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IMPROVING SOME INEQUALITIES ASSOCIATED WITH THE EULER–MASCHERONI CONSTANT

JENICA CRINGANU

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Abstract. The aim of this paper is to improve the results obtained by Chen and Mortici in 2013 about the inequalities for the Euler-Mascheroni constant.

1. Introduction

It is well known that the sequence

$$\gamma_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n,\ n \geqslant 1,$$

is convergent to a limit denoted $\gamma = 0,5772\ldots$ now known as Euler-Mascheroni constant. Many authors have obtained different estimations for $\gamma_n - \gamma$, for exemple the following increasingly better

$$\frac{1}{2(n+1)} < \gamma_n - \gamma < \frac{1}{2(n-1)}, n \geqslant 2 \quad ([9]),$$

$$\frac{1}{2(n+1)} < \gamma_n - \gamma < \frac{1}{2n}, n \geqslant 1 \quad ([12]),$$

$$\frac{1-\gamma}{n} < \gamma_n - \gamma < \frac{1}{2n}, n \geqslant 1 \quad ([2]),$$

$$\frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n}, n \geqslant 1 \quad ([6,7]),$$

$$\frac{1}{2n+\frac{2}{5}} < \gamma_n - \gamma < \frac{1}{2n+\frac{1}{3}}, n \geqslant 1 \quad ([10]),$$

$$\frac{1}{2n+\frac{2\gamma-1}{1-\gamma}} < \gamma_n - \gamma < \frac{1}{2n+\frac{1}{3}}, n \geqslant 1 \quad ([1,10]).$$

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The convergence of the sequence γ_n to γ is very slow. In 1993, DeTemple [5] studied a modified sequence which converges faster and he proved that

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2},\ n \geqslant 1,$$

where $R_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln(n + \frac{1}{2})$.
In 2010, Chen [3] proved that for all integers $n \geqslant 1$,

$$\frac{1}{24(n+a)^2} \leqslant R_n - \gamma < \frac{1}{24(n+b)^2},$$

with the best possible constants

$$a = \frac{1}{\sqrt{24[-\gamma + 1 - \ln(\frac{3}{2})]}} - 1 = 0,55106... \text{ and } b = \frac{1}{2}.$$

We define the sequence $S_n = \frac{2}{1} + \frac{2}{3} + ... + \frac{2}{2n-1} - \ln(4n)$, for $n \geqslant 1$.
Recently, Chen and Mortici [4] obtained the following bounds for $S_n - \gamma$

$$\frac{1}{24(n+a)^2} \leqslant S_n - \gamma < \frac{1}{24(n+b)^2},\ n \geqslant 1,$$

with the best possible constants

$$a = \frac{1}{\sqrt{24(2 - 2\ln 2 - \gamma)}} - 1 = 0,06858... \text{ and } b = 0.$$

In this paper we obtain a better estimation for the left inequality and for the right inequality we remark that $b = 0$ is the best constant using an elementary sequence method.
In 1997, Negoi [8] proved that the sequence $T_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln(n + \frac{1}{2} + \frac{1}{24n})$ is strictly increasing and convergent to γ. Moreover, he proved that

$$\frac{1}{48(n+1)^3} < \gamma - T_n < \frac{1}{48n^3},\ n \geqslant 1.$$

In 2013, in same paper [4], Chen and Mortici proved that for all integers $n \geqslant 1$,

$$\frac{1}{48(n+a)^3} \leqslant \gamma - T_n < \frac{1}{48(n+b)^3},$$

with the best possible constants

$$a = \frac{1}{\sqrt[3]{48[1 - \gamma + \ln(\frac{37}{24})]}} - 1 = 0,27380525... \text{ and } b = \frac{83}{360} = 0,23055555....$$

The second aim of this paper is to obtain a better estimation for the left inequality and for the right inequality we remark that $b = \frac{83}{360}$ is the best constant also using an elementary sequence method.

2. The main result

THEOREM 1. *(i) For every integer $n \geqslant 1$ we have*

$$S_n - \gamma < \frac{1}{24n^2};$$

(ii) For every $a > 0$ there exists $n_a \in N$ such that

$$\frac{1}{24(n+a)^2} < S_n - \gamma \quad \textit{for all } n \geqslant n_a.$$

Proof. We define the sequence

$$a_n = S_n - \gamma - \frac{1}{24(n+a)^2} = \frac{2}{1} + \frac{2}{3} + \dots + \frac{2}{2n-1} - \ln(4n) - \gamma - \frac{1}{24(n+a)^2}, \text{ for } a \geqslant 0,$$

and so $a_{n+1} - a_n = f(n)$, where

$$f(n) = \frac{2}{2n+1} - \ln(4n+4) + \ln(4n) - \frac{1}{24(n+a+1)^2} + \frac{1}{24(n+a)^2}.$$

The derivative of function f is equal to

$$f'(n) = \frac{P(n)}{12n(n+1)(2n+1)^2(n+a)^3(n+a+1)^3},$$

where

$$\begin{aligned} P(n) =& 48an^5 + (168a^2 + 120a - 7)n^4 + 2(120a^3 + 168a^2 + 45a - 7)n^3 + 180a^4 \\ &+ 360a^3 + 201a^2 + 15a - 8)n^2 + (72a^5 + 180a^4 + 144a^3 + 33a^2 - 3a - 1)n \\ &+ 12a^6 + 36a^5 + 36a^4 + 12a^3. \end{aligned}$$

(i) If $a = 0$ then $P(n) = -7n^4 - 14n^3 - 8n^2 - n < 0$, for all $n \geqslant 1$, and then f is strictly decreasing. We have $f(\infty) = 0$ and then $f(n) > 0$ for all $n \geqslant 1$, so that $(a_n)_{n \geqslant 1}$ is strictly increasing. Since (a_n) converges to zero it results that $a_n < 0$ for all $n \geqslant 1$, so that

$$S_n - \gamma < \frac{1}{24n^2}, \text{ for all } n \geqslant 1.$$

(ii) If $a > 0$ then there exists $n_a \in N$ such that $P(n) > 0$ for all $n \geqslant n_a$ and then f is strictly increasing on $[n_a, \infty)$. Since $f(\infty) = 0$ it results that $f(n) < 0$ for all $n \geqslant n_a$, so that $(a_n)_{n \geqslant n_a}$ is strictly decreasing. The sequence (a_n) converges to zero and then it results that $a_n > 0$ for all $n \geqslant n_a$, so that

$$\frac{1}{24(n+a)^2} < S_n - \gamma, \text{ for all } n \geqslant n_a. \quad \square$$

Now we find the constant n_a in some particular cases. For example, if $a = 0,06 = \frac{3}{50} < \frac{1}{\sqrt{24(2-2\ln 2-\gamma)}} - 1 = 0,06858...$, then

$$P(n) = \frac{72}{25}n^5 + \frac{503}{625}n^4 - \frac{22933}{3125}n^3 - \frac{491899}{78125}n^2 - \frac{40144813}{39062500}n + \frac{12059037}{3906250000} > 0,$$

for all $n \geqslant 2$, and so

$$\frac{1}{24(n+\frac{3}{50})^2} < S_n - \gamma < \frac{1}{24n^2}, \text{ for all } n \geqslant 2.$$

Now, if $a = 0,01 = \frac{1}{100}$, then

$$P(n) = \frac{12}{25}n^5 - \frac{7229}{1250}n^4 - \frac{163327}{12500}n^3 - \frac{39147691}{5000000}n^2 - \frac{1283192741}{1250000000}n + \frac{3090903}{250000000000} > 0,$$

for all $n \geqslant 15$, and so

$$\frac{1}{24(n+\frac{1}{100})^2} < S_n - \gamma < \frac{1}{24n^2}, \text{ for all } n \geqslant 15.$$

Let us remark that a direct calculus show that these inequalities hold and for $n \in \{9,10,11,12,13,14\}$, and then

$$\frac{1}{24(n+\frac{1}{100})^2} < S_n - \gamma < \frac{1}{24n^2}, \text{ for all } n \geqslant 9.$$

THEOREM 2. *(i) For every $0 < a \leqslant \frac{83}{360}$ we have*

$$\frac{1}{48(n+a)^3} < \gamma - T_n, \text{ for all } n \geqslant 1.$$

(ii) For every $a > \frac{83}{360}$ there exists $n_a \in N$ such that

$$\gamma - T_n < \frac{1}{48(n+a)^3}, \text{ for all } n \geqslant n_a.$$

Consequently $a = \frac{83}{360}$ is the best constant for the inequality of (i).

Proof. The sequence

$$a_n = \gamma - T_n - \frac{1}{48(n+a)^3} = \gamma - 1 - \frac{1}{2} - \ldots - \frac{1}{n} + \ln(n + \frac{1}{2} + \frac{1}{24n}) - \frac{1}{48(n+a)^3}, \text{ for } a > 0,$$

converges to zero and it is strictly increasing for $a \geqslant \frac{83}{360}$ and strictly decreasing for $a < \frac{83}{360}$. For this, we have $a_{n+1} - a_n = f(n)$, where

$$f(n) = -\frac{1}{n+1} + \ln\left(n + \frac{3}{2} + \frac{1}{24(n+1)}\right) - \ln(n + \frac{1}{2} + \frac{1}{24n}) - \frac{1}{48(n+a+1)^3} + \frac{1}{48(n+a)^3}.$$

The derivative of function f is equal to

$$f'(n) = \frac{P(n)}{16n(n+1)^2(24n^2+12n+1)(24n^2+60n+37)(n+a)^4(n+a+1)^4},$$

where

$$\begin{aligned} P(n) =& 32(360a-83)n^9 + (57600a^2+47872a-15963)n^8 + 32(3960a^3+7576a^2 \\ &+2180a-1183)n^7 + 4(40320a^4+121216a^3+99856a^2+7240a-12737)n^6 \\ &+2(64512a^5+269920a^4+360768a^3+160656a^2-14062a-20275)n^5 \\ &+4(16128a^6+91840a^5+170840a^4+131384a^3+31541a^2-9197a-4807)n^4 \\ &+(18432a^7+151424a^6+376832a^5+408320a^4+193708a^3+19306a^2 \\ &-15720a-5211)n^3+2(1152a^8+17024a^7+58656a^6+85344a^5+58832a^4 \\ &+16892a^3-180a^2-1378a-363)n^2+(3104a^8+17152a^7+35200a^6 \\ &+33728a^5+14944a^4+2220a^3-222a^2-148a-37)n+592(a^8+4a^7+6a^6 \\ &+4a^5+a^4). \end{aligned}$$

(i) If $a = \frac{83}{360}$ then

$$\begin{aligned} P(n) =& -\frac{27961}{15}n^8 - \frac{29697679}{4050}n^7 - \frac{4043881267}{243000}n^6 - \frac{8530365029807}{437400000}n^5 \\ &- \frac{324974432017513}{26244000000}n^4 - \frac{113729719684675837}{28343520000000}n^3 \\ &- \frac{205720432718683173 73}{40814668800000000}n^2 + \frac{1268317064074833048 97}{881596846080000000 0}n \\ &+ \frac{676283822104345478 77}{176319369216000000 00} < 0, \end{aligned}$$

for all $n \geqslant 1$, and then f is strictly decreasing. Since $f(\infty) = 0$ it results that $f(n) > 0$ for all $n \geqslant 1$, so that $(a_n)_{n\geqslant 1}$ is strictly increasing. Since (a_n) converges to zero it results that $a_n < 0$ for all $n \geqslant 1$, so that

$$\gamma - T_n < \frac{1}{48(n+\frac{83}{360})^3}, \text{ for all } n \geqslant 1.$$

If $a < \frac{83}{360}$ then $\frac{1}{48(n+\frac{83}{360})^3} < \frac{1}{48(n+a)^3}$ and so

$$\gamma - T_n < \frac{1}{48(n+a)^3}, \text{ for all } n \geqslant 1.$$

(ii) If $a > \frac{83}{360}$ then there exists $n_a \in N$ such that $P(n) > 0$ for all $n \geqslant n_a$ and then f is strictly increasing on $[n_a, \infty)$. Since $f(\infty) = 0$ it results that $f(n) < 0$ for all $n \geqslant n_a$, so that $(a_n)_{n \geqslant n_a}$ is strictly decreasing. By convergence to zero of (a_n) it results that $a_n > 0$ for all $n \geqslant n_a$, so that

$$\frac{1}{48(n+a)^3} < \gamma - T_n, \text{ for all } n \geqslant n_a. \quad \square$$

Finally we find the constant n_a in some particular cases. For example, if $a = 0,25 = \frac{1}{4} \in \left(\frac{83}{360}, \frac{1}{\sqrt[3]{48[1-\gamma+\ln(\frac{37}{24})]}} - 1\right)$, then

$$P(n) = 224n^9 - 395n^8 - 3284n^7 - 10538n^6 - \frac{55961}{4}n^5 - \frac{74275}{8}n^4 - \frac{93011}{32}n^3 - \frac{64655}{256}n^2 + \frac{97921}{2048}n + \frac{23125}{4096} > 0,$$

for all $n \geqslant 6$, and so

$$\frac{1}{48(n+\frac{1}{4})^3} < \gamma - T_n < \frac{1}{48(n+\frac{83}{360})^3}, \text{ for all } n \geqslant 6.$$

Let us remark that a direct calculus show that these inequalities hold and for $n \in \{4,5\}$, and then

$$\frac{1}{48(n+\frac{1}{4})^3} < \gamma - T_n < \frac{1}{48(n+\frac{83}{360})^3}, \text{ for all } n \geqslant 4.$$

If $a = 0,24 = \frac{6}{25} \in \left(\frac{83}{360}, \frac{1}{4}\right)$, then

$$P(n) = \frac{544}{5}n^9 - \frac{28899}{25}n^8 - \frac{16867936}{3125}n^7 - \frac{1074447604}{78125}n^6 - \frac{165272252126}{9765625}n^5 - \frac{2674071306628}{244140625}n^4 - \frac{21419721292923}{6103515625}n^3 - \frac{59638519190286}{152587890625}n^2 - \frac{4481000881739}{152587890625}n + \frac{708554863872}{152587890625} > 0,$$

for all $n \geqslant 15$, and so

$$\frac{1}{48(n+\frac{6}{25})^3} < \gamma - T_n < \frac{1}{48(n+\frac{83}{360})^3}, \text{ for all } n \geqslant 15.$$

Moreover, we remark that a direct calculus show that these inequalities hold and for $n \in \{8,9,10,11,12,13,14\}$, and then

$$\frac{1}{48(n+\frac{6}{25})^3} < \gamma - T_n < \frac{1}{48(n+\frac{83}{360})^3}, \text{ for all } n \geqslant 8.$$

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Jenica Cringanu
"Dunarea de Jos" University of Galati
Department of Mathematics and Computer Science
Domneasca No 111, Galati, Romania
e-mail: jcringanu@ugal.ro