

## APPROXIMATION PROPERTIES OF CERTAIN BERNSTEIN–STANCU TYPE OPERATORS

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(Communicated by I. Raşa)

*Abstract.* In this paper we introduce and investigate a new operator of Bernstein-Stancu type, based on  $q$ -polynomials. We study approximation properties for these operators based on Korovkin type approximation theorem and study some direct theorems. Also, the study contains numerical considerations regarding the constructed operators based on Maple algorithms.

### 1. Introduction

In 1968, Stancu [24] proposed the sequence of positive linear operators  $S_n^{<\alpha>} : C[0, 1] \rightarrow C[0, 1]$ , depending on a non-negative parameter  $\alpha$  given by

$$S_n^{<\alpha>}(f; x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) p_{n,k}^{<\alpha>}(x), \quad (1)$$

where  $p_{n,k}^{<\alpha>}(x) = \binom{n}{k} \frac{\prod_{i=0}^{k-1} (x + i\alpha) \prod_{j=0}^{n-k-1} (1 - x + j\alpha)}{\prod_{i=0}^{n-1} (1 + i\alpha)}$ ,  $x \in [0, 1]$ .

For  $\alpha = 0$  these operators reduces to the classical Bernstein operators.

In the papers [11], [12] and [23] starting with the Bernstein operators, the following Stancu type operators are constructed and studied:

$$C_n : C[0, 1] \rightarrow \Pi_n$$

$$C_n(f; x) = \sum_{k=0}^n \frac{k!}{n^k} \binom{n}{k} \mathbf{m}_{k,n} \left[ 0, \frac{1}{n}, \dots, \frac{k}{n}; f \right] x^k, \quad f \in C[0, 1], \quad (2)$$

where the real numbers  $(m_{k,n})_{k=0}^{\infty}$  are selected in order to preserve some important properties of Bernstein operators and  $\Pi_n$  is the linear space of all real polynomials of degree  $\leq n$ .

In the following we consider that  $\mathbf{m}_{0,n} = 1$ ,  $\lim_{n \rightarrow \infty} \mathbf{m}_{1,n} = 1$  and  $\mathbf{m}_{k,n} = \frac{(a_n)_k}{k!}$ ,  $a_n \in (0, 1]$ , where  $(x)_k = x(x+1)\dots(x+k-1)$  with  $(x)_0 = 1$  is the Pochhammer symbol.

*Mathematics subject classification* (2010): 41A36, 41A25.

*Keywords and phrases:* Bernstein-Stancu operator,  $q$ -integers, rate of convergence, moduli of continuity.

For the above special case of real sequence  $(m_{k,n})_{k=0}^{\infty}$ , the Bernstein-Stancu operators  $C_n$  were written in the Bernstein basis as follows (see [11], Theorem 10):

$$C_n(f; x) = \sum_{k=0}^n b_{n,k}(x) C_{k,n}[f] \tag{3}$$

where

$$C_{k,n}[f] = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} f\left(\frac{j}{n}\right) (a_n)_j (1 - a_n)_{k-j}$$

and

$$b_{n,k}(x) = \binom{n}{k} x^k (1 - x)^{n-k}.$$

We remark that if  $a_n \in (0, 1]$ , then  $C_n$  are linear positive operators.

The  $q$ -analogue of Stancu operators have attracted much interest, and a great number of interesting results have been obtained. Recently, G. Nowak [20] introduced a  $q$ -analogue of Stancu’s operators defined in (1). In 2010, O. Agratini [7] involving modulus of continuity and Lipschitz type maximal function obtained estimates for the rate of convergence of  $q$ -analogue of Stancu operators. Also, a probabilistic approach is given and some approximation properties are established. The approximation properties of  $q$ -Stancu operators were studied by Acar and Aral [1], Acu [4], Agrawal et al. [8], Aral et al. [10] and Nowak and Gupta [21]. The first results in this field have been achieved by A. Lupaş [17] and G.M. Phillips [22] who consider  $q$ -analogue of Bernstein operators. In the recent years, several researchers have made significant contribution in this area of approximation theory [2, 3, 5, 6, 9, 14, 16, 19].

First of all, we recall elements of  $q$ -Calculus, see, e.g., [15] and [18]. For any fixed real number  $q > 0$ , the  $q$ -integer  $[k]_q$ , for  $k \in \mathbb{N}$  is defined as

$$[k]_q = \begin{cases} (1 - q^k) / (1 - q), & q \neq 1 \\ k, & q = 1. \end{cases}$$

Set  $[0]_q = 0$ . The  $q$ -factorial  $[k]_q!$  and  $q$ -binomial coefficients  $\begin{bmatrix} n \\ k \end{bmatrix}_q$  are defined as follows

$$[k]_q! = \begin{cases} [k]_q [k - 1]_q \dots [1]_q, & k = 1, 2, \dots \\ 1, & k = 0 \end{cases}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n - k]_q!}, \text{ for } k \in \{0, 1, \dots, n\}.$$

The  $q$ -analogue of  $(x - a)^n$  is the polynomial

$$(x - a)_q^n = \begin{cases} 1, & n = 0 \\ (x - a)(x - qa) \dots (x - q^{n-1}a), & n \geq 1. \end{cases}$$

For  $f \in C[0, 1]$ ,  $q > 0$ ,  $\alpha \geq 0$  and  $n \in \mathbb{N}$ , Nowak (see [20]) in 2009 defined the  $q$ -Bernstein-Stancu operators as follows

$$S_n^{<q,\alpha>}(f;x) = \sum_{k=0}^n p_{n,k}^{<\alpha>}(x;q) f\left(\frac{[k]_q}{[n]_q}\right), x \in [0, 1],$$

with

$$p_{n,k}^{<\alpha>}(x;q) = \binom{n}{k}_q \frac{\prod_{i=0}^{k-1} (x + \alpha [i]_q) \prod_{j=0}^{n-k-1} (1 - q^j x + \alpha [j]_q)}{\prod_{i=0}^{n-1} (1 + \alpha [i]_q)}.$$

**THEOREM 1.1.** [20] *Let  $0 < q < 1$ ,  $\alpha \geq 0$ . Then*

$$\begin{aligned} S_n^{<q,\alpha>}(1;x) &= 1; \\ S_n^{<q,\alpha>}(t;x) &= x; \\ S_n^{<q,\alpha>}(t^2;x) &= \frac{1}{1 + \alpha} \left( x(x + \alpha) + \frac{x(1-x)}{[n]_q} \right), \end{aligned}$$

for all  $n \in \mathbb{N}$  and  $x \in [0, 1]$ .

For  $q = 1$ ,  $S_n^{<q,\alpha>}$  turns out to be the Bernstein-Stancu operators (1). For  $\alpha = 0$ ,  $S_n^{<q,\alpha>}$  reduces to  $q$ -Bernstein operators defined by Phillips [22]:

$$B_n^{<q>}(f;x) = \sum_{k=0}^n b_{n,k}(x;q) f\left(\frac{[k]_q}{[n]_q}\right), x \in [0, 1],$$

where  $b_{n,k}(x;q) = \binom{n}{k}_q x^k \prod_{j=0}^{n-k-1} (1 - q^j x)$ .

These operators verify

$$B_n^{<q>}(1;x) = 1, \quad B_n^{<q>}(t;x) = x, \quad B_n^{<q>}(t^2;x) = x^2 + \frac{x(1-x)}{[n]_q}. \tag{4}$$

**2. Construction of the generalized  $q$ -Bernstein-Stancu operators and the approximation properties**

Our aim is to introduce a  $q$ -analogue of the Bernstein-Stancu operators defined in (3). A  $q$ -analogue of  $C_n$  is in fact a  $q$ -deformation of Bernstein-Stancu operator. Furthermore, in this section we study some approximation properties of these operators.

Let us define on  $C[0,1]$  the linear positive operators  $C_n^{<q>}$  by

$$C_n^{<q>}(f;x) = \sum_{k=0}^n b_{n,k}(x;q) C_{k,n}[f;q], \text{ for all } x \in [0, 1], \tag{5}$$

with

$$C_{k,n}[f;q] = \sum_{j=0}^k c_{k,j} f\left(\frac{[j]_q}{[n]_q}\right), \quad 0 \leq k \leq n,$$

$$c_{k,j} = \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{\prod_{i=0}^{j-1} (a_n + [i]_q) \prod_{s=0}^{k-j-1} (1 - q^s a_n + [s]_q)}{\prod_{i=0}^{k-1} (1 + [i]_q)},$$

where  $a_n \in (0, 1], n \in \mathbb{N}$  and  $q > 0$ .

We note that  $c_{k,j} = p_{k,j}^{<1>}(a_n; q)$ . Therefore,  $C_{k,n}[f; q] = S_k^{<q, 1>}(\tilde{f}; a_n)$ , where  $\tilde{f}(t) = f\left(t \frac{[k]_q}{[n]_q}\right)$ .

LEMMA 2.1. *The  $q$ -Bernstein-Stancu operators  $C_n^{<q>}$  given by (5) verify the following identities*

- i)  $C_n^{<q>}(e_0; x) = 1,$
- ii)  $C_n^{<q>}(e_1; x) = a_n x,$
- iii)  $C_n^{<q>}(e_2; x) = \frac{a_n(a_n + 1)}{2} \left(1 - \frac{1}{[n]_q}\right) x^2 + \frac{a_n}{[n]_q} x,$

where  $e_j(x) = x^j, j = 0, 1, 2$  are the test functions.

*Proof.* From Theorem 1.1 we have

$$\begin{aligned} \sum_{j=0}^k c_{k,j} &= 1; \\ \sum_{j=0}^k c_{k,j} \frac{[j]_q}{[k]_q} &= a_n; \\ \sum_{j=0}^k c_{k,j} \frac{[j]_q^2}{[k]_q^2} &= \frac{1}{2} \left( a_n(a_n + 1) + \frac{a_n(1 - a_n)}{[k]_q} \right). \end{aligned}$$

Therefore

$$\begin{aligned} C_{k,n}[e_0; q] &= 1; \quad C_{k,n}[e_1; q] = \frac{[k]_q}{[n]_q} a_n; \\ C_{k,n}[e_2; q] &= \frac{1}{2} \frac{[k]_q^2}{[n]_q^2} \left( a_n(a_n + 1) + \frac{a_n(1 - a_n)}{[k]_q} \right). \end{aligned}$$

Using the properties (4) of the  $q$ -Bernstein operators, the values of the operator  $C_n^{<q>}$  for test functions are obtained.  $\square$

REMARK 2.1. We obtain the following values for the central moments of the  $q$ -Bernstein-Stancu operators

$$\begin{aligned} C_n^{<q>}(t - x; x) &= C_n^{<q>}(e_1; x) - x C_n^{<q>}(e_0; x) = (a_n - 1)x, \\ C_n^{<q>}((t - x)^2; x) &= C_n^{<q>}(e_2; x) - 2x C_n^{<q>}(e_1; x) + x^2 C_n^{<q>}(e_0; x) \\ &= \left( \frac{a_n(a_n + 1)([n]_q - 1)}{2[n]_q} - 2a_n + 1 \right) x^2 + \frac{a_n}{[n]_q} x. \end{aligned}$$

LEMMA 2.2. For  $f \in C[0, 1]$  we have  $\|C_n^{<q>}\| \leq \|f\|$ .

*Proof.* From the definition of the operator and using Lemma 2.1 we have

$$C_n^{<q>}(f; x) = \sum_{k=0}^n b_{n,k}(x; q) \sum_{j=0}^k c_{k,j} \left| f\left(\frac{[j]_q}{[n]_q}\right) \right| \leq \|f\| C_n^{<q>}(e_0; x) \leq \|f\|.$$

□

LEMMA 2.3. For all  $x \in [0, 1]$  we have

$$C_n^{<q>}\left((t-x)^2; x\right) \leq \frac{a_n}{[n]_q} x(1-x) + (1-a_n).$$

*Proof.* Using Remark 2.1, we get

$$\begin{aligned} C_n^{<q>}\left((t-x)^2; x\right) &= \left(\frac{a_n(a_n+1)([n]_q-1)}{2[n]_q} - 2a_n + 1\right)x^2 + \frac{a_n}{[n]_q}x \\ &= \frac{a_n}{[n]_q}x(1-x) + \left(\frac{a_n(a_n+1)([n]_q-1)}{2[n]_q} - 2a_n + 1 + \frac{a_n}{[n]_q}\right)x^2 \\ &= \frac{a_n}{[n]_q}x(1-x) + \frac{(a_n^2 - 3a_n + 2)[n]_q + a_n(1-a_n)}{2[n]_q}x^2 \\ &= \frac{a_n}{[n]_q}x(1-x) + \frac{(1-a_n)}{2} \left\{ (2-a_n) + \frac{a_n}{[n]_q} \right\} x^2 \\ &\leq \frac{a_n}{[n]_q}x(1-x) + (1-a_n). \end{aligned}$$

□

LEMMA 2.4. Let  $(q_n)_{n \geq 1}$ ,  $(a_n)_{n \geq 1}$  be real sequences such that  $0 < q_n < 1$ ,  $a_n \in (0, 1]$ ,  $n \in \mathbb{N}$ . If

$$\lim_{n \rightarrow \infty} q_n = 1, \lim_{n \rightarrow \infty} a_n = 1 \text{ and } \lim_{n \rightarrow \infty} [n]_{q_n}(1-a_n) = a \in \mathbb{R}, \tag{6}$$

then

$$\begin{aligned} \lim_{n \rightarrow \infty} [n]_{q_n} C_n^{<q_n>}(t-x; x) &= -ax, \\ \lim_{n \rightarrow \infty} [n]_{q_n} C_n^{<q_n>}\left((t-x)^2; x\right) &= x(1-x) + a \frac{x^2}{2}. \end{aligned}$$

*Proof.* The results follow from Remark 2.1. □

REMARK 2.2. The sequences  $(q_n)_{n \geq 1}$ ,  $(a_n)_{n \geq 1}$ ,  $a_n = q_n = 1 - \frac{1}{n}$  verify the conditions from the previous Lemma, namely

$$\lim_{n \rightarrow \infty} q_n = \lim_{n \rightarrow \infty} a_n = 1 \text{ and } \lim_{n \rightarrow \infty} [n]_{q_n}(1-a_n) = 1 - \frac{1}{e}.$$

### 3. Direct theorems

We investigate the approximation properties of these operators and we estimate the rate of convergence by using moduli of continuity. For the classical Bernstein-Stancu operators  $C_n$  similar results was obtained in [23].

**THEOREM 3.1.** *Let  $(q_n)_{n \geq 1}$ ,  $(a_n)_{n \geq 1}$  be real sequences such that  $q_n \in (0, 1)$ ,  $a_n \in (0, 1]$ . If  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} q_n = 1$  and  $f \in C[0, 1]$ , then*

$$\lim_{n \rightarrow \infty} C_n^{<q_n>}(f; x) = f(x) \text{ uniformly on } [0, 1].$$

*Proof.* Using Lemma 2.1 follows that

$$\lim_{n \rightarrow \infty} C_n^{<q_n>}(e_k; x) = e_k(x) \text{ uniformly on } [0, 1], \text{ for } k \in \{0, 1, 2\}.$$

Applying the Bohmann-Korovkin theorem, we get the result.  $\square$

The usual modulus of continuity for  $f \in C[0, 1]$  gives the maximum oscillation of  $f$  in any interval of length not exceeding  $\delta > 0$  and is defined as

$$\omega(f; \delta) = \sup_{|y-x| \leq \delta} |f(y) - f(x)|, \quad x, y \in [0, 1].$$

It is known that the modulus of continuity of  $f$  has the following properties

$$\omega(f; \lambda \delta) \leq (1 + \lambda) \omega(f; \delta)$$

and

$$|f(t) - f(x)| \leq \omega(f; \delta) \left( \frac{(t-x)^2}{\delta^2} + 1 \right).$$

Our next result is the following local theorem.

**THEOREM 3.2.** *Let  $(q_n)_{n \geq 1}$ ,  $(a_n)_{n \geq 1}$  be real sequences such that  $q_n \in (0, 1)$ ,  $a_n \in (0, 1]$ . If  $f \in C[0, 1]$ , then*

$$|C_n^{<q_n>}(f; x) - f(x)| \leq \frac{5}{4} \omega(f; \sqrt{\delta_n}),$$

where  $\delta_n = \frac{a_n}{[n]_{q_n}} + 4(1 - a_n)$ .

*Proof.* We have

$$|C_n^{<q_n>}(f; x) - f(x)| \leq C_n^{<q_n>}(|f(t) - f(x)|; x) \leq \omega(f; \delta) \left( 1 + \frac{1}{\delta^2} C_n^{<q_n>}((t-x)^2; x) \right).$$

Using Lemma 2.3, we can write

$$\begin{aligned} |C_n^{<q_n>}(f;x) - f(x)| &\leq \omega(f;\delta) \left[ 1 + \frac{1}{\delta^2} \left( \frac{a_n}{[n]_{q_n}} x(1-x) + (1-a_n) \right) \right] \\ &\leq \omega(f;\delta) \left[ 1 + \frac{1}{4\delta^2} \left( \frac{a_n}{[n]_{q_n}} + 4(1-a_n) \right) \right]. \end{aligned}$$

So, if we choose  $\delta = \sqrt{\delta_n}$ , we have the desired result.  $\square$

**THEOREM 3.3.** *Let  $(q_n)_{n \geq 1}$ ,  $(a_n)_{n \geq 1}$  be real sequences such that  $q_n \in (0, 1)$ ,  $a_n \in (0, 1]$ . If  $f \in C^1[0, 1]$ , then*

$$|C_n^{<q_n>}(f;x) - f(x)| \leq \lambda_n(x)|f'(x)| + 2\sqrt{\delta_n(x)}\omega(f', \sqrt{\delta_n(x)}),$$

where

$$\lambda_n(x) = (1 - a_n)x, \delta_n(x) = \frac{a_n}{[n]_{q_n}}\phi^2(x) + (1 - a_n) \text{ and } \phi^2(x) = x(1 - x). \tag{7}$$

*Proof.* Let  $f \in C^1[0, 1]$ . For any  $x, t \in [0, 1]$ , we have

$$f(t) - f(x) = f'(x)(t - x) + \int_x^t (f'(u) - f'(x)) du,$$

so, we get

$$C_n^{<q_n>}(f(t) - f(x);x) = f'(x)C_n^{<q_n>}(t - x;x) + C_n^{<q_n>}\left(\int_x^t (f'(u) - f'(x))du;x\right).$$

Using the following well known property of modulus of continuity

$$|f(y) - f(x)| \leq \omega(f;\delta) \left( \frac{|y-x|}{\delta} + 1 \right), \quad \delta > 0,$$

we have

$$\left| \int_x^t |f'(u) - f'(x)| du \right| \leq \omega(f';\delta) \left[ \frac{(t-x)^2}{\delta} + |t-x| \right].$$

Therefore,

$$\begin{aligned} |C_n^{<q_n>}(f;x) - f(x)| &\leq |f'(x)| \cdot |C_n^{<q_n>}(t - x;x)| \\ &\quad + \omega(f';\delta) \left\{ \frac{1}{\delta} C_n^{<q_n>}((t-x)^2;x) + C_n^{<q_n>}(|t-x|;x) \right\}. \end{aligned}$$

Using Cauchy-Schwartz inequality

$$C_n^{<q_n>}(|t-x|;x) \leq \sqrt{C_n^{<q_n>}(1;x)} \cdot \sqrt{C_n^{<q_n>}((t-x)^2;x)}$$

we obtain

$$|C_n^{<q_n>}(f;x) - f(x)| \leq |f'(x)| \cdot |C_n^{<q_n>}(t-x;x)| + \omega(f', \delta) \left\{ \frac{1}{\delta} \sqrt{C_n^{<q_n>}((t-x)^2;x)} + 1 \right\} \sqrt{C_n^{<q_n>}((t-x)^2;x)}.$$

Applying Lemma 2.3, we get

$$|C_n^{<q_n>}(f;x) - f(x)| \leq |f'(x)|(1-a_n)x + \omega(f', \delta) \cdot \left\{ \frac{1}{\delta} \sqrt{\frac{a_n}{[n]_{q_n}} \phi^2(x) + (1-a_n)} + 1 \right\} \cdot \sqrt{\frac{a_n}{[n]_{q_n}} \phi^2(x) + (1-a_n)}.$$

Choosing  $\delta = \sqrt{\delta_n(x)}$ , we find the desired inequality.  $\square$

Let

$$Lip_M(\gamma) = \{f \in C[0, 1], |f(t) - f(x)| \leq M|t-x|^\gamma\}, 0 < \gamma \leq 1$$

be the class of Lipschitz functions. The next result gives the rate of convergence of the operators  $C_n^{<q_n>}$  in terms of the Lipschitz class.

**THEOREM 3.4.** *Let  $(q_n)_{n \geq 1}, (a_n)_{n \geq 1}$  be real sequences such that  $q_n \in (0, 1), a_n \in (0, 1]$ . If  $f \in Lip_M(\gamma)$ , then*

$$|C_n^{<q_n>}(f;x) - f(x)| \leq M(\delta_n(x))^{\gamma/2},$$

where  $\delta_n(x)$  is defined in (7).

*Proof.* Since  $C_n^{<q_n>}(e_0; \cdot) = e_0$  and  $f \in Lip_M(\gamma)$ , we have

$$\begin{aligned} |C_n^{<q_n>}(f;x) - f(x)| &\leq \sum_{k=0}^n b_{n,k}(x; q_n) \sum_{j=0}^k c_{k,j} \left| f\left(\frac{[j]_{q_n}}{[n]_{q_n}}\right) - f(x) \right| \\ &\leq M \sum_{k=0}^n b_{n,k}(x; q_n) \sum_{j=0}^k c_{k,j} \left| \frac{[j]_{q_n}}{[n]_{q_n}} - x \right|^\gamma. \end{aligned}$$



Applying the Hölder’s inequality with  $p = \frac{2}{\gamma}$  and  $q = \frac{2}{2-\gamma}$ , we get

$$\begin{aligned} |C_n^{<q_n>}(f;x) - f(x)| &\leq M \sum_{k=0}^n b_{n,k}(x;q_n) \left( \sum_{j=0}^k c_{k,j} \right)^{\frac{2-\gamma}{2}} \left[ \sum_{j=0}^k c_{k,j} \left( \frac{[j]_{q_n}}{[n]_{q_n}} - x \right)^2 \right]^{\frac{\gamma}{2}} \\ &= M \sum_{k=0}^n b_{n,k}(x;q_n) \left[ \sum_{j=0}^k c_{k,j} \left( \frac{[j]_{q_n}}{[n]_{q_n}} - x \right)^2 \right]^{\frac{\gamma}{2}} \\ &= M \sum_{k=0}^n (b_{n,k}(x;q_n))^{\frac{2-\gamma}{2}} \left[ b_{n,k}(x;q_n) \sum_{j=0}^k c_{k,j} \left( \frac{[j]_{q_n}}{[n]_{q_n}} - x \right)^2 \right]^{\frac{\gamma}{2}} \\ &\leq M \left( \sum_{k=0}^n b_{n,k}(x;q_n) \right)^{\frac{2-\gamma}{2}} \left[ \sum_{k=0}^n b_{n,k}(x;q_n) \sum_{j=0}^k c_{k,j} \left( \frac{[j]_{q_n}}{[n]_{q_n}} - x \right)^2 \right]^{\frac{\gamma}{2}} \\ &= M \{C_n^{<q_n>}((t-x)^2;x)\}^{\frac{\gamma}{2}} \leq M(\delta_n(x))^{\gamma/2}. \end{aligned}$$

□

In order to give the next result we recall the definition of K-functional:

$$K_2(f, \delta) := \inf \{ \|f - g\| + \delta \|g''\| : g \in W^2 \},$$

where

$$W^2 = \{ g \in C[0, 1] : g'' \in C[0, 1] \},$$

$\delta \geq 0$  and  $\|\cdot\|$  is the uniform norm on  $C[0, 1]$ . The second order modulus of continuity is defined as follows

$$\omega_2(f, \sqrt{\delta}) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{x, x+2h \in [0, 1]} \{|f(x+2h) - 2f(x+h) + f(x)|\}.$$

It is well known that K-functional and the second order modulus of continuity  $\omega_2(f, \sqrt{\delta})$  are equivalent, namely

$$K_2(f, \delta) \leq C \omega_2(f, \sqrt{\delta}), \tag{8}$$

where  $\delta \geq 0$  and  $C > 0$ .

**THEOREM 3.5.** *If  $f \in C[0, 1]$  and  $(q_n)_{n \geq 1}, (a_n)_{n \geq 1}$  are real sequences such that  $q_n \in (0, 1), a_n \in (0, 1]$ , then*

$$|C_n^{<q_n>}(f;x) - f(x)| \leq C \omega_2(f, \sqrt{\delta_n(x)}) + \omega(f, \lambda_n(x)),$$

where  $\delta_n(x)$  and  $\lambda_n(x)$  are defined in (7).

*Proof.* In order to prove the inequality, we construct the following auxiliary operators:

$$\tilde{C}_n^{<q_n>}(f; x) = C_n^{<q_n>}(f; x) + f(x) - f(a_n x). \tag{9}$$

It is not difficult to see that

$$\begin{aligned} \tilde{C}_n^{<q_n>}(e_0; x) &= C_n^{<q_n>}(e_0; x) = 1 \\ \tilde{C}_n^{<q_n>}(e_1; x) &= C_n^{<q_n>}(e_1; x) + x - a_n x = x. \end{aligned}$$

From Taylor’s formula we have

$$g(t) = g(x) + g'(x)(t - x) + \int_x^t (t - u)g''(u)du, \quad g \in W^2.$$

Applying  $\tilde{C}_n^{<q_n>}$  to above relation, it follows

$$\tilde{C}_n^{<q_n>}(g; x) = g(x) + \tilde{C}_n^{<q_n>}\left(\int_x^t (t - u)g''(u)du; x\right).$$

Therefore

$$\tilde{C}_n^{<q_n>}(g; x) = g(x) + C_n^{<q_n>}\left(\int_x^t (t - u)g''(u)du; x\right) - \int_x^{a_n x} (a_n x - u)g''(u)du.$$

This implies that

$$\begin{aligned} |\tilde{C}_n^{<q_n>}(g; x) - g(x)| &\leq \left| C_n^{<q_n>}\left(\int_x^t (t - u)g''(u)du; x\right) \right| + \left| \int_x^{a_n x} (a_n x - u)g''(u)du \right| \\ &\leq C_n^{<q_n>}((t - x)^2; x) \|g''\| + (a_n x - x)^2 \|g''\|. \end{aligned}$$

From Lemma 2.3 it follows

$$|\tilde{C}_n^{<q_n>}(g; x) - g(x)| \leq \delta_n(x) \|g''\| + \lambda_n^2(x) \|g''\|.$$

Since  $(1 - a_n)\delta_n(x) = \frac{(1 - a_n)a_n}{[n]_q} \phi^2(x) + (1 - a_n)^2 \geq (1 - a_n)^2 \geq \lambda_n^2(x)$ , we get

$$|\tilde{C}_n^{<q_n>}(g; x) - g(x)| \leq (2 - a_n)\delta_n(x) \|g''\| \leq 2\delta_n(x) \|g''\|. \tag{10}$$

In view of (9) and Lemma 2.2 we obtain

$$|\tilde{C}_n^{<q_n>}(f; x)| \leq |C_n^{<q_n>}(f; x)| + |f(x)| + |f(a_n x)| \leq 3 \|f\|. \tag{11}$$

Now, for  $f \in C[0, 1]$  and  $g \in W^2$ , using (9), (10) and (11) we obtain

$$\begin{aligned} |C_n^{<q_n>}(f; x) - f(x)| &= |\tilde{C}_n^{<q_n>}(f; x) - f(x) + f(a_n x) - f(x)| \\ &\leq |\tilde{C}_n^{<q_n>}(f - g; x)| + |\tilde{C}_n^{<q_n>}(g; x) - g(x)| + |g(x) - f(x)| + |f(a_n x) - f(x)| \\ &\leq 4 \|f - g\| + 2\delta_n(x) \|g''\| + \omega(f, |(1 - a_n)x|). \end{aligned}$$

Taking the infimum on the right side over all  $g \in W^2$ , we have

$$|C_n^{<q_n>}(f; x) - f(x)| \leq 4K_2(f, \delta_n(x)) + \omega(f, |(1 - a_n)x|).$$

Finally, using the equivalence between  $K$ -functional and the second order modulus of continuity (8), we have

$$|C_n^{<q_n>}(f; x) - f(x)| \leq C\omega_2\left(f, \sqrt{\delta_n(x)}\right) + \omega(f, (1 - a_n)x),$$

which completes the proof.  $\square$

Let  $\phi(x) = \sqrt{x(1-x)}$  and  $f \in C[0, 1]$ . The Ditzian-Totik first order modulus of smoothness is given by (see [13])

$$\omega_\phi(f; t) = \sup_{0 < h \leq t} \left\{ \left| f\left(x + \frac{h\phi(x)}{2}\right) - f\left(x - \frac{h\phi(x)}{2}\right) \right|, x \pm \frac{h\phi(x)}{2} \in [0, 1] \right\}. \tag{12}$$

The corresponding  $K$ -functional to (12) is defined by

$$K_\phi(f; t) = \inf_{g \in W_\phi[0,1]} \{ \|f - g\| + t\|\phi g'\| \} \quad (t > 0), \tag{13}$$

where  $W_\phi[0, 1] = \{g : g \in AC[0, 1], \|\phi g'\| < \infty\}$  and  $AC[0, 1]$  is the class of all absolutely continuous functions on  $[0, 1]$ . It is well known ([13], p.11) that there exists a constant  $C > 0$  such that

$$K_\phi(f; t) \leq C\omega_\phi(f; t). \tag{14}$$

A direct approximation theorem by means of Ditzian-Totik modulus of smoothness is given in the next result.

**THEOREM 3.6.** *If  $f \in C[0, 1]$  and  $(q_n)_{n \geq 1}$ ,  $(a_n)_{n \geq 1}$  are real sequences such that  $q_n \in (0, 1)$ ,  $a_n \in (0, 1]$ , then*

$$|C_n^{<q_n>}(f; x) - f(x)| < C\omega_\phi\left(f; \frac{\sqrt{\delta_n(x)}}{\phi(x)}\right), \tag{15}$$

where  $\delta_n(x)$  is defined in (7) and  $C$  is a constant independent of  $n$  and  $x$ .

*Proof.* Using the relation

$$g(t) = g(x) + \int_x^t g'(u)du,$$

we get

$$|C_n^{<q_n>}(g; x) - g(x)| = \left| C_n^{<q_n>}\left(\int_x^t g'(u)du; x\right) \right|. \tag{16}$$

For any  $x, t \in (0, 1)$ , we find that

$$\begin{aligned} \left| \int_x^t g'(u) du \right| &\leq \|\phi g'\| \left| \int_x^t \frac{1}{\phi(u)} du \right| \leq \|\phi g'\| \left| \int_x^t \left( \frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1-u}} \right) du \right| \\ &\leq 2\|\phi g'\| \left( |\sqrt{t} - \sqrt{x}| + |\sqrt{1-t} - \sqrt{1-x}| \right) \\ &= 2\|\phi g'\| |t-x| \left( \frac{1}{\sqrt{t} + \sqrt{x}} + \frac{1}{\sqrt{1-t} + \sqrt{1-x}} \right) \\ &< 2\|\phi g'\| |t-x| \left( \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}} \right) \leq \frac{2\sqrt{2}\|\phi g'\| |t-x|}{\phi(x)}. \end{aligned} \quad (17)$$

Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |C_n^{<q_n>}(g;x) - g(x)| &< 2\sqrt{2}\|\phi g'\| \|\phi^{-1}(x) C_n^{<q_n>}(|t-x|;x) \\ &\leq 2\sqrt{2}\|\phi g'\| \|\phi^{-1}(x) \left( C_n^{<q_n>}((t-x)^2;x) \right)^{1/2}. \end{aligned}$$

Applying Lemma 2.3, we get

$$|C_n^{<q_n>}(g;x) - g(x)| < 2\sqrt{2} \frac{\|\phi g'\|}{\phi(x)} \sqrt{\delta_n(x)}. \quad (18)$$

Therefore we can write

$$\begin{aligned} |C_n^{<q_n>}(f;x) - f(x)| &\leq |C_n^{<q_n>}(f-g;x)| + |f(x) - g(x)| + |C_n^{<q_n>}(g;x) - g(x)| \\ &\leq 2\|f-g\| + \frac{2\sqrt{2}\|\phi g'\|}{\phi(x)} \sqrt{\delta_n(x)} \\ &\leq 2\sqrt{2} \left\{ \|f-g\| + \frac{\|\phi g'\|}{\phi(x)} \sqrt{\delta_n(x)} \right\}. \end{aligned} \quad (19)$$

Taking infimum on the right hand side of the above inequality over all  $g \in W_\phi[0, 1]$ , we get

$$|C_n^{<q_n>}(f;x) - f(x)| < CK_\phi \left( f; \frac{\sqrt{\delta_n(x)}}{\phi(x)} \right).$$

Using the relation (14) this theorem is proven.  $\square$

#### 4. Numerical examples

In order to show the relevance of the operators  $C_n^{<q>}$ , in this section are given some numerical examples regarding the approximation properties. Furthermore, we compare the convergence of the operators  $C_n^{<q>}$  with the  $q$ -Bernstein-Stancu operators proposed by Nowak [20]. From these results follows that for certain functions, the operators introduced in this paper  $C_n^{<q>}$  converge faster than the  $q$ -Bernstein-Stancu operators  $S_n^{<q,1>}$ .

EXAMPLE 4.1. Let  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = \frac{x^4}{12} - \frac{x^3}{6} + \frac{3x^2}{32}$ . For  $q = 0.8$ ,  $a_n = 1 - \frac{1}{n^2}$  the convergence of the operators  $C_n^{<q>}$  to the function  $f$  is illustrated in Figure 1. We note that if the sequence  $(a_n)$  converges to 1 the operators  $C_n^{<q>}$  are going to the graph of the function  $f$ .

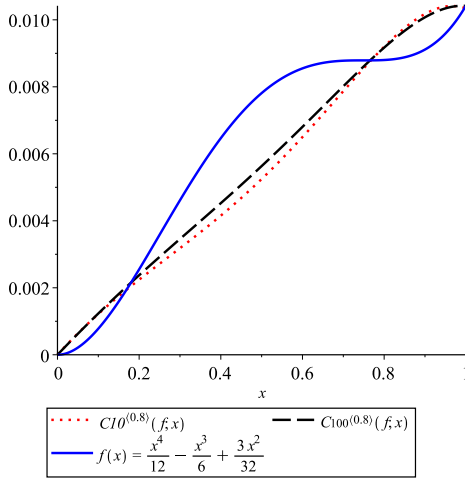


Figure 1: Approximation process by  $C_n^{<q>}$

EXAMPLE 4.2. We consider  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = -\frac{4x^4}{3} + \frac{5x^3}{3} - \frac{x^2}{2}$ . For  $n = 100$ ,  $a_n = 1 - \frac{1}{n^2}$  and  $q \in \{0.7; 0.8; 0.9\}$  the convergence of the operators  $C_n^{<q>}$  to the function  $f$  is illustrated in Figure 2. We note that if the sequence  $(q_n)$  converges to 1 the operators  $C_n^{<q>}$  are going to the graph of the function  $f$ .

EXAMPLE 4.3. We consider  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = \sin(2\pi x)$ . For  $q = 0.8$ ,  $n = 10$  and  $a_n = 1 - \frac{1}{n^2}$  the convergence of the operators  $S_n^{<q,1>}$ ,  $C_n^{<q>}$  to the function  $f$  is illustrated in Figure 3. From this graph follows that the operator  $C_n^{<q>}$  introduced in this paper converges faster than the Stancu operator  $S_n^{<q,1>}$  for this particular choice of function  $f$ . A similar example is given in Figure 4 for the function  $f(x) = x^3 e^{x+1}$ .

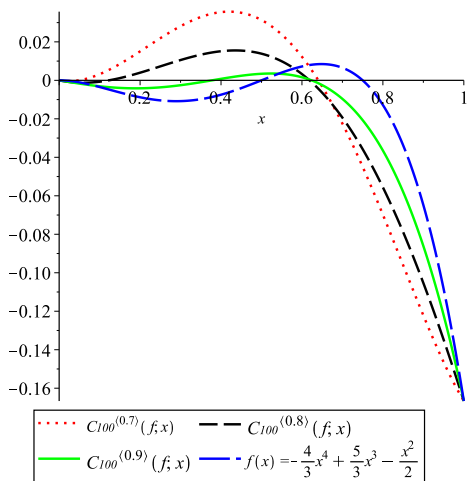


Figure 2: Approximation process by  $C_n^{<q>}$

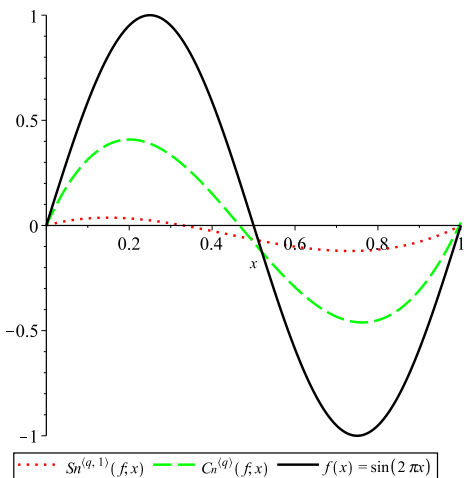


Figure 3: Approximation process by  $S_n^{<q,1>}$ ,  $C_n^{<q>}$

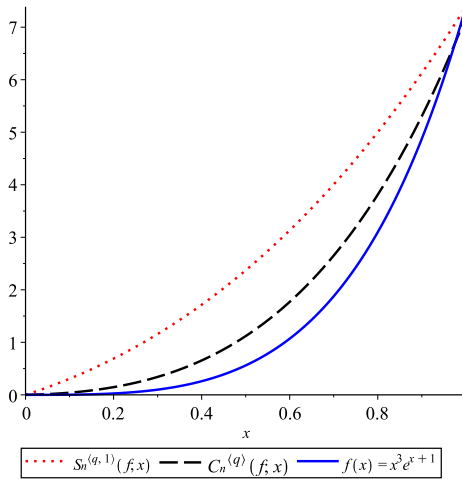


Figure 4: Approximation process by  $S_n^{<q,1>}$ ,  $C_n^{<q>}$

*Acknowledgement.* Project financed from Lucian Blaga University of Sibiu research grants LBUS-IRG-2018-04.

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(Received August 6, 2018)

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