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QUANTUM HERMITE–HADAMARD INEQUALITIES FOR DOUBLE INTEGRAL AND q–DIFFERENTIABLE CONVEX FUNCTIONS

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Abstract. In this paper, we establish some new quantum analogue of Hermite-Hadamard inequalities for double integral and refinements of Hermite-Hadamard inequality for q-differentiable convex functions.

1. Introduction

In mathematics, the quantum calculus is the study of calculus without limits and is sometimes called the q-calculus. In quantum calculus, we obtain q-analogues of mathematical objects that can be recaptured as $q \to 1$. The history of quantum calculus can be traced back to Euler (1707-1783), who first introduced the q-calculus in the tracks of Newton's work of infinite series. In the early twentieth century, Jackson [13] first defined and studied the q-integral in a systematic way. Later, the integral representations of q-gamma and q-beta functions were proposed by De Sole and Kac [5]. In recent years, the topic of q-calculus have been studied by several researchers and variety of new results can be found in the literature [1, 2, 4, 8, 9, 10, 12, 14, 16, 17, 19, 20, 23] and the references cited therein.

In 1893, Hadamard [11] investigated one of the fundamental inequalities in analysis, that is,

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which is now known as Hermite-Hadamard inequality. In 2014, Tariboon and Ntouyas [21] studied the extension to q-calculus of several important integral inequalities, from which they obtained the q-Hölder, q-Hermite-Hadamard, q-trapezoid, q-Ostrowski, q-Cauchy-Bunyakovsky-Schwarz, q-Grüss and q-Grüss-Čebyšev integral inequalities. In 2016, Alp et al. [3] proved the correct q-Hermite-Hadamard inequality, and then obtained some new q-Hermite-Hadamard inequalities and generalized q-Hermite-Hadamard inequalities. Using the left hand part of the correct q-Hermite-Hadamard inequality, they also obtained a new equality. Furthermore, they used the new equality to obtain q-midpoint type integral inequalities through q-differentiable convex and q-differentiable quasi-convex functions.

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In 1990, Dragomir [6] gave the following refinements of Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)dxdy \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1.2}$$

Since then, many researchers have developed various extensions and refinements of Hermite-Hadamard inequality.

The purpose of this paper is to present the q-calculus of Hermite-Hadamard inequalities for double integrals and refinements of Hermite-Hadamard inequality, obtained as special cases when $q \to 1$.

2. Preliminaries

In this section, we recall some previously known concepts and basic results. Throughout this section, we let $J=[a,b]\subset\mathbb{R}$ be an interval and q be a constant with $0<q<1$.

DEFINITION 2.1. Let $f : J \to \mathbb{R}$ be a continuous function and let $x \in J$. Then the q-derivative of f on J at x is defined as

$${}_aD_qf(x) = \frac{f(x)-f(qx+(1-q)a)}{(1-q)(x-a)}, \text{ for } x \neq a. \tag{2.1}$$

For $x=a$, we define ${}_aD_qf(a) = \lim_{x\to a} {}_aD_qf(x)$.

A function f is q-differentiable on J if ${}_aD_qf(x)$ exists for all $x\in J$. Moreover, if $a=0$ in (2.1), then ${}_0D_qf = D_qf$, where D_q is the well-known q-derivative of the function $f(x)$, which is defined by

$$D_qf(x) = \frac{f(x)-f(qx)}{(1-q)x}.$$

For more details, see [15].

In addition, we shall define higher-order q-derivatives of functions on J.

DEFINITION 2.2. Let $f : J \to \mathbb{R}$ be a continuous function. The second-order q-derivative of f on J, denote by ${}_aD_q^2f$ (provided that ${}_aD_qf$ is q-differentiable on J), is the function from $J\to\mathbb{R}$ defined by

$${}_aD_q^2f = {}_aD_q({}_aD_qf).$$

Similarly, provided that ${}_aD_q^{n-1}f$ is q-differentiable on J for some integer $n>2$, the n^{th}-order q-derivative of f on J is the function from $J\to\mathbb{R}$ defined by

$${}_aD_q^nf = {}_aD_q({}_aD_q^{n-1}f).$$

EXAMPLE 2.1. Define function $f : J \to \mathbb{R}$ by $f(x) = x^2 + 1$. Let $0 < q < 1$. Then for $x \neq a$, we have

$$
\begin{aligned}
{}_aD_q(x^2+1) &= \frac{(x^2+1) - \left[(qx+(1-q)a)^2+1\right]}{(1-q)(x-a)} = \frac{(1+q)x^2 - 2qax - (1-q)a^2}{(x-a)} \\
&= (1+q)x + (1-q)a. \qquad (2.2)
\end{aligned}
$$

For $x = a$, we have ${}_aD_q f(a) = \lim_{x \to a} {}_aD_q f(x) = 2a$.

DEFINITION 2.3. Let $f : J \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the q-integral on J is defined by

$$
\int_a^x f(t)\ {}_ad_q t = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \qquad (2.3)
$$

for $x \in J$.

If $a = 0$ in (2.3), then we have the classical q-integral of the function $f(x)$, which is defined by

$$
\int_0^x f(t)\ {}_0d_q t = (1-q)x \sum_{n=0}^{\infty} q^n f(q^n x)
$$

for $x \in [0, \infty)$; see [15] for more details.

EXAMPLE 2.2. Define function $f : J \to \mathbb{R}$ by $f(x) = 2x$. Let $0 < q < 1$. Then we have

$$
\begin{aligned}
\int_a^b f(x)\ {}_ad_q x = \int_a^b 2x\ {}_ad_q x &= 2(1-q)(b-a) \sum_{n=0}^{\infty} q^n (q^n b + (1-q^n)a) \\
&= \frac{2(b-a)(b+qa)}{1+q}.
\end{aligned}
$$

Note that if $q \to 1$, then we have the classical integration

$$
\int_a^b f(x)dx = \int_a^b 2xdx = b^2 - a^2.
$$

THEOREM 2.1. Let $f : J \to \mathbb{R}$ be a continuous function. Then we have the following:

(i) ${}_aD_q \int_a^x f(t)\ {}_ad_q t = f(x)$;

(ii) $\int_c^x {}_aD_q f(t)\ {}_ad_q t = f(x) - f(c)$ for $c \in (a, x)$.

THEOREM 2.2. Let $f, g : J \to \mathbb{R}$ be continuous functions and $\alpha \in \mathbb{R}$. Then we have the following:

(i) $\int_a^x [f(t) + g(t)]\ {}_ad_q t = \int_a^x f(t)\,{}_ad_q t + \int_a^x g(t)\,{}_ad_q t$;

(ii) $\int_a^x (\alpha f)(t)\ {}_a d_q t = \alpha \int_a^x f(t)\ {}_a d_q t$;

(iii) $\int_c^x f(t) {}_a D_q g(t)\ {}_a d_q t = (fg)|_c^x - \int_c^x g(qt+(1-q)a) {}_a D_q f(t)\ {}_a d_q t$ for $c \in (a,x)$.

For the proofs of theorem 2.1 and theorem 2.2, see [22].

THEOREM 2.3. Let $f : J \to \mathbb{R}$ be a convex differentiable function on (a,b) and $0 < q < 1$. Then we have

$$f\left(\frac{qa+b}{1+q}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\ {}_a d_q x \leqslant \frac{qf(a)+f(b)}{1+q}. \tag{2.4}$$

THEOREM 2.4. Let $f : J \to \mathbb{R}$ be a convex differentiable function on (a,b) and $0 < q < 1$. Then we have

$$f\left(\frac{a+qb}{1+q}\right) + \frac{(1-q)(b-a)}{1+q} f'\left(\frac{a+qb}{1+q}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\ {}_a d_q x \leqslant \frac{qf(a)+f(b)}{1+q}. \tag{2.5}$$

THEOREM 2.5. Let $f : J \to \mathbb{R}$ be a convex differentiable function on (a,b) and $0 < q < 1$. Then we have

$$f\left(\frac{a+b}{2}\right) + \frac{(1-q)(b-a)}{2(1+q)} f'\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\ {}_a d_q x \leqslant \frac{qf(a)+f(b)}{1+q}. \tag{2.6}$$

For the proof of theorem 2.3, theorem 2.4, and theorem 2.5, see [3].

LEMMA 2.1. Let $f : J \to \mathbb{R}$ be a convex continuous function on J and $0 < q < 1$. Then we have

$$\begin{aligned} & f\left(\frac{1}{(b-a)^2}\int_a^b \int_a^b (tx+(1-t)y)\ {}_a d_q x {}_a d_q y\right) \\ \leqslant & \frac{1}{(b-a)^2}\int_a^b \int_a^b f(tx+(1-t)y)\ {}_a d_q x {}_a d_q y. \end{aligned} \tag{2.7}$$

Proof. The lemma 2.1 follows directly from definition 2.3 and Jensen's inequality. □

3. Main results

In this section, we present the q-Hermite-Hadamard double integral inequality and refinements of q-Hermite-Hadamard inequalities on the interval $J = [a,b]$.

THEOREM 3.1. *Let $f : J \to \mathbb{R}$ be a convex continuous function on J and $0 < q < 1$. Then we have*

$$f\left(\frac{qa+b}{1+q}\right) \leqslant \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\ {}_ad_qx_ad_qy \leqslant \frac{1}{b-a}\int_a^b f(x)_ad_qx \leqslant \frac{qf(a)+f(b)}{1+q} \tag{3.1}$$

for all $t \in [0,1]$.

Proof. Since f is convex on J, it follows that

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y) \tag{3.2}$$

for all $x,y \in J$ and $t \in [0,1]$. Taking double q-integration on both sides of (3.2) on $J \times J$, we obtain

$$\int_a^b\int_a^b f(tx+(1-t)y)\ {}_ad_qx_ad_qy \leqslant \int_a^b\int_a^b [tf(x)+(1-t)f(y)]\ {}_ad_qx_ad_qy = (b-a)\int_a^b f(x)_ad_qx, \tag{3.3}$$

which proves the second part of (3.1) by using the right hand side of q-Hermite-Hadamard's inequality.

On the other hand, by lemma 2.1, we have

$$f\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b (tx+(1-t)y)\ {}_ad_qx_ad_qy\right) \leqslant \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\ {}_ad_qx_ad_qy.$$

Since

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b (tx+(1-t)y)_ad_qx_ad_qy = \frac{qa+b}{1+q},$$

this yields the first part of (3.1). □

REMARK 3.1. If $q \to 1$, then (3.1) reduces to (1.2), that is,

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)dxdy \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}.$$

COROLLARY 3.1. *Let $f : J \to \mathbb{R}$ be a convex continuous function on $[a,b]$ and $0 < q < 1$. Then we have*

$$f\left(\frac{qa+b}{1+q}\right) \leqslant \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)\ {}_ad_qx_ad_qy \leqslant \frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx \leqslant \frac{qf(a)+f(b)}{1+q}. \tag{3.4}$$

REMARK 3.2. If $q \to 1$, then (3.4) reduces to

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2},$$

which readily appeared in [18].

THEOREM 3.2. *Let $f : J \to \mathbb{R}$ be a convex continuous function on J and $0 < q < 1$. Then we have*

$$\begin{aligned}&\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+qy}{1+q}\right)\,{}_ad_qx\,{}_ad_qy\\ \leqslant&\frac{1}{(b-a)^2}\int_0^1\int_a^b\int_a^b f(tx+(1-t)y)\,{}_ad_qx\,{}_ad_qy\,{}_ad_qt \leqslant \frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx. \end{aligned} \tag{3.5}$$

Proof. Consider the mapping $g : J \to \mathbb{R}$ given by

$$g(t) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\,{}_ad_qx\,{}_ad_qy.$$

For all $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 1$, we have

$$\begin{aligned}g(\alpha t_1+\beta t_2) &= \frac{1}{(b-a)^2}\int_a^b\int_a^b f((\alpha t_1+\beta t_2)x+(1-(\alpha t_1+\beta t_2))y)\,{}_ad_qx\,{}_ad_qy\\ &\leqslant \frac{\alpha}{(b-a)^2}\int_a^b\int_a^b f(t_1x+(1-t_1)y)\,{}_ad_qx\,{}_ad_qy\\ &\quad+\frac{\beta}{(b-a)^2}\int_a^b\int_a^b f(t_2x+(1-t_2)y)\,{}_ad_qx\,{}_ad_qy\\ &= \alpha g(t_1)+\beta g(t_2),\end{aligned}$$

which proves that g is convex on $[0,1]$. Using theorem 2.3 for the convex function g, we have

$$\begin{aligned}&\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+qy}{1+q}\right)\,{}_ad_qx\,{}_ad_qy\\ =&g\left(\frac{1}{1+q}\right) \leqslant \int_0^1 g(t)\,{}_ad_qt = \frac{1}{(b-a)^2}\int_0^1\int_a^b\int_a^b f(tx+(1-t)y)\,{}_ad_qx\,{}_ad_qy\,{}_ad_qt\\ \leqslant&\frac{qg(0)+g(1)}{1+q} = \frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx.\end{aligned}$$

This completes the proof. □

REMARK 3.3. If $q \to 1$, then (3.5) reduces to

$$\begin{aligned}\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy &\leqslant \frac{1}{(b-a)^2}\int_0^1\int_a^b\int_a^b f(tx+(1-t)y)dxdydt\\ &\leqslant \frac{1}{b-a}\int_a^b f(x)dx,\end{aligned}$$

which readily appeared in [18].

THEOREM 3.3. *Let $f : J \to \mathbb{R}$ be a q-differentiable convex continuous function and $0 < q < 1$. Then the inequalities*

$$\begin{aligned} 0 &\leqslant \frac{1}{b-a}\int_a^b f(x)\ {}_a d_q x - \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\ {}_a d_q x_a d_q y \\ &\leqslant t\left[\frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a}\int_a^b f(qx+(1-q)a)\ {}_a d_q x\right] \end{aligned} \tag{3.6}$$

are valid for all $t \in [0,1]$.

Proof. Since f is convex on J, it follows that

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y)$$

for all $x, y \in J$ and $t \in [0,1]$. Taking double q-integration on both sides of the above inequality on $J \times J$, we obtain

$$\begin{aligned} \int_a^b\int_a^b f(tx+(1-t)y)\ {}_a d_q x_a d_q y &\leqslant \int_a^b\int_a^b [tf(x)+(1-t)f(y)]\ {}_a d_q x_a d_q y \\ &= (b-a)\int_a^b f(x)\ {}_a d_q x, \end{aligned}$$

which yields the first part of (3.6).

On the other hand, since f is q-differentiable convex on J and $f' \geqslant {}_a D_q f$, we have

$$f(tx+(1-t)y) - f(y) \geqslant t(x-y)\ {}_a D_q f(y)$$

for all $x, y \in J$ and $t \in [0,1]$. Taking double q-integration on both sides of the above inequality on $J \times J$, we obtain

$$\begin{aligned} &\int_a^b\int_a^b f(tx+(1-t)y)\ {}_a d_q x_a d_q y - (b-a)\int_a^b f(x)\ {}_a d_q x \\ \geqslant &t\int_a^b\int_a^b (x-y)_a D_q f(y)\ {}_a d_q x_a d_q y. \end{aligned} \tag{3.7}$$

Since

$$\begin{aligned} &\int_a^b\int_a^b (x-y)_a D_q f(y)\ {}_a d_q x_a d_q y \\ =&(b-a)\int_a^b f(qx+(1-q)a)\ {}_a d_q x - (b-a)^2\frac{f(a)+qf(b)}{1+q}, \end{aligned}$$

it follows from (3.7) that

$$
\begin{aligned}
&(b-a)\int_a^b f(x) - \int_a^b\int_a^b f(tx+(1-t)y)\ {}_ad_qx_ad_qy \\
\leqslant & t\left[(b-a)^2\frac{f(a)+qf(b)}{1+q} - (b-a)\int_a^b f(qx+(1-q)a)\ {}_ad_qx\right]
\end{aligned}
$$

for all $t \in [0,1]$, which is the second part of (3.6). □

REMARK 3.4. If $q \to 1$, then (3.6) reduces to

$$
\begin{aligned}
0 &\leqslant \frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)dxdy \\
&\leqslant t\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right],
\end{aligned}
$$

which readily appeared in [7, 18].

COROLLARY 3.2. *Let $f: J \to \mathbb{R}$ be a q-differentiable convex continuous function and $0<q<1$. Then we have*

$$
\begin{aligned}
0 &\leqslant \frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx - \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)\ {}_ad_qx_ad_qy \\
&\leqslant \frac{1}{2}\left[\frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a}\int_a^b f(qx+(1-q)a)\ {}_ad_qx\right]. \qquad (3.8)
\end{aligned}
$$

REMARK 3.5. If $q \to 1$, then (3.5) reduces to

$$
\begin{aligned}
0 &\leqslant \frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy \\
&\leqslant \frac{1}{2}\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right],
\end{aligned}
$$

which readily appeared in [18].

THEOREM 3.4. *Let $f: J \to \mathbb{R}$ be a q-differentiable convex continuous function which is define at the point $\frac{qa+b}{1+q} \in (a,b)$ and $0<q<1$. Then the inequalities*

$$
\begin{aligned}
0 &\leqslant \frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx - \frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{qa+b}{1+q}\right)\ {}_ad_qx \\
&\leqslant (1-t)\left[\frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a}\int_a^b f(qx+(1-q)a)\ {}_ad_qx\right] \qquad (3.9)
\end{aligned}
$$

are valid for all $t \in [0,1]$.

Proof. Since f is convex on J, it follow from theorem 2.3 that

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx &= \frac{t}{b-a}\int_a^b f(x)\ {}_ad_qx + \frac{1-t}{b-a}\int_a^b f(x)\ {}_ad_qx \\ &\geqslant \frac{t}{b-a}\int_a^b f(x)\ {}_ad_qx + (1-t)f\left(\frac{qa+b}{1+q}\right) \\ &\geqslant \frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{qa+b}{1+q}\right)\ {}_ad_qx\end{aligned}$$

for all $t\in[0,1]$, which the first part of (3.9).

On the other hand, since f is q-differentiable convex on J, we have

$$f\left(tx+(1-t)\frac{qa+b}{1+q}\right) - f(x) \geqslant (1-t)\left(\frac{qa+b}{1+q}-x\right)\ {}_aD_qf(x).$$

Taking q-integration on both sides of the above inequality on J, we obtain

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{qa+b}{1+q}\right)\ {}_ad_qx - \frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx \\ \geqslant &\frac{1}{b-a}\int_a^b (1-t)\left(\frac{qa+b}{1+q}-x\right)\ {}_aD_qf(x){}_ad_qx. \qquad (3.10)\end{aligned}$$

Since

$$\int_a^b \left(\frac{qa+b}{1+q}-x\right)\ {}_aD_qf(x){}_ad_qx = \int_a^b f(qx+(1-q)a){}_ad_qx - (b-a)\frac{f(a)+qf(b)}{1+q}. \qquad (3.11)$$

Using (3.10) and (3.11), we get the second part of (3.9). □

COROLLARY 3.3. *Let $f:J\to\mathbb{R}$ be a q-differentiable convex continuous function and $0<q<1$. Then we have*

$$\begin{aligned}0 &\leqslant \frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx - \frac{2}{b-a}\int_{\frac{a(1+2q)+b}{2(1+q)}}^{\frac{qa+b(2+q)}{2(1+q)}} f(x)\ {}_ad_qx \\ &\leqslant \frac{1}{2}\left[\frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a}\int_a^b f(qx+(1-q)a)\ {}_ad_qx\right]. \qquad (3.12)\end{aligned}$$

THEOREM 3.5. *Let $f:J\to\mathbb{R}$ be a q-differentiable convex continuous function which is define at the point $\frac{a+qb}{1+q}\in(a,b)$ and $0<q<1$. Then the inequalities*

$$\begin{aligned}&(1-t)\frac{(1-q)(b-a)}{1+q}f'\left(\frac{a+qb}{1+q}\right) \\ \leqslant &\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx - \frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+qb}{1+q}\right)\ {}_ad_qx \\ \leqslant &(1-t)\left[\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(qx+(1-q)a)\ {}_ad_qx\right] \qquad (3.13)\end{aligned}$$

are valid for all $t\in[0,1]$.

Proof. The proof uses theorem 2.4 and is similar to that of theorem 3.4. □

COROLLARY 3.4. *Let $f : J \to \mathbb{R}$ be a q-differentiable convex continuous function and $0 < q < 1$. Then we have*

$$\begin{aligned}
&\frac{(1-q)(b-a)}{2(1+q)} f'\left(\frac{a+qb}{1+q}\right) \\
\leqslant &\frac{1}{b-a}\int_a^b f(x)\ {}_a d_q x - \frac{2}{b-a}\int_{\frac{2a+q(a+b)}{2(1+q)}}^{\frac{a+b(2q+1)}{2(1+q)}} f(x)_a d_q x \\
\leqslant &\frac{1}{2}\left[\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(qx+(1-q)a)\ {}_a d_q x\right]. \qquad (3.14)
\end{aligned}$$

THEOREM 3.6. *Let $f : J \to \mathbb{R}$ be a q-differentiable convex continuous function which is define at the point $\frac{a+b}{2} \in (a,b)$ and $0 < q < 1$. Then the inequalities*

$$\begin{aligned}
&(1-t)\frac{(1-q)(b-a)}{2(1+q)} f'\left(\frac{a+b}{2}\right) \\
\leqslant &\frac{1}{b-a}\int_a^b f(x)\ {}_a d_q x - \frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right) {}_a d_q x \\
\leqslant &(1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(qx+(1-q)a)\ {}_a d_q x\right] \qquad (3.15)
\end{aligned}$$

are valid for all $t \in [0,1]$.

Proof. The proof uses theorem 2.5 and is similar to that of theorem 3.4. □

COROLLARY 3.5. *Let $f : J \to \mathbb{R}$ be a q-differentiable convex continuous function and $0 < q < 1$. Then we have*

$$\begin{aligned}
&\frac{(1-q)(b-a)}{4(1+q)} f'\left(\frac{a+b}{2}\right) \\
\leqslant &\frac{1}{b-a}\int_a^b f(x)_a d_q x - \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)_a d_q x \\
\leqslant &\frac{1}{2}\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(qx+(1-q)a)_a d_q x\right]. \qquad (3.16)
\end{aligned}$$

REMARK 3.6. If $q \to 1$, then (3.9), (3.13), and (3.15) reduce to

$$\begin{aligned}
0 &\leqslant \frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx \\
&\leqslant (1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right],
\end{aligned}$$

which readily appeared in [18].

REMARK 3.7. If $q \to 1$, then (3.12), (3.14), and (3.16) reduces to

$$0 \leqslant \frac{1}{b-a}\int_a^b f(x)dx - \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)dx \leqslant \frac{1}{2}\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right],$$

which readily appeared in [18].

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