Journal of
Mathematical
Inequalities
Volume 13, Number 3 (2019), 655–666 doi:10.7153/jmi-2019-13-43

L_p–DUAL AFFINE SURFACE AREAS FOR THE GENERAL L_p–CENTROID BODIES

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(*Communicated by J. Pečarić*)

Abstract. Lutwak and Zhang proposed the concept of L_p-centroid bodies. Further, Haberl and Schuster extended this notion to the general L_p-centroid bodies. In this paper, associated with the L_p-dual affine surface areas, we give the extremum values of polar for the general L_p-centroid bodies. Moreover, the L_p-dual affine surface area forms of the Brunn-Minkowski type inequality and a monotone inequality are established.

1. Introduction and main results

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. For the set of star bodies (about the origin), the set of star bodies whose centroid lie at the origin and the set of origin-symmetric star bodies in $\mathbb{R}^n$, we write $\mathscr{S}_o^n$, $\mathscr{S}_c^n$ and $\mathscr{S}_{os}^n$, respectively. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$ and $V(K)$ denote the n-dimensional volume of a body K. For the standard unit ball B in $\mathbb{R}^n$, its volume is written by $\omega_n = V(B)$.

The notion of centroid body was introduced by Petty ([18]). For a compact set K, the centroid body, ΓK, of K is an origin-symmetric convex body whose support function is defined by (see [8])

$$h(\Gamma K, u) = \frac{1}{V(K)} \int_K |u \cdot x| dx,$$

for all $u \in S^{n-1}$. The centroid body is one of the most important notions in the Brunn-Minkowski theory. In the recent 30 years, the centroid bodies have attracted increasing attention (see [8, 21]).

In 1997, Lutwak and Zhang ([14]) introduced the notion of L_p-centroid bodies. For each compact star-shaped (about the origin) K in $\mathbb{R}^n$ and real number $p \geqslant 1$,

Mathematics subject classification (2010): 52A20, 52A40, 52A39.
Keywords and phrases: General L_p-centroid body, L_p-dual affine surface area, extremum value, Brunn-Minkowski type inequality, monotone inequality.
Research is supported in part by the Natural Science Foundation of China (Grant No.11371224) and Innovation Foundation of Graduate Student of China Three Gorges University (Grant No. 2019SSPY145).
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the L_p-centroid body, $\Gamma_p K$, of K is an origin-symmetric convex body whose support function is defined by

$$h(\Gamma_p K,u)^p = \frac{1}{c_{n,p}V(K)}\int_K |u\cdot x|^p dx = \frac{1}{c_{n,p}(n+p)V(K)}\int_{S^{n-1}} |u\cdot v|^p \rho(K,v)^{n+p} dS(v), \tag{1.1}$$

for all $u \in S^{n-1}$. Here

$$c_{n,p} = \omega_{n+p}/\omega_2\omega_n\omega_{p-1},$$

and $dS(v)$ denotes the standard spherical Lebesgue measure on S^{n-1}. The normalization in (1.1) is chosen such that $\Gamma_p B = B$. Regarding the investigations of L_p-centroid bodies, we may refer to [1, 2, 3, 4, 15, 23, 24, 26, 27, 32].

In 2005, Ludwig ([16]) introduced a function $\varphi_\tau : \mathbb{R} \to [0,+\infty)$ by

$$\varphi_\tau(t) = |t| + \tau t, \tag{1.2}$$

with a parameter $\tau \in [-1,1]$.

Based on L_p-centroid bodies and definition (1.2), Ludwig ([16]) defined a corresponding notion of general L_p-centroid bodies (in fact, the author defined the general L_p-moment body which is a dilatation of the general L_p-centroid body. For the definition of general L_p-centroid body, can see [6]). For $K \in \mathscr{S}_o^n$, $p \geqslant 1$ and $\tau \in [-1,1]$, the general L_p-centroid body, $\Gamma_p^\tau K$, of K is a convex body whose support function is defined by

$$h(\Gamma_p^\tau K,u)^p = \frac{2}{c_{n,p}(\tau)V(K)}\int_K \varphi_\tau(u\cdot x)^p dx = \frac{2}{c_{n,p}(\tau)(n+p)V(K)}\int_{S^{n-1}} \varphi_\tau(u\cdot v)^p \rho(K,v)^{n+p} dS(v), \tag{1.3}$$

where

$$c_{n,p}(\tau) = c_{n,p}[(1+\tau)^p + (1-\tau)^p].$$

The normalization in (1.3) is chosen such that $\Gamma_p^\tau B = B$ for every $\tau \in [-1,1]$ and $\Gamma_p^0 K = \Gamma_p K$. For the more investigations of general L_p-centroid bodies, see [6, 19, 25].

In 2009, Haberl and Schuster ([11]) introduced the notion of asymmetric L_p-centroid bodies (they actually defined the asymmetric L_p-moment body which is a dilatation of the asymmetric L_p-centroid body) as follows: For $K \in \mathscr{S}_o^n$, $p \geqslant 1$, the asymmetric L_p-centroid body, $\Gamma_p^+ K$, of K is the convex body whose support function is defined by

$$h(\Gamma_p^+ K,u)^p = \frac{2}{c_{n,p}V(K)}\int_K (u\cdot x)_+^p dx = \frac{2}{c_{n,p}(n+p)V(K)}\int_{S^{n-1}} (u\cdot v)_+^p \rho(K,v)^{n+p} dS(v), \tag{1.4}$$

where $(u\cdot x)_+ = \max\{u\cdot x, 0\}$ for all $u \in \mathscr{S}^{n-1}$. From (1.4) we see that $\Gamma_p^+ B = B$. In [11] Haberl and Schuster also defined

$$\Gamma_p^- K = \Gamma_p^+(-K). \tag{1.5}$$

From the definitions of $\Gamma_p^{\pm} K$ and (1.3), Haberl and Schuster ([11]) deduced that for $K \in \mathscr{S}_o^n$, $p \geqslant 1$ and $\tau \in [-1,1]$,

$$\Gamma_p^{\tau} K = f_1(\tau)\cdot \Gamma_p^+ K +_p f_2(\tau)\cdot \Gamma_p^- K, \tag{1.6}$$

where '$+_p$' denotes the L_p-Minkowski addition, and

$$f_1(\tau) = \frac{(1+\tau)^p}{(1+\tau)^p + (1-\tau)^p}, \quad f_2(\tau) = \frac{(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p}. \tag{1.7}$$

Obviously, by (1.7), we deduce that

$$f_1(-\tau) = f_2(\tau), \quad f_2(-\tau) = f_1(\tau), \tag{1.8}$$

$$f_1(\tau) + f_2(\tau) = 1. \tag{1.9}$$

Setting $\tau = 0$ in (1.6) and combining with (1.7), we see that

$$\Gamma_p K = \frac{1}{2}\cdot \Gamma_p^+ K +_p \frac{1}{2}\cdot \Gamma_p^- K. \tag{1.10}$$

If $\tau = \pm 1$ in (1.6), then by (1.7), $\Gamma_p^{+1} K = \Gamma_p^+ K$, $\Gamma_p^{-1} K = \Gamma_p^- K$. From (1.4), (1.6) and (1.8), we easily obtain for $\tau \in [-1,1]$ (see [25])

$$\Gamma_p^{-\tau} K = \Gamma_p^{\tau}(-K) = -\Gamma_p^{\tau} K. \tag{1.11}$$

In 2010, Wang, Yuan and He ([30]) showed a type of L_p-dual affine surface areas. In 2015, Pei and Wang ([20]) made the following improvement: For $K \in \mathscr{S}_o^n$ and $p > 0$, the L_p-dual affine surface area, $\widetilde{\Omega}_p(K)$, of K is defined by

$$n^{-\frac{p}{n}} \widetilde{\Omega}_p(K)^{\frac{n+p}{n}} = \sup\{n\widetilde{V}_p(K, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathscr{S}_{os}^n\}. \tag{1.12}$$

Here the $\widetilde{V}_p(M,N)$ denotes the L_p-dual mixed volume of $M, N \in \mathscr{S}_o^n$. When $Q \in \mathscr{S}_c^n$, the L_p-dual affine surface area was given by Wang and Wang (see [28]). For the studies of L_p-dual affine surface area, some results have been obtained in these articles (see [5, 22, 29]).

In this paper, associated with the L_p-dual affine surface area, we continuously study general L_p-centroid bodies. Firstly, combined with (1.12), we obtain the extremum values for the L_p-dual affine surface areas of polar of the general L_p-centroid bodies.

THEOREM 1.1. *For $K \in \mathscr{S}_o^n$, $1 \leqslant p < n$ and $\tau \in [-1,1]$, then*

$$\widetilde{\Omega}_p(\Gamma_p^* K) \leqslant \widetilde{\Omega}_p(\Gamma_p^{\tau,*} K) \leqslant \widetilde{\Omega}_p(\Gamma_p^{\pm,*} K). \tag{1.13}$$

If K is not origin-symmetric and p is not an odd integer, then there is equality in the left inequality if and only if $\tau = 0$ and equality in the right inequality if and only if $\tau = \pm 1$. Here, $\Gamma_p^* M$ denotes the polar body of $\Gamma_p M$.

Then, we establish the following L_p-dual affine surface area version of Brunn-Minkowski inequality for the polar of the general L_p-centroid bodies.

THEOREM 1.2. *For* $K,L \in \mathscr{S}_o^n$, $1 \leqslant p < n$ *and* $\tau \in [-1,1]$, *then*

$$\widetilde{\Omega}_p(\Gamma_p^{\tau,*}(K\widehat{+}_pL))^{-\frac{p(n+p)}{n(n-p)}} \geqslant \widetilde{\Omega}_p(\Gamma_p^{\tau,*}K)^{-\frac{p(n+p)}{n(n-p)}} + \widetilde{\Omega}_p(\Gamma_p^{\tau,*}L)^{-\frac{p(n+p)}{n(n-p)}}, \tag{1.14}$$

with equality if and only if $\Gamma_p^{\tau,*}K$ *and* $\Gamma_p^{\tau,*}L$ *are dilates.*

Finally, we give a monotone inequality for the polar of the general L_p-centroid bodies.

THEOREM 1.3. *For* $K,L \in \mathscr{S}_o^n$, $1 \leqslant p < n$ *and* $\tau \in [-1,1]$, *if* $K \subseteq L$, *then*

$$\frac{\widetilde{\Omega}_p(\Gamma_p^{\tau,*}K)^{\frac{n+p}{n}}}{V(K)^{\frac{n-p}{p}}} \leqslant \frac{\widetilde{\Omega}_p(\Gamma_p^{\tau,*}L)^{\frac{n+p}{n}}}{V(L)^{\frac{n-p}{p}}}, \tag{1.15}$$

equality holds when $K = L$.

2. Notation and background material

In order to complete the proofs of Theorems 1.1-1.3, we will require the following notions.

2.1. Support function, radial function and polar of convex bodies

Let $\mathbb{R}$ be the set of real numbers. If $K \in \mathscr{K}^n$, then the support function of K, $h_K = h(K,\cdot) : \mathbb{R}^n \to \mathbb{R}$, is defined by (see [8])

$$h(K,x) = \max\{x\cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x\cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$.

For K is a compact star shaped (about the origin) in $\mathbb{R}^n$, the radial function of K, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \setminus \{0\} \to [0,+\infty)$, is defined by (see [21])

$$\rho(K,x) = \max\{\lambda \geqslant 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

If ρ_K is positive and continuous, K will be called a star body (with respect to the origin).

If E is a nonempty subset and contains the origin in $\mathbb{R}^n$, then the polar set, E^*, of E is defined by (see [8, 21])

$$E^* = \{x \in \mathbb{R}^n : x\cdot y \leqslant 1,\ y \in E\}.$$

Meanwhile, it is easy to get that $(K^*)^* = K$ for $K \in \mathscr{K}_o^n$. Here $\mathscr{K}_o^n$ denotes the set of convex bodies containing the origin in their interiors in $\mathbb{R}^n$.

From the above definitions, we see that if $K \in \mathscr{K}_o^n$, then (see [8, 21])

$$h(K^*,\cdot) = \frac{1}{\rho(K,\cdot)}, \qquad \rho(K^*,\cdot) = \frac{1}{h(K,\cdot)}. \tag{2.1}$$

Associated with (2.1), if $K,L \in \mathscr{K}_o^n$ and $K \subseteq L$, then $K^* \supseteq L^*$.

2.2. L_p-Minkowski combination and L_p-harmonic radial combination

For $K,L \in \mathscr{K}_o^n$, $p \geqslant 1$ and $\lambda,\mu \geqslant 0$ (not both zero), the L_p-Minkowski combination, $\lambda \cdot K +_p \mu \cdot L \in \mathscr{K}_o^n$, of K and L is defined by (see [7, 21])

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K,\cdot)^p + \mu h(L,\cdot)^p, \tag{2.2}$$

where $\lambda \cdot K$ denotes the L_p-Minkowski scalar multiplication and we easily obtain $\lambda \cdot K = \lambda^{\frac{1}{p}} K$.

For $K,L \in \mathscr{S}_o^n$, $p \geqslant 1$ and $\lambda,\mu \geqslant 0$ (not both zero), the L_p-harmonic radial combination, $\lambda \times K \widetilde{+}_{-p} \mu \times L \in \mathscr{S}_o^n$, of K and L is defined by (see [13])

$$\rho(\lambda \times K \widetilde{+}_{-p} \mu \times L, \cdot)^{-p} = \lambda \rho(K,\cdot)^{-p} + \mu \rho(L,\cdot)^{-p}, \tag{2.3}$$

where the operation '$\widetilde{+}_{-p}$' is called L_p-harmonic radial addition, $\lambda \times K$ denotes the L_p-harmonic radial scalar multiplication and we easily obtain $\lambda \times K = \lambda^{-\frac{1}{p}} K$.

From (2.1), (2.2) and (2.3), we easily get that if $K,L \in \mathscr{K}_o^n$, $p \geqslant 1$, and $\lambda,\mu \geqslant 0$ (not both zero), then (see [13])

$$(\lambda \cdot K +_p \mu \cdot L)^* = \lambda \times K^* \widetilde{+}_{-p} \mu \times L^*. \tag{2.4}$$

2.3. L_p-dual mixed volume

For $K,L \in \mathscr{S}_o^n$, then for $p > 0$ and $\lambda,\mu \geqslant 0$ (not both zero), the L_p-radial combination, $\lambda \star K \widetilde{+}_p \mu \star L$, of K and L is given by (see [9, 21])

$$\rho(\lambda \star K \widetilde{+}_p \mu \star L, \cdot)^p = \lambda \rho(K,\cdot)^p + \mu \rho(L,\cdot)^p,$$

where $\lambda \star K$ denotes the L_p-radial scalar multiplication and we easily obtain $\lambda \star K = \lambda^{\frac{1}{p}} K$.

Associated with the L_p-radial combinations of star bodies, the notion of L_p-dual mixed volume as follows: For $K,L \in \mathscr{S}_o^n$, $p > 0$ and $\varepsilon > 0$, the L_p-dual mixed volume, $\widetilde{V}_p(K,L)$, of K and L is given by (see [10, 31])

$$\frac{n}{p}\widetilde{V}_p(K,L) = \lim_{\varepsilon \to 0^+} \frac{V(K \widetilde{+}_p \varepsilon \star L) - V(K)}{\varepsilon}.$$

From above definition, the integral representation of L_p-dual mixed volume can be given by (see [10])

$$\widetilde{V}_p(K,L) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^{n-p} \rho(L,u)^p dS(u), \tag{2.5}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}.

From (2.5), we easily know that

$$\widetilde{V}_p(K,K) = V(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^n dS(u).$$

2.4. L_p-harmonic Blaschke combination

For $K,L\in\mathscr{S}_o^n$, $p\geqslant 1$ and $\lambda,\mu\geqslant 0$ (not both zero), the L_p-harmonic Blaschke combination, $\lambda\circ K\widehat{+}_p\mu\circ L\in\mathscr{S}_o^n$, of K and L is defined by (see [17])

$$\frac{\rho(\lambda\circ K\widehat{+}_p\mu\circ L,\cdot)^{n+p}}{V(\lambda\circ K\widehat{+}_p\mu\circ L)}=\lambda\frac{\rho(K,\cdot)^{n+p}}{V(K)}+\mu\frac{\rho(L,\cdot)^{n+p}}{V(L)},$$

where the operation '$\widehat{+}_p$' is called L_p-harmonic Blaschke addition, $\lambda\circ K$ denotes L_p-harmonic Blaschke scalar multiplication and we easily obtain $\lambda\circ K=\lambda^{\frac{1}{p}}K$. When $\lambda=\mu=1$, $K\widehat{+}_pL$ is called L_p-harmonic Blaschke sum.

3. Proofs of main theorems

In this section, we will prove Theorems 1.1-1.3. To complete the proof of Theorem 1.1, we require the following lemmas.

LEMMA 3.1. *If $K,L\in\mathscr{S}_o^n$, $1\leqslant p<n$ and $\lambda,\mu\geqslant 0$ (not both zero), then for any $Q\in\mathscr{S}_o^n$,*

$$\widetilde{V}_p(\lambda\times K\widetilde{+}_{-p}\mu\times L,Q^*)^{-\frac{p}{n-p}}\geqslant\lambda\widetilde{V}_p(K,Q^*)^{-\frac{p}{n-p}}+\mu\widetilde{V}_p(L,Q^*)^{-\frac{p}{n-p}},$$

with equality if and only if K and L are dilates.

Proof. For $K,L\in\mathscr{S}_o^n$ and $1\leqslant p<n$, thus $-\frac{n-p}{p}<0$. By (2.5) and the Minkowski integral inequality (see[12]), for any $Q\in\mathscr{S}_o^n$, we have

$$\begin{aligned}
&\widetilde{V}_p(\lambda\times K\widetilde{+}_{-p}\mu\times L,Q^*)^{-\frac{p}{n-p}}\\
=&\left[\frac{1}{n}\int_{S^{n-1}}\rho(\lambda\times K\widetilde{+}_{-p}\mu\times L,u)^{n-p}\rho(Q^*,u)^p dS(u)\right]^{-\frac{p}{n-p}}\\
=&\left[\frac{1}{n}\int_{S^{n-1}}\left(\rho(\lambda\times K\widetilde{+}_{-p}\mu\times L,u)^{-p}\rho(Q^*,u)^{-\frac{p^2}{n-p}}\right)^{-\frac{n-p}{p}}dS(u)\right]^{-\frac{p}{n-p}}\\
=&\left\{\frac{1}{n}\int_{S^{n-1}}\left[\left(\lambda\rho(K,u)^{-p}+\mu\rho(L,u)^{-p}\right)\rho(Q^*,u)^{-\frac{p^2}{n-p}}\right]^{-\frac{n-p}{p}}dS(u)\right\}^{-\frac{p}{n-p}}\\
\geqslant&\lambda\left[\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n-p}\rho(Q^*,u)^p dS(u)\right]^{-\frac{p}{n-p}}\\
&+\mu\left[\frac{1}{n}\int_{S^{n-1}}\rho(L,u)^{n-p}\rho(Q^*,u)^p dS(u)\right]^{-\frac{p}{n-p}}\\
=&\lambda\widetilde{V}_p(K,Q^*)^{-\frac{p}{n-p}}+\mu\widetilde{V}_p(L,Q^*)^{-\frac{p}{n-p}}.
\end{aligned}$$

According to the equality condition of Minkowski integral inequality, we see that equality holds if and only if K and L are dilates. □

LEMMA 3.2. *If $K, L \in \mathscr{S}_o^n$, $1 \leqslant p < n$ and $\lambda, \mu \geqslant 0$ (not both zero), then*

$$\widetilde{\Omega}_p(\lambda \times K \widetilde{+}_{-p} \mu \times L)^{-\frac{p(n+p)}{n(n-p)}} \geqslant \lambda \widetilde{\Omega}_p(K)^{-\frac{p(n+p)}{n(n-p)}} + \mu \widetilde{\Omega}_p(L)^{-\frac{p(n+p)}{n(n-p)}}, \tag{3.1}$$

with equality if and only if K and L are dilates.

Proof. Since $1 \leqslant p < n$, thus $-\frac{n-p}{p} < 0$. Combined with Lemma 3.1 and (1.12), we have

$$\begin{aligned}
&\widetilde{\Omega}_p(\lambda \times K \widetilde{+}_{-p} \mu \times L)^{-\frac{p(n+p)}{n(n-p)}} \\
=&\left[\sup\left\{n^{\frac{n+p}{n}} \widetilde{V}_p(\lambda \times K \widetilde{+}_{-p} \mu \times L, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathscr{S}_{os}^n\right\}\right]^{-\frac{p}{n-p}} \\
=&\inf\left\{n^{-\frac{p(n+p)}{n(n-p)}} \widetilde{V}_p(\lambda \times K \widetilde{+}_{-p} \mu \times L, Q^*)^{-\frac{p}{n-p}} V(Q)^{-\frac{p^2}{n(n-p)}} : Q \in \mathscr{S}_{os}^n\right\} \\
\geqslant&\inf\left\{n^{-\frac{p(n+p)}{n(n-p)}} \left[\lambda \widetilde{V}_p(K, Q^*)^{-\frac{p}{n-p}} + \mu \widetilde{V}_p(L, Q^*)^{-\frac{p}{n-p}}\right] V(Q)^{-\frac{p^2}{n(n-p)}} : Q \in \mathscr{S}_{os}^n\right\} \\
\geqslant&\lambda \inf\left\{\left[n^{\frac{n+p}{n}} \widetilde{V}_p(K, Q^*) V(Q)^{\frac{p}{n}}\right]^{-\frac{p}{n-p}} : Q \in \mathscr{S}_{os}^n\right\} \\
&+ \mu \inf\left\{\left[n^{\frac{n+p}{n}} \widetilde{V}_p(L, Q^*) V(Q)^{\frac{p}{n}}\right]^{-\frac{p}{n-p}} : Q \in \mathscr{S}_{os}^n\right\} \\
=&\lambda \left[\sup\left\{n^{\frac{n+p}{n}} \widetilde{V}_p(K, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathscr{S}_{os}^n\right\}\right]^{-\frac{p}{n-p}} \\
&+ \mu \left[\sup\left\{n^{\frac{n+p}{n}} \widetilde{V}_p(L, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathscr{S}_{os}^n\right\}\right]^{-\frac{p}{n-p}} \\
=&\lambda \widetilde{\Omega}_p(K)^{-\frac{p(n+p)}{n(n-p)}} + \mu \widetilde{\Omega}_p(L)^{-\frac{p(n+p)}{n(n-p)}}.
\end{aligned}$$

Thus

$$\widetilde{\Omega}_p(\lambda \times K \widetilde{+}_{-p} \mu \times L)^{-\frac{p(n+p)}{n(n-p)}} \geqslant \lambda \widetilde{\Omega}_p(K)^{-\frac{p(n+p)}{n(n-p)}} + \mu \widetilde{\Omega}_p(L)^{-\frac{p(n+p)}{n(n-p)}}.$$

This yields (3.1). According to the equality condition of Lemma 3.1, we see that equality holds in (3.1) if and only if K and L are dilates. □

LEMMA 3.3. ([25]) *If $K \in \mathscr{S}_o^n$, $p \geqslant 1$ and p is not odd integer, then $\Gamma_p^+ K = \Gamma_p^- K$ if and only if K is origin-symmetric.*

LEMMA 3.4. ([25]) *If $K \in \mathscr{S}_o^n$, $p \geqslant 1$ and p is not odd integer, then for $\tau \in [-1, 1]$ and $\tau \neq 0$, $\Gamma_p^\tau K = \Gamma_p^{-\tau} K$ if and only if K is origin-symmetric.*

LEMMA 3.5. ([25]) *If $K \in \mathscr{S}_o^n$, $p \geqslant 1$ and $\tau \in [-1, 1]$, then*

$$\Gamma_p K = \frac{1}{2} \cdot \Gamma_p^\tau K +_p \frac{1}{2} \cdot \Gamma_p^{-\tau} K.$$

Proof of Theorem 1.1. From (1.6) and (2.4), we have

$$\Gamma_p^{\tau,*}K = f_1(\tau)\times\Gamma_p^{+,*}K\widetilde{+}_{-p}f_2(\tau)\times\Gamma_p^{-,*}K.$$

Combining with (3.1), we have

$$\begin{aligned}\widetilde{\Omega}_p(\Gamma_p^{\tau,*}K)^{-\frac{p(n+p)}{n(n-p)}} &= \widetilde{\Omega}_p(f_1(\tau)\times\Gamma_p^{+,*}K\widetilde{+}_{-p}f_2(\tau)\times\Gamma_p^{-,*}K)^{-\frac{p(n+p)}{n(n-p)}}\\ &\geqslant f_1(\tau)\widetilde{\Omega}_p(\Gamma_p^{+,*}K)^{-\frac{p(n+p)}{n(n-p)}} + f_2(\tau)\widetilde{\Omega}_p(\Gamma_p^{-,*}K)^{-\frac{p(n+p)}{n(n-p)}}. \qquad (3.2)\end{aligned}$$

Since $\Gamma_p^{+,*}K = -\Gamma_p^{-,*}K$ and $Q\in\mathscr{S}_{os}^n$, then $\rho(Q,u)=\rho(-Q,u)=\rho(Q,-u)$ for all $u\in S^{n-1}$, thus by (2.5) we get that

$$\widetilde{V}_p(\Gamma_p^{-,*}K,Q^*) = \widetilde{V}_p(-\Gamma_p^{+,*}K,Q^*) = \widetilde{V}_p(\Gamma_p^{+,*}K,Q^*).$$

Therefore, from definition (1.12), it follows that

$$\widetilde{\Omega}_p(\Gamma_p^{+,*}K) = \widetilde{\Omega}_p(\Gamma_p^{-,*}K). \qquad (3.3)$$

Combining with (3.2), (3.3) and (1.9), we can get

$$\widetilde{\Omega}_p(\Gamma_p^{\tau,*}K)^{-\frac{p(n+p)}{n(n-p)}} \geqslant \widetilde{\Omega}_p(\Gamma_p^{\pm,*}K)^{-\frac{p(n+p)}{n(n-p)}},$$

i.e.,

$$\widetilde{\Omega}_p(\Gamma_p^{\tau,*}K) \leqslant \widetilde{\Omega}_p(\Gamma_p^{\pm,*}K). \qquad (3.4)$$

According to the equality condition of inequality (3.1), we know that equality holds in (3.4) if and only if $\Gamma_p^{+,*}K$ and $\Gamma_p^{-,*}K$ are dilates. Since $\Gamma_p^{+,*}K = -\Gamma_p^{-,*}K$, this means $\Gamma_p^{+,*}K = \Gamma_p^{-,*}K$. Hence, from Lemma 3.3, we see that if K is not origin-symmetric, then equality holds in (3.4) if and only if $\tau=\pm 1$.

Now, we prove the left inequality of (1.13). From Lemma 3.5 and (2.4), we have

$$\Gamma_p^{*}K = \frac{1}{2}\times\Gamma_p^{\tau,*}K\widetilde{+}_{-p}\frac{1}{2}\times\Gamma_p^{-\tau,*}K.$$

Combining with (3.1), we have

$$\begin{aligned}\widetilde{\Omega}_p(\Gamma_p^{*}K)^{-\frac{p(n+p)}{n(n-p)}} &= \widetilde{\Omega}_p\left(\frac{1}{2}\times\Gamma_p^{\tau,*}K\widetilde{+}_{-p}\frac{1}{2}\times\Gamma_p^{-\tau,*}K\right)^{-\frac{p(n+p)}{n(n-p)}}\\ &\geqslant \frac{1}{2}\widetilde{\Omega}_p(\Gamma_p^{\tau,*}K)^{-\frac{p(n+p)}{n(n-p)}} + \frac{1}{2}\widetilde{\Omega}_p(\Gamma_p^{-\tau,*}K)^{-\frac{p(n+p)}{n(n-p)}}. \qquad (3.5)\end{aligned}$$

Due to $\Gamma_p^{-\tau,*}K = -\Gamma_p^{\tau,*}K$ by (1.11), similar to the proof of (3.3), we have

$$\widetilde{\Omega}_p(\Gamma_p^{\tau,*}K) = \widetilde{\Omega}_p(\Gamma_p^{-\tau,*}K). \qquad (3.6)$$

From (3.5) and (3.6), we deduce

$$\widetilde{\Omega}_p(\Gamma_p^{*}K) \leqslant \widetilde{\Omega}_p(\Gamma_p^{\tau,*}K). \qquad (3.7)$$

Using $\Gamma_p^{\tau,*}K = -\Gamma_p^{-\tau,*}K$ and the equality condition of inequality (3.1), we know that equality holds in (3.7) if and only if $\Gamma_p^{\tau,*}K = \Gamma_p^{-\tau,*}K$. By Lemma 3.4, we see that if K is not origin-symmetric, then equality holds in (3.7) if and only if $\tau=0$. □

LEMMA 3.6. ([6]) *If* $K,L\in\mathscr{S}_o^n$, $p\geqslant 1$ *and* $\tau\in[-1,1]$, *then*

$$\rho(\Gamma_p^{\tau,*}(K\widehat{+}_pL),u)^{-p}=\rho(\Gamma_p^{\tau,*}K,u)^{-p}+\rho(\Gamma_p^{\tau,*}L,u)^{-p}. \tag{3.8}$$

LEMMA 3.7. *If* $K,L\in\mathscr{S}_o^n$, $1\leqslant p<n$ *and* $\tau\in[-1,1]$, *then for any* $Q\in\mathscr{S}_o^n$,

$$\widetilde{V}_p(\Gamma_p^{\tau,*}(K\widehat{+}_pL),Q^*)^{-\frac{p}{n-p}}\geqslant\widetilde{V}_p(\Gamma_p^{\tau,*}K,Q^*)^{-\frac{p}{n-p}}+\widetilde{V}_p(\Gamma_p^{\tau,*}L,Q^*)^{-\frac{p}{n-p}}, \tag{3.9}$$

with equality if and only if $\Gamma_p^{\tau,*}K$ *and* $\Gamma_p^{\tau,*}L$ *are dilates.*

Proof. Since $1\leqslant p<n$, thus $-\frac{n-p}{p}<0$. Hence by (2.5), (3.8) and the Minkowski integral inequality (see[12]), for any $Q\in\mathscr{S}_o^n$, we have that

$$\begin{aligned}
&\widetilde{V}_p(\Gamma_p^{\tau,*}(K\widehat{+}_pL),Q^*)^{-\frac{p}{n-p}}\\
=&\left[\frac{1}{n}\int_{S^{n-1}}\rho(\Gamma_p^{\tau,*}(K\widehat{+}_pL),u)^{n-p}\rho(Q^*,u)^p dS(u)\right]^{-\frac{p}{n-p}}\\
=&\left[\frac{1}{n}\int_{S^{n-1}}\left(\rho(\Gamma_p^{\tau,*}(K\widehat{+}_pL),u)^{-p}\rho(Q^*,u)^{-\frac{p^2}{n-p}}\right)^{-\frac{n-p}{p}}dS(u)\right]^{-\frac{p}{n-p}}\\
=&\left\{\frac{1}{n}\int_{S^{n-1}}\left[\left(\rho(\Gamma_p^{\tau,*}K,u)^{-p}+\rho(\Gamma_p^{\tau,*}L,u)^{-p}\right)\rho(Q^*,u)^{-\frac{p^2}{n-p}}\right]^{-\frac{n-p}{p}}dS(u)\right\}^{-\frac{p}{n-p}}\\
\geqslant&\left[\frac{1}{n}\int_{S^{n-1}}\rho(\Gamma_p^{\tau,*}K,u)^{n-p}\rho(Q^*,u)^p dS(u)\right]^{-\frac{p}{n-p}}\\
&+\left[\frac{1}{n}\int_{S^{n-1}}\rho(\Gamma_p^{\tau,*}L,u)^{n-p}\rho(Q^*,u)^p dS(u)\right]^{-\frac{p}{n-p}}\\
=&\widetilde{V}_p(\Gamma_p^{\tau,*}K,Q^*)^{-\frac{p}{n-p}}+\widetilde{V}_p(\Gamma_p^{\tau,*}L,Q^*)^{-\frac{p}{n-p}}.
\end{aligned}$$

According to the equality condition of Minkowski integral inequality, we see that equality holds in (3.9) if and only if $\Gamma_p^{\tau,*}K$ and $\Gamma_p^{\tau,*}L$ are dilates. □

Proof of Theorem 1.2. Since $-\frac{n-p}{p}<0$, thus by definition (1.12) and inequality (3.9), we obtain

$$\begin{aligned}
&\widetilde{\Omega}_p(\Gamma_p^{\tau,*}(K\widehat{+}_pL))^{-\frac{p(n+p)}{n(n-p)}}\\
=&\left[\sup\left\{n^{\frac{n+p}{n}}\widetilde{V}_p(\Gamma_p^{\tau,*}(K\widehat{+}_pL),Q^*)V(Q)^{\frac{p}{n}}:Q\in\mathscr{S}_{os}^n\right\}\right]^{-\frac{p}{n-p}}\\
=&\inf\left\{n^{-\frac{p(n+p)}{n(n-p)}}\widetilde{V}_p(\Gamma_p^{\tau,*}(K\widehat{+}_pL),Q^*)^{-\frac{p}{n-p}}V(Q)^{-\frac{p^2}{n(n-p)}}:Q\in\mathscr{S}_{os}^n\right\}\\
\geqslant&\inf\left\{n^{-\frac{p(n+p)}{n(n-p)}}\left[\widetilde{V}_p(\Gamma_p^{\tau,*}K,Q^*)^{-\frac{p}{n-p}}+\widetilde{V}_p(\Gamma_p^{\tau,*}L,Q^*)^{-\frac{p}{n-p}}\right]V(Q)^{-\frac{p^2}{n(n-p)}}:Q\in\mathscr{S}_{os}^n\right\}\\
\geqslant&\inf\left\{n^{\frac{n+p}{n}}\widetilde{V}_p(\Gamma_p^{\tau,*}K,Q^*)V(Q)^{\frac{p}{n}}:Q\in\mathscr{S}_{os}^n\right\}^{-\frac{p}{n-p}}
\end{aligned}$$

$$+\inf\left\{n^{\frac{n+p}{n}}\widetilde{V}_p(\Gamma_p^{\tau,*}L,Q^*)V(Q)^{\frac{p}{n}}:Q\in\mathscr{S}_{os}^n\right\}^{-\frac{p}{n-p}}$$
$$=\left[\sup\left\{n^{\frac{n+p}{n}}\widetilde{V}_p(\Gamma_p^{\tau,*}K,Q^*)V(Q)^{\frac{p}{n}}:Q\in\mathscr{S}_{os}^n\right\}\right]^{-\frac{p}{n-p}}$$
$$+\left[\sup\left\{n^{\frac{n+p}{n}}\widetilde{V}_p(\Gamma_p^{\tau,*}L,Q^*)V(Q)^{\frac{p}{n}}:Q\in\mathscr{S}_{os}^n\right\}\right]^{-\frac{p}{n-p}}$$
$$=\widetilde{\Omega}_p(\Gamma_p^{\tau,*}K)^{-\frac{p(n+p)}{n(n-p)}}+\widetilde{\Omega}_p(\Gamma_p^{\tau,*}L)^{-\frac{p(n+p)}{n(n-p)}}.$$

This yields inequality (1.14).

By the equality condition of (3.9), we see that equality holds in (1.14) if and only if $\Gamma_p^{\tau,*}K$ and $\Gamma_p^{\tau,*}L$ are dilates. □

Letting $\tau=0$ in Theorem 1.2, we get another Brunn-Minkowski type inequality with respect to L_p-harmonic Blaschke combination:

COROLLARY 3.1. *For $K,L\in\mathscr{S}_o^n$ and $1\leqslant p<n$, then*

$$\widetilde{\Omega}_p(\Gamma_p^*(K\widehat{+}_pL))^{-\frac{p(n+p)}{n(n-p)}}\geqslant\widetilde{\Omega}_p(\Gamma_p^*K)^{-\frac{p(n+p)}{n(n-p)}}+\widetilde{\Omega}_p(\Gamma_p^*L)^{-\frac{p(n+p)}{n(n-p)}},$$

with equality if and only if Γ_p^*K and Γ_p^*L are dilates.

Proof of Theorem 1.3. For $K,L\in\mathscr{S}_o^n$, $1\leqslant p<n$ and $\tau\in[-1,1]$. If $K\subseteq L$, then

$$\rho(K,\cdot)\leqslant\rho(L,\cdot), \tag{3.10}$$

with equality if and only if $K=L$.

From (1.3) and (2.1), we have

$$\rho(\Gamma_p^{\tau,*}K,u)^{-p}=\frac{2}{c_{n,p}(\tau)(n+p)V(K)}\int_{S^{n-1}}\varphi_\tau(u\cdot v)^p\rho(K,v)^{n+p}dS(v). \tag{3.11}$$

By (2.5), (3.10) and (3.11), we can get

$$V(K)^{-\frac{n-p}{p}}\widetilde{V}_p(\Gamma_p^{\tau,*}K,Q^*)$$
$$=V(K)^{-\frac{n-p}{p}}\frac{1}{n}\int_{S^{n-1}}\rho(\Gamma_p^{\tau,*}K,u)^{n-p}\rho(Q^*,u)^pdS(u)$$
$$=V(K)^{-\frac{n-p}{p}}\frac{1}{n}\int_{S^{n-1}}\left[\rho(\Gamma_p^{\tau,*}K,u)^{-p}\rho(Q^*,u)^{-\frac{p^2}{n-p}}\right]^{-\frac{n-p}{p}}dS(u)$$
$$=\frac{1}{n}\int_{S^{n-1}}\left[\frac{2}{c_{n,p}(\tau)(n+p)}\int_{S^{n-1}}\varphi_\tau(u\cdot v)^p\rho(K,v)^{n+p}dS(v)\rho(Q^*,u)^{-\frac{p^2}{n-p}}\right]^{-\frac{n-p}{p}}dS(u)$$
$$\leqslant\frac{1}{n}\int_{S^{n-1}}\left[\frac{2}{c_{n,p}(\tau)(n+p)}\int_{S^{n-1}}\varphi_\tau(u\cdot v)^p\rho(L,v)^{n+p}dS(v)\rho(Q^*,u)^{-\frac{p^2}{n-p}}\right]^{-\frac{n-p}{p}}dS(u)$$
$$=V(L)^{-\frac{n-p}{p}}\widetilde{V}_p(\Gamma_p^{\tau,*}L,Q^*).$$

Combined with (1.12), we easily get

$$\begin{aligned}\frac{\widetilde{\Omega}_p(\Gamma_p^{\tau,*}K)^{\frac{n+p}{n}}}{V(K)^{\frac{n-p}{p}}} &= V(K)^{-\frac{n-p}{p}}\sup\left\{n^{\frac{n+p}{n}}\widetilde{V}_p(\Gamma_p^{\tau,*}K,Q^*)V(Q)^{\frac{p}{n}} : Q\in\mathscr{S}_{os}^n\right\}\\ &\leqslant V(L)^{-\frac{n-p}{p}}\sup\left\{n^{\frac{n+p}{n}}\widetilde{V}_p(\Gamma_p^{\tau,*}L,Q^*)V(Q)^{\frac{p}{n}} : Q\in\mathscr{S}_{os}^n\right\}\\ &= \frac{\widetilde{\Omega}_p(\Gamma_p^{\tau,*}L)^{\frac{n+p}{n}}}{V(L)^{\frac{n-p}{p}}}.\end{aligned}$$

According to the equality condition of (3.10), we see that equality holds in (1.15) when $K=L$. □

Letting $\tau=0$ in Theorem 1.3, we get another monotone inequality for the polar of the L_p-centroid bodies:

COROLLARY 3.2. *For $K,L\in\mathscr{S}_o^n$ and $1\leqslant p<n$, if $K\subseteq L$, then*

$$\frac{\widetilde{\Omega}_p(\Gamma_p^*K)^{\frac{n+p}{n}}}{V(K)^{\frac{n-p}{p}}} \leqslant \frac{\widetilde{\Omega}_p(\Gamma_p^*L)^{\frac{n+p}{n}}}{V(L)^{\frac{n-p}{p}}},$$

equality holds when $K=L$.

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(Received May 30, 2018)

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