2. [8] The Riemann-Liouville fractional derivative of order $\alpha \geqslant 0$ is defined by $(D_{a^+}^0 f) \equiv f$ and

$$(D_{a^+}^{\alpha} f)(t) = (D^m I_{a^+}^{m-\alpha} f)(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_a^t (t-s)^{m-\alpha-1} f(s) ds,$$

for $\alpha > 0$, where m is the smallest integer greater or equal to α.

LEMMA 2.3. *[8] Assume that $u \in C(a,b) \cap L(a,b)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(a,b) \cap L(a,b)$. Then*

$$I_{a^+}^{\alpha}(D_{a^+}^{\alpha} u)(t) = u(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \cdots + c_n(t-a)^{\alpha-n},$$

where $c_i \in \mathbb{R}, i = 1, 2, \cdots, n$, and $n = [\alpha] + 1$.

LEMMA 2.4. *For $1 < \alpha \leqslant 2, 0 \leqslant \beta < 1$, we have*

$$(D_{a^+}^{\beta}(s-a)^{\alpha-1})(t) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(t-a)^{\alpha-\beta-1}.$$

3. Main results

We begin by writing problems (1.12)-(1.13) in its equivalent integral form.

LEMMA 3.1. *We have that $u \in C[a,b]$ is a solution to the boundary value problem (1.12)-(1.13) if and only if u satisfies the integral equation*

$$u(t) = \int_a^b G(t,s) q(s) u(s) ds + T(t) \int_a^b \left(\sum_{i=1}^{m-2} b_i G(\xi,s) q(s) u(s) \right) ds, \tag{3.1}$$

where Green's function $G(t,s)$ *is defined by*

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}}-(t-s)^{\alpha-1}, & a\leqslant s\leqslant t\leqslant b,\\ \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}}, & a\leqslant t\leqslant s\leqslant b.\end{cases}$$

$$T(t)=\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}-\sum_{i=1}^{m-2}b_i(\xi_i-a)^{\alpha-\beta-1}},\quad a\leqslant t\leqslant b.$$

Proof. From Lemma 2.3, $u\in C[a,b]$ is a solution to the boundary value problem (1.12)-(1.13) if and only if

$$u(t)=c_1(t-a)^{\alpha-1}+c_2(t-a)^{\alpha-2}-(I_{a^+}^{\alpha}qu)(t),$$

for some real constants c_1,c_2. Using the boundary condition $u(a)=0$, we obtain $c_2=0$. Therefore

$$u(t)=c_1(t-a)^{\alpha-1}-(I_{a^+}^{\alpha}qu)(t).$$

We apply the operator $D_{a^+}^{\beta}$ to both side of above equation, we obtain

$$\begin{aligned}(D_{a^+}^{\beta}u)(t)&=c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(t-a)^{\alpha-\beta-1}-(I_{a^+}^{\alpha-\beta}qu)(t)\\&=c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(t-a)^{\alpha-\beta-1}-\frac{1}{\Gamma(\alpha-\beta)}\int_a^t(t-s)^{\alpha-\beta-1}q(s)u(s)ds,\end{aligned}$$

the boundary condition $(D_{a^+}^{\beta}u)(b)=\sum_{i=1}^{m-2}b_i(D_{a^+}^{\beta}u)(\xi_i)$ imply that

$$\begin{aligned}&c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(b-a)^{\alpha-\beta-1}-\frac{1}{\Gamma(\alpha-\beta)}\int_a^b(b-s)^{\alpha-\beta-1}q(s)u(s)ds\\&=\sum_{i=1}^{m-2}b_i\left[c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(\xi_i-a)^{\alpha-\beta-1}-\frac{1}{\Gamma(\alpha-\beta)}\int_a^{\xi_i}(\xi_i-s)^{\alpha-\beta-1}q(s)u(s)ds\right],\end{aligned}$$

thus

$$\begin{aligned}c_1=&\frac{1}{[(b-a)^{\alpha-\beta-1}-\sum_{i=1}^{m-2}b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)}\int_a^b(b-s)^{\alpha-\beta-1}q(s)u(s)ds\\&-\frac{1}{[(b-a)^{\alpha-\beta-1}-\sum_{i=1}^{m-2}b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)}\sum_{i=1}^{m-2}b_i\int_a^{\xi_i}(\xi_i-s)^{\alpha-\beta-1}q(s)u(s)ds.\end{aligned}$$

By the relation

$$\begin{aligned}&\frac{1}{(b-a)^{\alpha-\beta-1}-\sum_{i=1}^{m-2}b_i(\xi_i-a)^{\alpha-\beta-1}}\\&=\frac{1}{(b-a)^{\alpha-\beta-1}}+\frac{\sum_{i=1}^{m-2}b_i(\xi_i-a)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}[(b-a)^{\alpha-\beta-1}-\sum_{i=1}^{m-2}b_i(\xi_i-a)^{\alpha-\beta-1}]},\end{aligned}$$

we obtain

$$
\begin{aligned}
c_1 =&\frac{1}{\Gamma(\alpha)}\int_a^b \frac{(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}}q(s)u(s)ds+\frac{\sum_{i=1}^{m-2} b_i\int_a^b \frac{(\xi_i-a)^{\alpha-\beta-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}}q(s)u(s)ds}{[(b-a)^{\alpha-\beta-1}-\sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)}\\
&-\frac{\sum_{i=1}^{m-2} b_i\int_a^{\xi_i}(\xi_i-s)^{\alpha-\beta-1}q(s)u(s)ds}{[(b-a)^{\alpha-\beta-1}-\sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)},
\end{aligned}
$$

therefore

$$
\begin{aligned}
u(t) =&c_1(t-a)^{\alpha-1}-(I_{a^+}^{\alpha}qu)(t)\\
=&\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}\int_a^b \frac{(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}}q(s)u(s)ds-\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}q(s)u(s)ds\\
&+\frac{(t-a)^{\alpha-1}\sum_{i=1}^{m-2} b_i\int_a^b \frac{(\xi_i-a)^{\alpha-\beta-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}}q(s)u(s)ds}{[(b-a)^{\alpha-\beta-1}-\sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)}\\
&-\frac{(t-a)^{\alpha-1}\sum_{i=1}^{m-2} b_i\int_a^{\xi_i}(\xi_i-s)^{\alpha-\beta-1}q(s)u(s)ds}{[(b-a)^{\alpha-\beta-1}-\sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)}\\
=&\int_a^b G(t,s)q(s)u(s)ds+T(t)\int_a^b\left(\sum_{i=1}^{m-2} b_iG(\xi,s)q(s)u(s)\right)ds,
\end{aligned}
$$

which concludes the proof. □

LEMMA 3.2. *The Green's function G defined in Lemma 3.1 satisfies the following properties:*

(**i**) $0\leqslant G(t,s)\leqslant G(s,s)=\dfrac{(s-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}\Gamma(\alpha)}$,

(**ii**) *For any $s\in[a,b]$,*

$$
\max_{s\in[a,b]} G(s,s)=G(s^*,s^*)=(\alpha-\beta-1)^{\alpha-\beta-1}\frac{(\alpha-1)^{\alpha-1}(b-a)^{\alpha-1}}{(2\alpha-\beta-2)^{2\alpha-\beta-2}\Gamma(\alpha)},
$$

where $s^*=\frac{\alpha-\beta-1}{2\alpha-\beta-2}a+\frac{\alpha-1}{2\alpha-\beta-2}b$.

Proof. (**i**) Let us define two functions

$$
\begin{aligned}
g_1(t,s)&=\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}}-(t-s)^{\alpha-1},\ a\leqslant s\leqslant t\leqslant b,\\
g_2(t,s)&=\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}},\qquad a\leqslant t\leqslant s\leqslant b.
\end{aligned}
$$

Obviously, $g_2(t,s)$ is an increasing function in t and $0\leqslant g_2(t,s)\leqslant g_2(s,s)$. Now we turn our attention to the function $g_1(t,s)$. By the relation $\alpha>\beta+1, 2-\alpha\geqslant 0$, we have $0 < \left(\frac{b-s}{b-a}\right)^{\alpha-\beta-1} \leqslant 1$, $0 < \frac{1}{(t-a)^{2-\alpha}} \leqslant \frac{1}{(t-s)^{2-\alpha}}$, so we obtain

$$\frac{\partial g_1(t,s)}{\partial t} = (\alpha-1)\left[\left(\frac{b-s}{b-a}\right)^{\alpha-\beta-1}\frac{1}{(t-a)^{2-\alpha}} - \frac{1}{(t-s)^{2-\alpha}}\right] \leqslant 0.$$

Hence, for a given $s \in [a,b]$, $g_1(t,s)$ is an non-increasing function of $t \in [s,b]$. Therefore, we have

$$g_1(b,s) \leqslant g_1(t,s) \leqslant g_1(s,s).$$

As

$$\begin{aligned} g_1(b,s) &= \frac{(b-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}} - (b-s)^{\alpha-1} \\ &= (b-a)^{\beta}(b-s)^{\alpha-\beta-1} - (b-s)^{\alpha-1} \\ &= (b-a)^{\beta}(b-s)^{\alpha-1}\left[\frac{1}{(b-s)^{\beta}} - \frac{1}{(b-a)^{\beta}}\right] \\ &\geqslant 0, \end{aligned}$$

so we get

$$0 \leqslant g_1(t,s) \leqslant g_1(s,s),$$

thus

$$0 \leqslant G(t,s) \leqslant G(s,s).$$

(ii) Let $\varphi(s) = (s-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}$, $s \in [a,b]$, then

$$\varphi'(s) = (s-a)^{\alpha-2}(b-s)^{\alpha-\beta-2}[(\alpha-1)(b-s) - (\alpha-\beta-1)(s-a)],\ s \in (a,b),$$

moreover,

$$\varphi'(s) = 0,\ s \in (a,b) \Leftrightarrow s = s^* = \frac{\alpha-\beta-1}{2\alpha-\beta-2}a + \frac{\alpha-1}{2\alpha-\beta-2}b.$$

It is easy to check that $\varphi''(s) < 0, s \in (a,b)$, therefore,

$$\max_{s\in[a,b]} \varphi(s) = \varphi(s^*) = (\alpha-\beta-1)^{\alpha-\beta-1}\frac{(\alpha-1)^{\alpha-1}(b-a)^{2\alpha-\beta-2}}{(2\alpha-\beta-2)^{2\alpha-\beta-2}},$$

hence

$$\max_{s\in[a,b]} G(s,s) = G(s^*,s^*) = (\alpha-\beta-1)^{\alpha-\beta-1}\frac{(\alpha-1)^{\alpha-1}(b-a)^{\alpha-1}}{(2\alpha-\beta-2)^{2\alpha-\beta-2}\Gamma(\alpha)}.$$

□

Now, we are ready to prove our first Lyapunov-type inequality.

THEOREM 3.3. *If a nontrivial continuous solution of the fractional boundary value problem*

$$(D_{a^+}^{\alpha}u)(t)+q(t)u(t)=0, \quad a<t<b,\ 1<\alpha\leqslant 2,$$
$$u(a)=0, \quad (D_{a^+}^{\beta}u)(b)=\sum_{i=1}^{m-2} b_i(D_{a^+}^{\beta}u)(\xi_i),$$

exists, then

$$\int_a^b (s-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}|q(s)|ds$$
$$\geqslant (b-a)^{\alpha-\beta-1}\cdot\frac{(b-a)^{\alpha-\beta-1}-\sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}+(b-a)^{\alpha-1}\sum_{i=1}^{m-2} b_i-\sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}}\Gamma(\alpha).$$

Proof. Let $B=C[a,b]$ be the Banach space endowed with norm $\|u\|=\sup_{t\in[a,b]}|u(t)|$. It follows from Lemma 3.1 that a solution u to the boundary value problem satisfies the integral equation

$$u(t)=\int_a^b G(t,s)q(s)u(s)ds+T(t)\int_a^b\left(\sum_{i=1}^{m-2} b_iG(\xi,s)q(s)u(s)\right)ds.$$

Now, using Lemma 3.2 (i), we obtain

$$\|u\|\leqslant\|u\|\int_a^b |G(s,s)||q(s)|ds+\|u\|\sum_{i=1}^{m-2} b_iT(b)\int_a^b |G(s,s)||q(s)|ds,$$

which yields

$$\|u\|\leqslant\|u\|\int_a^b\left(1+\sum_{i=1}^{m-2} b_iT(b)\right)G(s,s)|q(s)|ds.$$

Therefore, if u is a nontrivial continuous solution to (1.12)-(1.13), we have

$$\int_a^b (s-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}|q(s)|ds\geqslant\frac{(b-a)^{\alpha-\beta-1}\Gamma(\alpha)}{1+\sum_{i=1}^{m-2} b_iT(b)}$$
$$=(b-a)^{\alpha-\beta-1}\cdot\frac{(b-a)^{\alpha-\beta-1}-\sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}+(b-a)^{\alpha-1}\sum_{i=1}^{m-2} b_i-\sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}}\Gamma(\alpha).$$

□

Now, from Theorem 3.3 and Lemma 3.2 (ii), if problem (1.12)-(1.13) has a nontrivial continuous solution, then we have the following result.

COROLLARY 3.4. *If a nontrivial continuous solution of the fractional boundary value problem*

$$(D_{a^+}^{\alpha}u)(t)+q(t)u(t)=0, \quad a<t<b,\ 1<\alpha\leqslant 2,$$
$$u(a)=0, \quad (D_{a^+}^{\beta}u)(b)=\sum_{i=1}^{m-2} b_i(D_{a^+}^{\beta}u)(\xi_i),$$

exists, then

$$\int_a^b |q(s)|ds \geqslant \frac{\Gamma(\alpha)}{[(\alpha-1)(b-a)]^{\alpha-1}} \cdot \frac{(2\alpha-\beta-2)^{2\alpha-\beta-2}}{(\alpha-\beta-1)^{\alpha-\beta-1}} \cdot \frac{(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1} + (b-a)^{\alpha-1}\sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}}.$$

Let $\beta = 0$ in Theorem 3.3, we obtain

COROLLARY 3.5. *If a nontrivial continuous solution of the fractional boundary value problem*

$$(D_{a^+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \leqslant 2,$$
$$u(a) = 0, \quad u(b) = \sum_{i=1}^{m-2} b_i u(\xi_i),$$

exists, then

$$\int_a^b (s-a)^{\alpha-1}(b-s)^{\alpha-1}|q(s)|ds \geqslant (b-a)^{\alpha-1} \cdot \frac{(b-a)^{\alpha-1} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-1}}{(b-a)^{\alpha-1}(1+\sum_{i=1}^{m-2} b_i) - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-1}}\Gamma(\alpha).$$

Let $\beta = 0$ in Corollary 3.4, we have the following result.

COROLLARY 3.6. *If a nontrivial continuous solution of the fractional boundary value problem*

$$(D_{a^+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \leqslant 2,$$
$$u(a) = 0, \quad u(b) = \sum_{i=1}^{m-2} b_i u(\xi_i),$$

exists, then

$$\int_a^b |q(s)|ds \geqslant \left(\frac{4}{b-a}\right)^{\alpha-1} \cdot \frac{(b-a)^{\alpha-1} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-1}}{(b-a)^{\alpha-1}(1+\sum_{i=1}^{m-2} b_i) - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-1}}\Gamma(\alpha).$$

REMARK 3.7. Let $b_1 = b_2 = \cdots = b_{m-2} = 0$ in Corollary 3.6, then we obtain (1.5).

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